

ON THE HERMITE-HADAMARD INTEGRAL INEQUALITY IN THE CONTEXT OF INVERSE FUNCTIONS

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Abstract

The Hermite-Hadamard integral inequality is one of the most well-known integral inequalities in the theory of convex functions. This article contributes to the subject by examining the Hermite-Hadamard integral inequality in the context of inverse functions. In addition, alternative proofs are presented to highlight the subtlety and complexity of the problem.

1. Introduction

Convex and concave functions are among the most fundamental and extensively studied classes of functions in mathematics. Their formal

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definitions are described below. Let $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$ such that $b > a$. Then $f : [a, b] \rightarrow \mathbb{R}$ is a convex function if, for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

Similarly, $f : [a, b] \rightarrow \mathbb{R}$ is a concave function if, for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$, we have

$$\lambda f(x) + (1 - \lambda)f(y) \leq f(\lambda x + (1 - \lambda)y).$$

These classes of functions give rise to an important family of results known as convex integral inequalities, which have numerous theoretical and practical applications.

The most prominent examples include the Jensen integral inequality and the Hermite-Hadamard integral inequality. The latter, in particular, provides elegant lower and upper bounds for the integral mean of a convex function, as described below. Let $a, b \in \mathbb{R}$ such that $b > a$. Then, assuming that $f : [a, b] \rightarrow \mathbb{R}$ is a convex function, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.$$

Similarly, assuming that $f : [a, b] \rightarrow \mathbb{R}$ is a concave function, we have

$$\frac{f(a) + f(b)}{2} \leq \frac{1}{b-a} \int_a^b f(x)dx \leq f\left(\frac{a+b}{2}\right).$$

Convex integral inequalities have inspired a wide range of generalizations, refinements, and extensions in modern analysis. They continue to play a central role in diverse areas of mathematics, including numerical analysis, probability theory, and information theory. For further developments and detailed discussions, see [11, 12, 13, 14, 1, 8, 2, 15, 16, 19, 17, 9, 10, 5, 6, 7, 4, 3, 18].

This article continues in this direction. It is devoted to the study of the Hermite-Hadamard integral inequality in the context of inverse functions, f^{-1} . Alongside the main result, it provides several alternative proofs. These demonstrate the complexity of the problem, the variety of techniques that can be employed and the nuanced relationship between convexity and inversion. A connection is made with the Young integral inequality. The case of the product of two inverse functions is also considered.

The rest of the article is organized as follows: The next section is dedicated to the main theorem. Section 3 is devoted to some complementary results. Section 4 provides a brief conclusion, along with perspectives for future research.

2. Main Theorem

Our main theorem is given below, followed by multiple proofs.

Theorem 2.1. *Let $a, b \in \mathbb{R}$ with $b > a$, and $f : \mathbb{R} \rightarrow [a, b]$ be a strictly increasing convex function. Then we have*

$$\frac{f^{-1}(a) + f^{-1}(b)}{2} \leq \frac{1}{b-a} \int_a^b f^{-1}(x) dx \leq f^{-1}\left(\frac{a+b}{2}\right).$$

Proof. The main contribution of this article is the provision of multiple proofs for the lower and upper bounds established in this theorem.

Proof 1 (Lower and upper bounds). If f is a convex function, then, for any $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

For any $s, t \in [a, b]$, if we take $x = f^{-1}(s)$ and $y = f^{-1}(t)$, then we have

$$f(\lambda f^{-1}(s) + (1 - \lambda)f^{-1}(t)) \leq \lambda f(f^{-1}(s)) + (1 - \lambda)f(f^{-1}(t)),$$

so that

$$f(\lambda f^{-1}(s) + (1 - \lambda)f^{-1}(t)) \leq \lambda s + (1 - \lambda)t.$$

Since f is strictly increasing, f^{-1} is also strictly increasing. Therefore, we have

$$f^{-1}(f(\lambda f^{-1}(s) + (1 - \lambda)f^{-1}(t))) \leq f^{-1}(\lambda s + (1 - \lambda)t),$$

so that

$$\lambda f^{-1}(s) + (1 - \lambda)f^{-1}(t) \leq f^{-1}(\lambda s + (1 - \lambda)t).$$

This inequality implies that f^{-1} is concave. Applying the Hermite-Hadamard integral inequality to the concave function f^{-1} , we directly obtain

$$\frac{f^{-1}(a) + f^{-1}(b)}{2} \leq \frac{1}{b - a} \int_a^b f^{-1}(x) dx \leq f^{-1}\left(\frac{a + b}{2}\right).$$

The desired bounds are obtained.

Proof 2 (upper bound). An alternative proof to the upper bound is now given. Since f is convex, the Jensen integral inequality applied to f gives

$$\begin{aligned} f\left(\frac{1}{b - a} \int_a^b f^{-1}(x) dx\right) &\leq \frac{1}{b - a} \int_a^b f(f^{-1}(x)) dx = \frac{1}{b - a} \int_a^b x dx \\ &= \frac{1}{b - a} \left[\frac{x^2}{2} \right]_a^b = \frac{1}{b - a} \frac{b^2 - a^2}{2} = \frac{a + b}{2}. \end{aligned}$$

Since f is strictly increasing, f^{-1} is also strictly increasing. Therefore, we have

$$f^{-1}\left(f\left(\frac{1}{b-a}\int_a^b f^{-1}(x)dx\right)\right) \leq f^{-1}\left(\frac{a+b}{2}\right),$$

so that

$$\frac{1}{b-a}\int_a^b f^{-1}(x)dx \leq f^{-1}\left(\frac{a+b}{2}\right).$$

The desired bound is obtained.

Proof 3 (lower bound). An alternative proof to the lower bound is now given assuming that f is differentiable. Using the change of variables $x = f(u)$ and making an integration by parts, we get

$$\begin{aligned} \frac{1}{b-a}\int_a^b f^{-1}(x)dx &= \frac{1}{b-a}\int_{f^{-1}(a)}^{f^{-1}(b)} uf'(u)du \\ &= \frac{1}{b-a}\left([uf(u)]_{f^{-1}(a)}^{f^{-1}(b)} - \int_{f^{-1}(a)}^{f^{-1}(b)} f(u)du\right) \\ &= \frac{1}{b-a}\left(bf^{-1}(b) - af^{-1}(a) - \int_{f^{-1}(a)}^{f^{-1}(b)} f(u)du\right). \end{aligned}$$

Since f is convex, applying the right-hand side of the Hermite-Hadamard integral inequality to f at $f^{-1}(a)$ and $f^{-1}(b)$, we get

$$\frac{1}{f^{-1}(b) - f^{-1}(a)} \int_{f^{-1}(a)}^{f^{-1}(b)} f(u)du \leq \frac{f(f^{-1}(a)) + f(f^{-1}(b))}{2} = \frac{a+b}{2},$$

so that, since $f^{-1}(b) > f^{-1}(a)$,

$$\int_{f^{-1}(a)}^{f^{-1}(b)} f(u)du \leq \frac{a+b}{2} (f^{-1}(b) - f^{-1}(a)).$$

Therefore, combining the above inequalities, we obtain

$$\begin{aligned}
\frac{1}{b-a} \int_a^b f^{-1}(x) dx &\geq \frac{1}{b-a} \left(bf^{-1}(b) - af^{-1}(a) - \frac{a+b}{2} (f^{-1}(b) - f^{-1}(a)) \right) \\
&= \frac{1}{b-a} \left(\frac{b-a}{2} (f^{-1}(b) + f^{-1}(a)) \right) = \frac{f^{-1}(b) + f^{-1}(a)}{2}.
\end{aligned}$$

The desired bound is obtained.

This concludes the proof of the theorem. $\square$

One can prove that Theorem 2.1 also holds if f is a strictly decreasing concave function rather than a strictly increasing convex function.

In Theorem 2.1, we assumed $f : \mathbb{R} \rightarrow [a, b]$. However, the result extends without difficulty to functions $f : [c, d] \rightarrow [a, b]$, where $c, d \in \mathbb{R} \cup \{-\infty, \infty\}$ with $d > c$.

The natural concave counterpart of Theorem 2.1 is the theorem below.

Theorem 2.2. *Let $a, b \in \mathbb{R}$ with $b > a$, and $f : \mathbb{R} \rightarrow [a, b]$ be a strictly increasing concave function. Then we have*

$$f^{-1}\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f^{-1}(x) dx \leq \frac{f^{-1}(a) + f^{-1}(b)}{2}.$$

The proof is omitted, as it follows along the same lines as that of Theorem 2.1, with appropriate modifications to account for concavity instead of convexity.

One can prove that Theorem 2.2 also holds if f is a strictly decreasing convex function rather than a strictly increasing concave function.

3. Complements

3.1. An alternative to the Young integral inequality

The theorem below presents a lower bound involving the main integral term of the Young integral inequality.

Theorem 3.1. *Let $a, b, c > 0$ with $a < c$, and $f : [0, c] \rightarrow \mathbb{R}$ be a strictly increasing convex function satisfying $f(0) = 0$ and $b < f(c)$.*

Then we have

$$\max\left(af\left(\frac{a}{2}\right) + \frac{b}{2}f^{-1}(b), ab\right) \leq \int_0^a f(x)dx + \int_0^b f^{-1}(x)dx.$$

Proof. It follows from Theorem 2.1 applied to $a = 0$ that

$$\frac{f^{-1}(0) + f^{-1}(b)}{2} \leq \frac{1}{b-0} \int_0^b f^{-1}(x)dx.$$

Since $f^{-1}(0) = 0$ and $b > 0$, we have

$$\frac{b}{2}f^{-1}(b) \leq \int_0^b f^{-1}(x)dx.$$

The left-hand side of the Hermite-Hadamard integral inequality applied to the convex function f with “ $b = a$ ” and “ $a = 0$ ” gives

$$f\left(\frac{a+0}{2}\right) \leq \frac{1}{a-0} \int_0^a f(x)dx,$$

so that, since $a > 0$,

$$af\left(\frac{a}{2}\right) \leq \int_0^a f(x)dx.$$

Therefore, combining the above inequalities, we obtain

$$af\left(\frac{a}{2}\right) + \frac{b}{2} f^{-1}(b) \leq \int_0^a f(x)dx + \int_0^b f^{-1}(x)dx.$$

On the other hand, the Young integral inequality gives

$$ab \leq \int_0^a f(x)dx + \int_0^b f^{-1}(x)dx.$$

We immediately derive

$$\max\left(af\left(\frac{a}{2}\right) + \frac{b}{2} f^{-1}(b), ab\right) \leq \int_0^a f(x)dx + \int_0^b f^{-1}(x)dx.$$

This concludes the proof. $\square$

Note that, in the setting of Theorem 3.1, with the same arguments, we also have the following upper bound:

$$\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx \leq bf^{-1}\left(\frac{b}{2}\right) + \frac{a}{2} f(a).$$

3.2. An alternative upper bound

The theorem below provides an alternative upper bound to that in Theorem 2.1.

Theorem 3.2. *Let $a, b \in \mathbb{R}$ with $b > a$, and $f : \mathbb{R} \rightarrow [a, b]$ be a strictly increasing differentiable convex function. Then we have*

$$\begin{aligned} & \frac{1}{b-a} \int_0^b f^{-1}(x)dx \\ & \leq \frac{1}{b-a} \left(bf^{-1}(b) - af^{-1}(a) - f\left(\frac{f^{-1}(a) + f^{-1}(b)}{2}\right) (f^{-1}(b) - f^{-1}(a)) \right). \end{aligned}$$

Proof. Assuming that f is strictly increasing, using the change of variables $x = f(u)$ and making an integration by parts, we get

$$\begin{aligned}
\frac{1}{b-a} \int_a^b f^{-1}(x) dx &= \frac{1}{b-a} \int_{f^{-1}(a)}^{f^{-1}(b)} u f'(u) du \\
&= \frac{1}{b-a} \left([uf(u)]_{f^{-1}(a)}^{f^{-1}(b)} - \int_{f^{-1}(a)}^{f^{-1}(b)} f(u) du \right) \\
&= \frac{1}{b-a} \left(bf^{-1}(b) - af^{-1}(a) - \int_{f^{-1}(a)}^{f^{-1}(b)} f(u) du \right).
\end{aligned}$$

Since f is convex, applying the left-hand side of the Hermite-Hadamard integral inequality to f at $f^{-1}(a)$ and $f^{-1}(b)$, we have

$$f\left(\frac{f^{-1}(a) + f^{-1}(b)}{2}\right) \leq \frac{1}{f^{-1}(b) - f^{-1}(a)} \int_{f^{-1}(a)}^{f^{-1}(b)} f(u) du,$$

so that, since $f^{-1}(b) > f^{-1}(a)$,

$$f\left(\frac{f^{-1}(a) + f^{-1}(b)}{2}\right) (f^{-1}(b) - f^{-1}(a)) \leq \int_{f^{-1}(a)}^{f^{-1}(b)} f(u) du.$$

Therefore, combining the above inequalities, we obtain

$$\begin{aligned}
&\frac{1}{b-a} \int_a^b f^{-1}(x) dx \\
&\leq \frac{1}{b-a} \left(bf^{-1}(b) - af^{-1}(a) - f\left(\frac{f^{-1}(a) + f^{-1}(b)}{2}\right) (f^{-1}(b) - f^{-1}(a)) \right).
\end{aligned}$$

This completes the proof. $\square$

Due to their inherently different nature, the obtained upper bound is not directly comparable with that in Theorem 2.1. The two results should be viewed as complementary.

3.3. The case of the product of two inverse functions

The theorem below can be presented as a one-sided Hermite-Hadamard integral inequality for the product of two inverse functions. In a sense, it generalizes the lower bound in Theorem 2.1.

Theorem 3.3. *Let $a, b \in \mathbb{R}$ with $b > a$, and $f, g : [0, \infty) \rightarrow [a, b]$ be strictly increasing convex functions. Then we have*

$$\frac{1}{b-a} \int_a^b f^{-1}(x)g^{-1}(x)dx \geq \frac{(f^{-1}(a)+f^{-1}(b))(g^{-1}(a)+g^{-1}(b))}{4}.$$

Proof. Since f and g are strictly increasing, f^{-1} and g^{-1} are also strictly increasing, so of the same monotonicity. The Chebyshev integral inequality implies that

$$\frac{1}{b-a} \int_a^b f^{-1}(x)g^{-1}(x)dx \geq \left(\frac{1}{b-a} \int_a^b f^{-1}(x)dx \right) \left(\frac{1}{b-a} \int_a^b g^{-1}(x)dx \right).$$

Applying the left-hand side of the inequality in Theorem 2.1 to f and g , noting that the lower bounds are non-negative, yields

$$\begin{aligned} \left(\frac{1}{b-a} \int_a^b f^{-1}(x)dx \right) \left(\frac{1}{b-a} \int_a^b g^{-1}(x)dx \right) &\geq \frac{f^{-1}(a)+f^{-1}(b)}{2} \times \frac{g^{-1}(a)+g^{-1}(b)}{2} \\ &= \frac{(f^{-1}(a)+f^{-1}(b))(g^{-1}(a)+g^{-1}(b))}{4}. \end{aligned}$$

Therefore, combining the above inequalities, we derive

$$\frac{1}{b-a} \int_a^b f^{-1}(x)g^{-1}(x)dx \geq \frac{(f^{-1}(a)+f^{-1}(b))(g^{-1}(a)+g^{-1}(b))}{4}.$$

This completes the proof. $\square$

The natural concave counterpart of Theorem 3.3 is the theorem below.

Theorem 3.4. *Let $a, b \in \mathbb{R}$ with $b > a$, and $f, g : [0, \infty) \rightarrow [a, b]$ be strictly increasing concave functions. Then we have*

$$\frac{1}{b-a} \int_a^b f^{-1}(x)g^{-1}(x)dx \geq f^{-1}\left(\frac{a+b}{2}\right)g^{-1}\left(\frac{a+b}{2}\right).$$

The proof is omitted, as it follows along the same lines as that of Theorem 3.3, with appropriate modifications to account for concavity instead of convexity.

Finally, note that it is possible to consider more than two inverse functions in the product, provided Theorems 3.3 and 3.4 are adapted.

4. Conclusion

This work extends the Hermite-Hadamard integral inequality to inverse functions, revealing new bounds that complement classical results. The alternative proofs presented here demonstrate the complexity of the problem and the range of approaches that can be taken. Future research could explore extensions to broader classes of functions, such as non-monotonic or multivariate cases, and potential applications in related areas of analysis and optimization.

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