

REVISIT ON SOME DEGENERATE FUNCTIONS

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Abstract

Recently, several important degenerate functions have been introduced in the literature, including the degenerate sigmoid function in 2023, and various degenerate hyperbolic functions and their reciprocal forms in 2024. The main contribution of this article is a thorough revision of their domains of definition, as well as a reconsideration of some of their key properties. Numerical examples are given to illustrate the theory.

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1. Introduction

Sigmoid functions play an important role in several areas of science and engineering. This is particularly the case in machine learning, artificial neural networks and statistical modelling. They are known for their smooth “S-shaped” curve, which facilitates tasks, such as classification, activation and approximation. Traditional sigmoid functions, such as the logistic and hyperbolic tangent functions, have been extensively applied due to their desirable properties. These include differentiability and boundedness. The essentials of this topic can be found in [7, 6, 4, 9].

In recent years, efforts have been made to generalize or modify sigmoid functions. The aim is to increase their flexibility and to adapt them to more specialized applications. Notable studies on this subject can be found in [8, 3, 1, 2, 5]. Among these advances, a novel form known as the *degenerate sigmoid function* was developed in [1]. This function introduces a parameter λ which controls its shape and behavior, providing a new level of control and adaptability over the classical forms. The degenerate sigmoid is defined as.

$$S_\lambda(t) = \frac{(1 + \lambda t)^{1/\lambda}}{1 + (1 + \lambda t)^{1/\lambda}}. \quad (1)$$

The possible values for λ and t will be discussed in more detail later. The formulation of the degenerate sigmoid function reduces to the classical sigmoid function when $\lambda \rightarrow 0$ due to the limit $(1 + \lambda t)^{1/\lambda} \rightarrow e^t$ and to the constant $1/2$ when $\lambda \rightarrow \infty$. In a sense, it bridges the gap between a smooth, nonlinear transition and a flat response. The parameter λ thus plays a key role in determining the smoothness of the curve, making $S_\lambda(t)$ suitable for a wide range of applications where tunable nonlinearity is desired. The study in [1] is only theoretical, but opens the door to practice. Its application to machine learning, artificial

neural networks, and statistical modelling is currently being pursued by the authors.

On closer inspection, certain theoretical aspects in [1] require clarification. One such issue is the precise domain of definition for $S_\lambda(t)$, which is essential to ensure that the function remains well defined and numerically stable. This is crucial because incorrect or incomplete domain assumptions can lead to misinterpretation or computational errors in applications. Furthermore, we observe a similar omission in the recent introduction of *hyperbolic degenerate functions* and their reciprocal forms in [2], where the domain constraints and boundary behaviours are not fully addressed. Nevertheless, plots of the functions are drawn for some values, including negative values of t , which may be surprising at first sight.

The aim of this article is to rigorously fill these gaps by providing corrected formulations of the domains of definition, detailed mathematical analysis and complete proofs. Some numerical examples illustrate the theory. In doing so, we establish a more solid theoretical foundation for these degenerate functions and also enhance their potential for future applied work.

The corrections are given in the next section, i.e., Section 2. The last section, i.e., Section 3 gives a conclusion.

2. Corrections

2.1. Domain of definition

In [1], it is noted that the degenerate function given by Equation (1) is defined for $\lambda \in (0, \infty)$ and $t \in (-\infty, \infty)$. This definition includes possible values such as $1 + \lambda t < 0$, which makes the power transformation $(1 + \lambda t)^{1/\lambda}$ invalid if λ takes a decimal value. It therefore needs a correction, as proposed below.

Proposition 2.1. *The degenerate sigmoid function $S_\lambda(t)$ defined in Equation (1) is well defined in the mathematical sense if and only if λ and t satisfy the following conditions:*

$$\lambda \in (-\infty, \infty) \setminus \{0\} \text{ and } t \in (-\infty, \infty) \text{ such that}$$

$\lambda t \in (-1, \infty)$, or $1/\lambda \in \mathbb{Z} \setminus \{0\}$, or $\lambda t = -1$ and $\lambda > 0$ (giving the basic $S_\lambda(t) = 0$).

Proof of Proposition 2.1. For any

$$\lambda \in (-\infty, \infty) \setminus \{0\} \text{ and } t \in (-\infty, \infty) \text{ such that}$$

$$\lambda t \in (-1, \infty), \text{ or } 1/\lambda \in \mathbb{Z} \setminus \{0\}, \text{ or } \lambda t = -1 \text{ and } \lambda > 0,$$

the expression $(1 + \lambda t)^{1/\lambda}$ is mathematically valid: the term $1 + \lambda t$ is always non-negative where it must be, and it may be negative when $1/\lambda \in \mathbb{Z} \setminus \{0\}$. The denominator is always different from 0. As a result, $S_\lambda(t)$ is well defined. This concludes the proof. \square

In the rest of this section, we will denote \mathcal{D} the domain of definition of $S_\lambda(t)$, that is,

$$\mathcal{D} = \{(\lambda, t) \in (-\infty, \infty) \setminus \{0\} \times (-\infty, \infty) \text{ such that } \lambda t \in (-1, \infty), \text{ or } 1/\lambda \in \mathbb{Z} \setminus \{0\}, \text{ or } \lambda t = -1 \text{ and } \lambda > 0\}.$$

This formulation extends the original domain considered in [1], which excluded certain cases with negative values of λ . For instance, some arbitrary numerical examples are as follows:

$$S_{-1}(-1) = \frac{1}{3}, \quad S_{-2}(-1) = \frac{1}{1 + \sqrt{3}}, \quad S_{-2}(-2) = \frac{1}{1 + \sqrt{5}}.$$

They are now valid in the extended domain. Moreover, when $1/\lambda \in \mathbb{Z} \setminus \{0\}$, $S_\lambda(t)$ is well defined for $t \in (-\infty, \infty)$, without restriction. As illustrative examples, we have

$$S_{-1/2}(-1) = \frac{4}{13}, \quad S_{-1/3}(-2) = \frac{27}{152}, \quad S_{-1/4}(2) = \frac{16}{17}.$$

This careful re-examination of the domain of definition is essential and leads to a necessary revision of several key propositions presented in [1], as elaborated in the next section.

2.2. Corrections of key propositions

To begin with, in [1, Equation (10)], it is written that

$$S_\lambda(t) + S_\lambda(-t) = 1.$$

However, due to the revised domain of definition, this equality cannot be true. We address this problem in the proposition below:

Proposition 2.2. *For any $(\lambda, t) \in \mathcal{D}$, we have*

$$S_\lambda(t) + S_{-\lambda}(-t) = 1.$$

Proof of Proposition 2.2. We start with the expression of $S_{-\lambda}(-t)$ and proceed using standard mathematical developments, as follows:

$$\begin{aligned} S_{-\lambda}(-t) &= \frac{[1 + (-\lambda)(-t)]^{-1/\lambda}}{1 + [1 + (-\lambda)(-t)]^{-1/\lambda}} = \frac{(1 + \lambda t)^{-1/\lambda}}{1 + (1 + \lambda t)^{-1/\lambda}} \\ &= \frac{(1 + \lambda t)^{1/\lambda}}{(1 + \lambda t)^{1/\lambda}} \times \frac{(1 + \lambda t)^{-1/\lambda}}{1 + (1 + \lambda t)^{-1/\lambda}} = \frac{1}{(1 + \lambda t)^{1/\lambda} + 1} \\ &= \frac{1 + (1 + \lambda t)^{1/\lambda} - (1 + \lambda t)^{1/\lambda}}{(1 + \lambda t)^{1/\lambda} + 1} = 1 - \frac{(1 + \lambda t)^{1/\lambda}}{1 + (1 + \lambda t)^{1/\lambda}} \\ &= 1 - S_\lambda(t). \end{aligned} \tag{2}$$

Rearranging this identity gives us

$$S_\lambda(t) + S_{-\lambda}(-t) = 1.$$

This ends the proof of the proposition. \square

Compared to [1, Equation (10)], the term $S_\lambda(-t)$ is thus replaced by $S_{-\lambda}(-t)$, which is possible thanks to the new consideration of \mathcal{D} . Proposition 2.2 is interesting because it reflects a symmetry in the function $S_\lambda(t)$ with respect to both the parameter λ and the variable t . It also shows a kind of duality: flipping both λ and t preserves a complementary relationship.

Let us now focus on another aspect. In [1, Equation (11)], it is written that

$$S_\lambda(t)S_\lambda(-t) = (1 + \lambda t)S'_\lambda(t).$$

Again, because of the domain of definition, this relation cannot be true. We revisit it in the proposition below.

Proposition 2.3. *For any $(\lambda, t) \in \mathcal{D}$, we have*

$$S_\lambda(t)S_{-\lambda}(-t) = (1 + \lambda t)S'_\lambda(t).$$

Proof of Proposition 2.3. First, using standard differentiation rules, we get

$$\begin{aligned} S'_\lambda(t) &= \frac{(1 + \lambda t)^{1/\lambda-1} [1 + (1 + \lambda t)^{1/\lambda}] - (1 + \lambda t)^{1/\lambda} (1 + \lambda t)^{1/\lambda-1}}{[1 + (1 + \lambda t)^{1/\lambda}]^2} \\ &= \frac{(1 + \lambda t)^{1/\lambda-1}}{[1 + (1 + \lambda t)^{1/\lambda}]^2}. \end{aligned}$$

On the other hand, using an expression in Equation (2), we immediately have

$$S_{-\lambda}(-t) = \frac{1}{(1 + \lambda t)^{1/\lambda} + 1},$$

so that

$$S_\lambda(t)S_{-\lambda}(-t) = \frac{(1 + \lambda t)^{1/\lambda}}{1 + (1 + \lambda t)^{1/\lambda}} \times \frac{1}{(1 + \lambda t)^{1/\lambda} + 1}$$

$$\begin{aligned}
&= \frac{(1 + \lambda t)^{1/\lambda}}{[1 + (1 + \lambda t)^{1/\lambda}]^2} = (1 + \lambda t) \frac{(1 + \lambda t)^{1/\lambda-1}}{[1 + (1 + \lambda t)^{1/\lambda}]^2} \\
&= (1 + \lambda t) S'_\lambda(t).
\end{aligned}$$

This ends the proof of the proposition. \square

Compared to [1, Equation (11)], and as it was also the case for [1, Equation (10)], the term $S_\lambda(-t)$ is thus replaced by $S_{-\lambda}(-t)$, which is possible thanks to the new consideration of \mathcal{D} . Proposition 2.3 is interesting because it reveals a deep interplay between the functions $S_\lambda(t)$ and $S_{-\lambda}(t)$ and their derivatives, highlighting a structural symmetry.

A last property needs to be revised. In [1, Equation (12)], it is written that

$$S'_\lambda(t) = S'_\lambda(-t).$$

Again, because of the domain of definition, this relation cannot be true. We revisit it in the proposition below.

Proposition 2.4. *For any $(\lambda, t) \in \mathcal{D}$, we have*

$$S'_\lambda(t) = S'_{-\lambda}(-t).$$

Proof of Proposition 2.4. As in the proof of Proposition 2.3, using standard differentiation rules, we obtain

$$S'_\lambda(t) = \frac{(1 + \lambda t)^{1/\lambda-1}}{[1 + (1 + \lambda t)^{1/\lambda}]^2}.$$

By some mathematical developments, we get

$$S'_{-\lambda}(-t) = \frac{[1 + (-\lambda)(-t)]^{-1/\lambda-1}}{[1 + (1 + (-\lambda)(-t))^{-1/\lambda}]^2} = \frac{(1 + \lambda t)^{-1/\lambda-1}}{[1 + (1 + \lambda t)^{-1/\lambda}]^2}$$

$$\begin{aligned}
&= \frac{(1 + \lambda t)^{2/\lambda}}{(1 + \lambda t)^{2/\lambda}} \times \frac{(1 + \lambda t)^{-1/\lambda-1}}{[1 + (1 + \lambda t)^{-1/\lambda}]^2} = \frac{(1 + \lambda t)^{1/\lambda-1}}{[(1 + \lambda t)^{1/\lambda} + 1]^2} \\
&= S'_\lambda(t).
\end{aligned}$$

This ends the proof of the proposition. \square

Compared to [1, Equation (12)], the term $S'_\lambda(-t)$ is thus replaced by $S'_{-\lambda}(-t)$, which is possible thanks to the new consideration of \mathcal{D} . It again reveals an original reflection symmetry in the underlying structure of $S_\lambda(t)$.

By revising these propositions, we rehabilitate and extend the scope of the proposed degenerate sigmoid function. The propositions in [1] not revised previously are mathematically valid.

2.3. A secondary correction

In the article [2], following the same approach as in [1], several new hyperbolic degenerate functions are introduced, including the hyperbolic cosine, hyperbolic sine, and hyperbolic tangent degenerate functions, defined as follows:

$$U_\lambda(t) = \frac{(1 + \lambda t)^{1/\lambda} + (1 + \lambda t)^{-1/\lambda}}{2}, \quad (3)$$

$$V_\lambda(t) = \frac{(1 + \lambda t)^{1/\lambda} - (1 + \lambda t)^{-1/\lambda}}{2}, \quad (4)$$

and

$$W_\lambda(t) = \frac{(1 + \lambda t)^{1/\lambda} - (1 + \lambda t)^{-1/\lambda}}{(1 + \lambda t)^{1/\lambda} + (1 + \lambda t)^{-1/\lambda}}, \quad (5)$$

respectively.

The corresponding reciprocal forms are also presented, given by

$$X_\lambda(t) = \frac{2}{(1 + \lambda t)^{1/\lambda} + (1 + \lambda t)^{-1/\lambda}}, \quad (6)$$

$$Y_\lambda(t) = \frac{2}{(1 + \lambda t)^{1/\lambda} - (1 + \lambda t)^{-1/\lambda}}, \quad (7)$$

and

$$Z_\lambda(t) = \frac{(1 + \lambda t)^{1/\lambda} + (1 + \lambda t)^{-1/\lambda}}{(1 + \lambda t)^{1/\lambda} - (1 + \lambda t)^{-1/\lambda}}, \quad (8)$$

respectively.

It is noted that the functions are defined for $\lambda \in (0, \infty)$ and $t \in (-\infty, \infty)$. Again, this domain of definition includes possible values such that $1 + \lambda t < 0$, which makes the power transformation $(1 + \lambda t)^{1/\lambda}$ invalid if $1/\lambda$ is decimal. In the light of our result in Proposition 2.1, the domain of definition must be

$\lambda \in (-\infty, \infty) \setminus \{0\}$ and $t \in (-\infty, \infty)$ such that $\lambda t \in (-1, \infty)$, or $1/\lambda \in \mathbb{Z} \setminus \{0\}$, or $\lambda t = -1$ and $\lambda > 0$,

which corresponds to \mathcal{D} . This allows possible negative values of λ , which were excluded in [2]. As illustrative examples for $U_\lambda(t)$ only, we have

$$U_{-1/2}(-1) = \frac{97}{72}, \quad U_{-1/3}(-2) = \frac{8177}{3375}, \quad U_{-1/4}(2) = \frac{257}{32}.$$

The same numerical work can be made for $V_\lambda(t)$, $W_\lambda(t)$, $X_\lambda(t)$, $Y_\lambda(t)$, and $Z_\lambda(t)$.

Although there is an issue with the domain of definition, [2, Figures 1-6] display plots of the considered degenerate functions for both positive

and negative values of t . This is possible only because λ is chosen such that $1/\lambda \in \mathbb{Z} \setminus \{0\}$; otherwise, the functions would not be well-defined and could not be plotted. This remark serves to clarify the mathematical framework underlying these degenerate functions, justify the validity of the presented plots, and support the propositions presented in [2].

3. Conclusion

In conclusion, this article provides a necessary correction to the domain of definition for new degenerate functions, filling theoretical gaps previously overlooked in both [1] and [2]. By providing a rigorous mathematical framework, we ensure that the functions are well defined and numerically stable for a wider range of applications. These clarifications strengthen the theoretical understanding. It also allows for more reliable practical implementations of these degenerate functions in future work.

References

- [1] T. A. Akugre, K. Nantomah and M. M. Iddrisu, On certain properties of a degenerate sigmoid function, *Eur. J. Math. Anal.* 3 (2023), 17.
DOI: <https://doi.org/10.28924/ada/ma.3.17>
- [2] T. A. Akugre, K. Nantomah and M. M. Iddrisu, On some properties of the degenerate hyperbolic functions, *J. Math. Anal. Model.* 5(1) (2024), 26-40.
DOI: <https://doi.org/10.48185/jmam.v5i1.961>
- [3] Y. J. Bagul and C. Chesneau, Sigmoid functions for the smooth approximation to the solute value function, *Moroccan J. of Pure and Appl. Anal (NJPAA)* 7(1) (2021), 12-19.
DOI: <https://doi.org/10.2478/mjpaa-2021-0002>
- [4] C. M. Bishop, *Pattern Recognition and Machine Learning*, Springer, New York, 2016.
- [5] J. Dombi and T. Jonas, The generalized sigmoid function and its connection with logical operators, *Int. J. Approx. Reasoning* 143 (2022), 121-138.
DOI: <https://doi.org/10.1016/j.ijar.2022.01.006>

- [6] J. Han and C. Moraga, The Influence of the Sigmoid Function Parameters on the Speed of Backpropagation Learning. In: J. Mira, F. Sandoval (eds) From Natural to Artificial Neural Computation. IWANN 1995, Lecture Notes in Computer Science, Vol 930, Springer, Berlin, Heidelberg, 1995.
DOI: https://doi.org/10.1007/3-540-59497-3_175
- [7] D. W. Hosmer and S. Lemeshow, Applied Logistic Regression, Wiley, New York, Second Edition, 2000.
- [8] N. Kyurkchiev and S. Markov, Sigmoid functions: Some approximation and modelling aspects: Some moduli in programming environment MATHEMATICA, Saarbrücken, Germany: LAP LAMBERT Academic Publishing, 2015.
- [9] K. P. Murphy, Machine Learning: A Probabilistic Perspective, MIT Press, 2012.

