

ON SOME PROPERTIES OF HOMOGENEOUS REAL REDUCTIVE HEISENBERG-TYPE GROUPS

M. E. Egwe and J. I. Opadara

Department of Mathematics

University of Ibadan

Ibadan

Nigeria

e-mail: murphy.egwe@ui.edu.ng

jossylib@gmail.com

Abstract

In this paper, we consider homogeneous groups of Heisenberg type, which are generally reductive, generated with a heat kernels. We derive an equation to the generating heat kernel, which takes the form

$$K(x, a) = C \langle x\gamma, a \rangle^2 + A,$$

where $x\gamma$ and a are elements of the Heisenberg group and C and A are constants. The generating kernel is then used to construct the Lie algebra associated with the Heisenberg group. Using the Lie algebra, we define a homogeneous space associated with the Heisenberg group, which is a Riemannian manifold of Euclidian type. We provide a detailed analysis of the geometry of this homogeneous space and its relationship to the Lie algebra. In particular, we show how the Heisenberg group can be used to construct an invariant metric and study its properties.

2020 Mathematics Subject Classification: 22Exx, 17B66, 17B30, 17B20, 32W30.

Keywords and phrases: homogeneous groups, Riemannian manifold, Heisenberg-type, Lie algebra, heat kernels.

Communicated by: Francisco Bulnes.

Received December 03, 2024; Revised March 26, 2025

1. Preliminary

In this section the main components and points of this work, as well as the underlying setting is outlined. Also a general outline for realization of the setting and examples will be discussed.

Now let us describe some important tools needed for the development of our theory. Assuming that $(\mathfrak{M}, a, \sigma)$ is a space of measurable metric, satisfying the following conditions:

(a) (\mathfrak{M}, a) is a metric space which is locally compact with $a(\cdot, \cdot)$ as the distance where σ is a positive Radon measure which makes the following volume doubling conditions:

$$0 < \sigma(\mathcal{B}(p, 2q)) < 2^n \sigma(\mathcal{B}(p, q)) < \infty \text{ for } p \in \mathfrak{M} \text{ and } q > 0 \quad (1.1)$$

valid, where $(\mathcal{B}(p, q))$ is the open ball with center p and of radius q and $n > 0$ is a constant which stands for a dimension. Here $(\mathfrak{M}, a, \sigma)$ is a homogeneous space.

(b) We assume that the reverse of the doubling condition is valid, that is, \exists a constant $\alpha > 0$ such that

$$\sigma(\mathcal{B}(p, 2q)) \geq 2^\alpha \sigma(\mathcal{B}(p, q)) \text{ for } p \in \mathfrak{M} \text{ and } \frac{\text{diam}\mathfrak{M}}{3} \geq q > 0 \quad (1.2)$$

We shall prove later that this condition is a consequence of the doubling condition above (1.1) if \mathfrak{M} is connected.

(c) We shall also stipulate the following non-collapsing condition: That is, there exists a constant $m > 0$ such that

$$\inf_{p \in \mathfrak{M}} \sigma(\mathcal{B}(p, 1)) \geq m, \text{ for all } p \in \mathfrak{M}. \quad (1.3)$$

The case where $\sigma(\mathfrak{M}) < \infty$ will be shown later that the inequality above follows by (1.2). Hence, the case when $\sigma(\mathfrak{M}) = \infty$ is an additional assumption.

Since the paper is only based on the spaces of homogeneous functions, of Heisenberg type, it make sense that our assumptions are purely local, and to assume only doubling for the balls whose radius are bounded by some constant, in particular, which would considerably enlarge the range of examples. And the assumptions on the heat kernel $(h_m(p, l))$ are local.

The most important assumption is that the local geometry of the space $(\mathfrak{M}, a, \sigma)$ is related to an essentially self-adjoint positive operator \mathcal{L} on $\mathcal{L}^2(\mathfrak{M}, a\sigma)$ such that the associated semigroup $R_m = e^{-m\mathcal{L}}$ consists of integral operators with heat kernel $h_m(p, l)$ satisfying the conditions:

(d) The Gaussian upper bound:

$$h_m(p, l) \leq \frac{Ce^{\{-bd^2 \frac{h_m(p, l)}{m}\}}}{(\sigma(\mathcal{B}(p, \sqrt{m}))\sigma(\mathcal{B}(l, \sqrt{m})))^{1/2}} \cdot \text{ for } p, l \in \mathfrak{M}, 1 > m > 0. \quad (1.4)$$

One can observe that the combination of results in ([3], [4] and [11]), gives that this estimate and the doubling condition in (1.1) together with the opinion that $e^{-m\mathcal{L}}$ is a holomorphic semigroup on $\mathcal{L}^2(\mathfrak{M}, a\sigma)$, that is $e^{-r\mathcal{L}}$ exists where $r \in \mathbb{C}$ and $\text{Re } r \geq 0$, mean that $e^{-r\mathcal{L}}$ is an integral operator with kernel $h_r(p, l)$ obeying the estimation below: For any $r = m + iu$, $1 \geq m > 0$, $u \in \mathbb{R}$, and $p, l \in \mathfrak{M}$,

$$|h_r(p, l)| \leq \frac{C \exp\{-b \text{Re} \frac{d^2(p, l)}{r}\}}{(\sigma(\mathcal{B}(p, \sqrt{m}))\sigma(\mathcal{B}(l, \sqrt{m})))^{1/2}} \text{ for } p, l \in \mathfrak{M}, 1 > m > 0. \quad (1.5)$$

(e) Hölder continuity: There exists a constant $\gamma > 0$ such that

$$|h_m(p, l) - h_m(p, l')| \leq k \left(\frac{a(l, l')}{m^{1/2}} \right)^\gamma \frac{\exp\{-\frac{ba^2(p, l)}{m}\}}{(\sigma(\mathcal{B}(p, \sqrt{m}))\sigma(\mathcal{B}(l, \sqrt{m})))^{1/2}}, \quad (1.6)$$

$\forall p, l, l' \in \mathfrak{M}$ and $1 \geq m > 0$, where $a(l, l') \leq m^{1/2}$.

(f) Markov property:

$$\int_{\mathfrak{M}} h_m(p, l) a\sigma(l) \equiv 1 \text{ for } m > 0, \quad (1.7)$$

which by analytic continuation implies that,

$$\int_{\mathfrak{M}} h_r(p, l) a\sigma(l) \equiv 1 \text{ for } r = m + iu, \ m > 0. \quad (1.8)$$

The parameters $k, b > 0$ above represents the structural constants that could have consequence on other constant in sequel.

The main results in this paper will be inferred from the above conditions.

2. The Concept of Homogeneous Groups that are Naturally Reductive

Assume that (\mathfrak{M}, a) is a connected homogeneous manifold of dimension n . In addition, let G be a Lie group which effectively and transitively act from the left on \mathfrak{M} as a group of isometric. We shall denote by D , the subgroup which is isotropic at some point x of \mathfrak{M} . Let \mathfrak{g} and \mathfrak{d} be the Lie algebras of G and D , respectively.

Suppose that \mathfrak{e} is a vector space complement to \mathfrak{d} in \mathfrak{g} for which $\mathbb{F}d(D)\mathfrak{e} \subseteq \mathfrak{e}$. Then \mathfrak{e} may be identified with $T_x\mathfrak{M}$ through the map $F \rightarrow F_x^*$, where F_x^* is the Killing vector on (\mathfrak{M}, a) which is generated by the subgroup $\{e^{mF}\}$ acting on \mathfrak{M} . Let \langle, \rangle represent the inner product on \mathfrak{e} inveigle by the metric a .

Definition 2.1. The manifold (\mathfrak{M}, a) is said to be naturally reductive if \exists a Lie group \mathcal{G} and a subspace \mathfrak{e} such that

$$\langle [F, S]_{\mathfrak{e}}, R \rangle + \langle S, [F, R]_{\mathfrak{e}} \rangle = 0, \ F, S, R \in \mathfrak{e}, \quad (2.1)$$

where $[F, S]_{\mathfrak{e}}$ stands for the projection of $[F, S]$ on \mathfrak{e} related to the attributes above.

The following theorem may be used to define the reductive manifolds geometrically.

Theorem 2.2 [14]. *The homogeneous manifold (\mathfrak{M}, a) is naturally reductive if and only if the geodesic through x and tangent to $F \in \mathfrak{e} \cong U_x \mathfrak{M}$ is the curve $(e^{mF})x$, orbit of the subgroup e^{mF} of \mathcal{G} , $\forall F$.*

Clearly, it is of necessity to first determine all the transitive isometry groups of \mathcal{G} of \mathfrak{M} and consider all the complements of \mathfrak{f} in \mathfrak{c} that are invariant under $\mathbb{F}d(D)$ before one can conclude that \mathfrak{M} is naturally reductive.

Theorem 2.3. *Suppose that (\mathfrak{M}, a) is a connected, simply connected and complete Riemannian manifold.*

Then (\mathfrak{M}, a) is a naturally reductive and is homogeneous space if and only if \exists a tensor field \mathbb{U} of type $(2, 4) \ni$

$$(\mathbb{F}X) = \begin{cases} \text{(i)} & \sigma(\mathbb{U}_F S, R) + (S, \mathbb{U}_F R) = 0, \\ \text{(ii)} & (\Delta_F \mathfrak{F})_{SR} = [\mathbb{U}_F, \mathfrak{F}_{SR}] - \mathfrak{F}_{\mathbb{U}_F S R} - \mathfrak{F}_{S \mathbb{U}_F R}, \\ \text{(iii)} & (\Delta_F \mathbb{U})_S = [\mathbb{U}_F, \mathbb{U}_S] - \mathbb{U}_{\mathbb{U}_F S}, \end{cases}$$

and

$$\mathbb{U}_F S + \mathbb{U}_S F = 0, \quad (2.2)$$

where $F, S, R \in \mathfrak{T}(\mathfrak{M})$. Δ being the Levi Civita connection and \mathfrak{F} means the Riemann curvature tensor.

The proof of this theorem shall be discussed after the following estimates:

The existence of the tensor \mathbb{U} satisfying the Ambrose-Singer's conditions $(\mathbb{F}X)$ and is equivalent to the homogeneity of the manifold. One could note that if $\tilde{\Delta} = \Delta - \mathbb{U}$, conditions (ii) and (iii) of $(\mathbb{F}X)$ are

identical to $\tilde{\Delta}\tilde{\mathfrak{F}} = \tilde{\Delta}\mathbb{U} = 0$ and condition (i) implies that $\tilde{\Delta}$ is a metric connection. Now it will only be proved that (2.2) is virtually equal to the naturally reductive property. But before then, the construction of a transitive and effective group G acting on \mathfrak{M} which is isometry when a tensor \mathbb{U} is given needs to be recalled. Therefore, the concentration of some facts will be concern the Lie algebra \mathfrak{g} of Lie group G .

Now assume that $\pi : \theta(\epsilon) \rightarrow \mathfrak{M}$ is the principal bundle of orthogonal frames of \mathfrak{M} . The first condition of $(\mathbb{F}X)$ means that the linear connection $\tilde{\Delta} = \Delta - \mathbb{U}$ is metric as well instigates an infinitesimal connection on $\theta(\mathfrak{M})$. If $v = (x, v_1, v_2, \dots, v_m)$ is a point of $\theta(\mathfrak{M})$ and denote the holonomy bundle of $\tilde{\Delta}$ through v by $\tilde{\mathfrak{F}}(v)$. It can be seen that $\tilde{\mathfrak{F}}(v)$ is a principal subbundle of $\theta(\mathfrak{M})$ which have a structure group as holonomy group $\tilde{\varphi}(v)$ of $\tilde{\Delta}$ and this formed a subgroup of $V(m)$. Now let denote by \mathbb{F}^* the fundamental vertical vector field that equivalents to a member \mathbb{F} of the Lie algebra $\mathfrak{so}(m)$ of $V(m)$. Moreover, let $\mathbb{B}(\eta)$ represents the standard horizontal vector field corresponds to $\tilde{\Delta}$. and equivalent to the vector η of \mathbb{R}^m . Thus the Lie algebra \mathfrak{g} of the Lie group G is the subalgebra of $\mathfrak{T}(\tilde{\mathfrak{F}}(v))$ spanned by the restriction of the vector fields $\mathbb{F}_1^*, \mathbb{F}_2^*, \mathbb{F}_3^*, \dots, \mathbb{F}_q^*, \mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \dots, \mathbb{B}_m$ to $\tilde{\mathfrak{F}}(v)$. Thus, $(\mathbb{F}_1, \mathbb{F}_2, \mathbb{F}_3, \dots, \mathbb{F}_q)$ form a basis of the Lie algebra of $\tilde{\varphi}(v)$ while $\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \dots, \mathbb{B}_m$ are the horizontal vector fields equivalent to the natural basis of \mathbb{R}^m .

The point $x = \theta(v)$ is the point of the isotropy subgroup D which is connected subgroup of the Lie group G with Lie algebra \mathfrak{d} spanned by $\mathbb{F}_1^*, \mathbb{F}_2^*, \dots, \mathbb{F}_m^*$. It could be noted that the Lie algebra \mathfrak{d} and the holonomy algebra $\tilde{\Delta}$ are isomorphic.

Now, we are ready to write down the Lie brackets for the basis members of \mathfrak{g} in explicit manner. Hence we have that

$$[\mathbb{F}^*, \mathbb{F}'^*] = [\mathbb{F}^*, \mathbb{F}']^* \text{ for } \mathbb{F}, \mathbb{F}' \in \mathfrak{so}(m), \quad (2.3)$$

$$[\mathbb{F}^*, \mathbb{B}(\eta)] = \mathbb{B}(\mathbb{F}(\eta)) \text{ for } \mathbb{F} \in \mathfrak{so}(m) \text{ and } \eta \in \mathbb{R}^m, \quad (2.4)$$

In addition, for $r = (y, r_1, r_2, r_3, \dots, r_m) \in \tilde{\mathfrak{F}}(v)$, we have

$$[\mathbb{B}(\eta_1), \mathbb{B}(\eta_2)]_r = -\mathbb{B}(r^{-1}(\tilde{N}_{r(\eta_1)}r(\eta_2))) + (r^{-1}o(\tilde{\mathfrak{F}}_{r(\eta_1)}r(\eta_2))or)^*,$$

$$\forall \eta_1, \eta_2 \in \mathbb{R}^m, \quad (2.5)$$

for r is the isometry

$$r : \mathbb{R}^m \rightarrow \mathbb{U}_y\mathfrak{M} : (\eta^1, \eta^2, \dots, \eta^m) \mapsto \sum_{j=1}^m \eta^j r_j, \quad (2.6)$$

and \mathbb{R}^m is equipped with the standard metric.

Where $\tilde{\mathfrak{S}}$ stands for the curvature tensor of $\tilde{\Delta}$ which can be written as

$$\tilde{\mathfrak{S}}_{FS} = \tilde{\Delta}_{[F, S]} - [\tilde{\Delta}_F, \tilde{\Delta}_S], \quad \forall F, S \in \mathcal{T}(\mathfrak{M}), \quad (2.7)$$

and the tensor \tilde{N} which is the tension tensor of $\tilde{\Delta}$ is given as

$$\tilde{N}_F S = \tilde{\Delta}_F S - \tilde{\Delta}_S F - [F, S] = \mathbb{U}_S F - \mathbb{U}_F S, \quad \forall F, S \in \mathcal{T}(\mathfrak{M}), \quad (2.8)$$

If \mathfrak{e} is the vector subspace of Lie algebra \mathfrak{g} spanned by \mathbb{B}_i (where $i = 1, 2, 3, \dots, n$).

Now from the fact that D is connected combined with (3.4) we have that

$$\mathbb{F}d(D)\mathfrak{e} \subseteq \mathfrak{e}. \quad (2.9)$$

The condition above means that the homogeneous Riemannian space \mathfrak{M} is reductive.

Lastly, note that $(\mathbb{B}_1, \mathbb{B}_2, \mathbb{B}_3, \dots, \mathbb{B}_m)$ forms an orthonormal basis of \mathfrak{e} with regards to the inner product induced from the metric α . Thus the graph

$$\mathbb{B}or^{-1} : \mathbb{U}_y \mathfrak{M} \rightarrow \mathfrak{e} : \mathbb{F} \mapsto \mathbb{B}(r^{-1}(f)) \quad (2.10)$$

is an isometry $\forall r = (y, r_1, r_2, \dots, r_m)$ of $\varphi(\mathfrak{M})$.

Proof of Theorem 2.3. Suppose that the homogeneous space (\mathfrak{M}, α) is naturally reductive and suppose $\mathfrak{g} = \mathfrak{e} \oplus \mathfrak{d}$ is the naturally reductive decomposition of the Lie algebra \mathfrak{g} of the transitive group of Lie group G .

Take $\tilde{\Delta}$ as the corresponding canonical connection and set $\mathbb{U} = \Delta - \tilde{\Delta}$. Then $\mathbb{U}_F S + \mathbb{U}_S F = 0$ which implies that \mathbb{U} satisfies the condition (FX) since $\tilde{\mathfrak{F}}$ and \mathbb{U} are parallel with regards to $\tilde{\Delta}$.

Conversely, suppose that \mathbb{U} is a $(2, 4)$ -tensor field that satisfies the properties of the theorem. Now, set $\tilde{\Delta} = \Delta - \mathbb{U}$ again. Thus Ambrose and Singer theorem implies that the Lie group $G = \tilde{\mathfrak{F}}(v)$ is a transitive group isometries of \mathfrak{M} acting effectively on \mathfrak{M} . In addition, recall that there is a reductive decomposition given by $\mathfrak{g} = \mathfrak{e} \oplus \mathfrak{d}$. A member of \mathfrak{e} is of the form $\mathbb{B}(\mathfrak{d})$, for $\mathfrak{d} \in \mathbb{R}^m$ and

$$\begin{aligned} ([\mathbb{B}(\eta_1), \mathbb{B}(\eta_2)]_e)|_r &= -\mathbb{B}(r^{-1}(\tilde{N}_{r(\eta_1)} r(\eta_2))) \\ &= 4(\mathbb{B} \circ r^{-1})(\mathbb{U}_{r(\eta_1)} r(\eta_2)), \end{aligned}$$

where \mathbb{U} is evaluated at $y = \varphi(r)$, we thus have

$$\begin{aligned} &\langle [\mathbb{B}(\eta_1), \mathbb{B}(\eta_2)]_e, \mathbb{B}(\eta_3) \rangle + \langle \mathbb{B}(\eta_2), [\mathbb{B}(\eta_1), \mathbb{B}(\eta_3)]_e \rangle \\ &= 4\alpha_y(\mathbb{U}_{r(\eta_1)} r(\eta_2), r(\eta_3)) + 4\alpha_y(r(\eta_2), \mathbb{U}_{r(\eta_1)} r(\eta_3)). \end{aligned}$$

Since \mathbb{U}_F is skew-symmetric for all F , then (2.2) is obtained, hence \mathfrak{M} is naturally reductive. \square

Theorem 2.2 and Equation (2.3) give rise to the following corollary.

Corollary 2.4. Suppose (\mathfrak{M}, a) is a connected, simply connected homogeneous manifold. There exists \mathbb{U} of type (2, 4)-tensor field that satisfies the axioms (FX) such that $\mathbb{U}_F S + \mathbb{U}_S F = 0 \ \forall \ F \in \mathfrak{T}(\mathfrak{M})$, if and only if the geodesic tangent to $F \in \mathfrak{e} \cong \mathbb{U}_x \mathfrak{M}$ at x is the curve $(e^{wF})_x$ for all F .

3. Lie Groups of Heisenberg Type

In this part, we discuss briefly some general properties of Heisenberg group. Without loss of generality, we focus on the naturally reductive case.

Now, let us begin with the definition of a group of such type. Now suppose that K and J are two real vector spaces with n and m dimensions respectively (for $m \geq 1$), where both were equipped with the same inner product, denoted as $\langle \cdot, \cdot \rangle$. Moreover, suppose that $z : b \rightarrow z(b)p$, $b \in J$ and $p \in K$ is a linear map with conditions that

$$|z(b)p| = |p||b|, \text{ where } p \in K \text{ and } b \in J, \quad (3.1)$$

$$z(b)^2 = -|b|^2 I \text{ for } b \in J. \quad (3.2)$$

By polarizing the conditions, it implies that

$$\langle z(b)p, (c)p \rangle = \langle b, c \rangle |p|^2,$$

$$\langle z(b)p, z(b)l \rangle = |b|^2 \langle p, l \rangle, \ \forall \ p, l \in K \text{ and } b, c \in J.$$

In what follows we define the Lie algebra \mathfrak{g} as the direct sum of K and J with the bracket defined by

$$[b + p, c + l] = [p, l] \in J, \quad (3.3)$$

$$\langle [p, l], b \rangle = \langle z(b)p, l \rangle, \quad (3.4)$$

for $p, l \in K$ and $b, c \in J$. Hence, \mathfrak{g} is called the Lie algebra of Heisenberg type.

The simply connected, connected Lie group G whose Lie algebra is \mathfrak{g} is known as a generalized Heisenberg type group.

Remark 3.1. The Lie algebra \mathfrak{g} has an inner product which makes K and J orthogonal. That is,

$$\langle b + p, c + l \rangle = \langle b, c \rangle + \langle p, l \rangle \quad \forall p, l \in K \text{ and } b, c \in J.$$

Lemma 3.2. *Suppose that $(\mathfrak{M}, \langle, \rangle)$ together with the inner product \langle, \rangle is naturally reductive, then the dimension of J is 1 or 3.*

Proof. We refer to Theorem 2.3 that there is a tensor \mathbb{U} of type (2, 4) which gives that $\mathbb{U}_F S + \mathbb{U}_S F = 0$ and satisfies the $(\mathbb{F}X)$ conditions. In addition, fix the Ricci tensor of the manifold to be ℓ .

Then from condition (iii) of $(\mathbb{F}X)$, we have

$$(\Delta_F, \ell)_{SR} = -\ell \mathbb{U}_F R S - \ell S \mathbb{U}_F R. \quad (3.5)$$

Consider the Civita condition given by

$$\begin{cases} \Delta_F S = \frac{2}{4} [p, l], \\ \Delta_b F = \Delta_F b = -\frac{2}{4} z(b)p, \\ \Delta_b c = 0, \end{cases} \quad (3.6)$$

where $b, c \in J$ and $p, l \in K$.

Related to the Ricci tensor we have that

$$\begin{cases} \ell_{pl} = -\frac{2m}{4} \langle p, l \rangle, \\ \ell_{bc} = \frac{2n}{8} \langle b, c \rangle, \\ \ell_{pb} = 0. \end{cases} \quad (3.7)$$

Observing (3.6) and (3.7) it can be seen that all the components of $\Delta\ell$ become zero except

$$(\Delta_p \ell)_{cr} = -\frac{2n+4m}{16} \langle z(c)p, r \rangle. \quad (3.8)$$

Therefore (3.5) will follow if and only if

$$\langle \mathbb{U}_p r, c \rangle = \frac{2}{4} \langle z(c)p, r \rangle,$$

and

$$\langle \mathbb{U}_b r, c \rangle = 0.$$

Since \mathbb{U}_h is symmetrically skewed for all $h \in \mathfrak{g}$ we have $\mathbb{U}_b c \in J$ and

$$\mathbb{U}_p c = -\mathbb{U}_c p = -\frac{2}{4} z(c)p. \quad (3.9)$$

Now setting $\tilde{\Delta} = \Delta - \mathbb{U}$, it follows from (3.6) and (3.9) that

$$\begin{cases} \tilde{\Delta}_b p = -z(b)p, \\ \tilde{\Delta}_b c = -\mathbb{U}_b c. \end{cases}$$

Hence, we have

$$\langle (\tilde{\Delta}_b \mathbb{U})_{lr}, c \rangle = \frac{2}{4} \{ \langle z(c)z(b)l, r \rangle + \langle z(c)l, z(b)r \rangle + \langle z(\mathbb{U}_b c)l, r \rangle \}.$$

Since $z(b)$ is symmetrically skewed, $\forall b \in J$ and $\tilde{\Delta}\mathbb{U} = 0$, it implies that

$$z(\mathbb{U}_b c) = z(b)z(c) - z(c)z(b). \quad (3.10)$$

Using the polarization as in (3.2) we have

$$z(b)z(c) + z(c)z(b) = -2|b||c|I = -2|b.c|I = -2\langle b, c \rangle I. \quad (3.11)$$

Therefore (3.10) becomes

$$z(\mathbb{U}_b c) = 2\{z(b)z(c) + \langle b, c \rangle I\}, \quad (3.12)$$

which leads finally to

$$z(\mathbb{U}_b c)^2 = -2|\mathbb{U}_b c|^2 I.$$

Using (3.11) and (3.12) combined with the above facts, we have that

$$|\mathbb{U}_b c|^2 = 4(|b|^2|c|^2 - \langle b, c \rangle^2). \quad (3.13)$$

Similarly, we have

$$\langle \mathbb{U}_b c, b \rangle = \langle \mathbb{U}_b c, c \rangle = 0. \quad (3.14)$$

Thus, with the fact that $k_b c = \frac{2}{4} \mathbb{U}_b c$, $\forall b, c \in J$, we have from (3.13) and (3.14) that k is a two-fold vector cross product on J . Therefore, this gives that dimension of J is 1, and when $\mathbb{U}_b c = 0$ or otherwise dimension of J is 3 or 7.

What is left now is to prove that dimension of J cannot be 7 (i.e., $\dim J \neq 7$). To do this we have that if $\mathbb{W} = \mathbb{R} \oplus J$ with the multiplication

$$\begin{cases} 1b = b1 = 1, & 1 \in \mathbb{R}, \\ bc = k_b c - \langle b, c \rangle 1 & b, c \in J, \end{cases}$$

θ -dimensional composition algebra can be derived from the above and the inner product on J can be extended to \mathbb{W} by writing $|1| = 1$.

Now consider J being orthogonal to \mathbb{R} . Thus (3.12) means that \mathbb{W} is associative.

Define the linear map $\tilde{z} : \mathbb{W} \rightarrow \text{End}(K)$ by

$$\tilde{z}(1) = I, \text{ and } \tilde{z}(b) = z(b), \quad \forall b \in J. \quad (3.15)$$

This shows that \tilde{z} is injective. Furthermore, $\tilde{z}(bc) = \tilde{z}(b)\tilde{z}(c)$ since

$$\tilde{z}(bc) = \tilde{z}(k_b c) - \langle b, c \rangle I = \frac{2}{4} \tilde{z}(\mathbb{U}_b c) - \langle b, c \rangle I = z(b)z(c),$$

Therefore, \tilde{z} is a monomorphism between the algebra \mathbb{W} and $\text{End}(K)$ and hence \mathbb{W} is associative from the Equations (3.12) and (3.15). This does not include the case that dimension of J is 7 since any 8-dimensional composition algebra is not associative. \square

Theorem 3.3. *The homogeneous manifold $(\mathfrak{M}, \langle, \rangle)$ together with the inner product \langle, \rangle is naturally reductive if and only if \mathfrak{M} is Heisenberg type group or quaternionic analogue.*

Proof. By Clifford modulus' classification, (cf [9]) that for dimension of J to be 1, the equivalent groups G are the Heisenberg Groups and for dimension of J to be 3 the groups G are the quaternionic analogues.

To complete the proof we need to clarify that in these cases there is tensor \mathbb{U} which satisfies the required properties. Thus, we define \mathbb{U} as follows:

$$\mathbb{U}_p l = -\mathbb{U}_{lp} = \frac{2}{4} [p, l],$$

$$\mathbb{U}_p b = -\mathbb{U}_{bp} = -\frac{2}{4} z(b)p = -\frac{2}{4} bp,$$

$$\mathbb{U}_b c = 4 \{ bc + \langle b, c \rangle I \}.$$

Next we shall discuss briefly the geodesic and killing vector fields on reductive Heisenberg Lie groups.

It is believed that geodesic on the group (G, \langle, \rangle) of Heisenberg type orbits are of one-parameter subgroups of isometries of (G, \langle, \rangle) . To see this we shall employ the killing vector fields strategy.

Here, we first find a global coordinate system $(u_1, u_2, \dots, u_n; v_1, v_2, \dots, v_m)$ on G . Now (p_1, p_2, \dots, p_n) and (b_1, b_2, \dots, b_m) be orthogonal frames of K and J , respectively. For a point $x \in G$, we have that

$$\begin{cases} u_j(x) = u_j(e^{(p(x)+b(x))}) = \langle p(x), p_j \rangle, & \text{where } j = 1, 2, \dots, n, \\ v_k(x) = v_k(e^{(p(x)+b(x))}) = \langle b(x), b_k \rangle, & \text{where } k = 1, 2, \dots, m. \end{cases}$$

Therefore,

$$\begin{cases} \frac{\partial}{\partial u_j} = p_j - \frac{2}{4} \sum_{k,j} \mathbb{F}_{ij}^k u_i b_k, \\ \frac{\partial}{\partial v_k} = b_k. \end{cases}$$

where \mathbb{F}_{ij}^k stands for the structure constants of \mathfrak{g} . That is

$$[p_j, p_i] = \sum_k \mathbb{F}_{ij}^k b_k. \quad (3.16)$$

□

Moreover, taking \mathbb{F} to be a skew-symmetric endomorphism of K and \mathbb{B} be the skew-symmetric endomorphism of J such that

$$\mathbb{F}i(b) - i(b)\mathbb{F} = i(\mathbb{B}(b)), \quad b \in J, \quad (3.17)$$

and putting $\mathbb{F}(p_j) = \sum_i b_{ij} p_j$ and $\mathbb{B}(b_k) = \sum_\gamma c_{\gamma k} b_\gamma$, then the following

theorem follows:

Theorem 3.4. *The killing vector field η of $\langle G, \langle, \rangle \rangle$ is the equation given as*

$$\eta = \sum_j \eta_j \frac{\partial}{\partial u_j} + \sum_k \eta_k \frac{\partial}{\partial v_k},$$

for

$$\eta_j = \sum_i b_{ji} u_i + \zeta_j, \quad \zeta_j, \text{ being constants.}$$

and

$$\eta_k = \sum_{\gamma} c_{k\gamma} v_{\gamma} + \frac{2}{4} \sum_{j,i} \mathbb{F}_{ji}^k u_j t_i + \zeta_k, \quad \zeta_k \text{ being constants.} \quad (3.18)$$

Proof. One can express the killing vector equations as follows:

$$\begin{cases} a(\Delta_{p_j} \eta, p_i) + a(\Delta_{p_i} \eta, p_j) = 0, \\ a(\Delta_{b_k} \eta, b_{\gamma}) + a(\Delta_{b_{\gamma}} \eta, b_k) = 0, \\ a(\Delta_{b_k} \eta, p_j) + a(\Delta_{p_j} \eta, b_k) = 0. \end{cases} \quad (3.19)$$

By setting ℓ to be Ricci tensor of G . We have specifically that

$$\ell([\eta, b_k], p_j) + \ell([\eta, p_j], p_k) = 0. \quad (3.20)$$

Applying (3.6) and (3.7) the following conditions will be obtained which are corresponding to (3.19) and (3.20)

$$\begin{cases} p_j(\eta_i) + p_i(\eta_j) = 0, \\ b_k(\eta_{\gamma}) + b_{\gamma}(\eta_k) = 0, \\ b_k(\eta_j) = 0, \\ p_j(\tilde{\eta}_k) + \sum_i \eta_i \langle [p_j, p_t], b_k \rangle = 0, \end{cases} \quad (3.21)$$

where

$$\tilde{\eta}_k = \eta_k - \frac{2}{4} \sum_{j,i} \mathbb{F}_{ij}^k u_i \eta_j. \quad (3.22)$$

Differentiating the first and the third conditions, we obtain

$$\frac{\partial \eta_j}{\partial v_k} = 0 \text{ and } \frac{\partial \eta_j}{\partial u_i} + \frac{\partial \eta_i}{\partial u_j} = 0.$$

Thus,

$$\eta_j = \sum_i b_{ji} u_i + \zeta_j,$$

for $b_{ji} + b_{ij} = 0$ and b_{ji}, ζ_j being constants. In a similar manner, from second condition we have

$$\tilde{\eta}_k = \sum_{\gamma} c_{k\gamma}(u) v_{\gamma} + \xi_k(u), \quad (3.23)$$

where $c_{k\gamma} + c_{\gamma k} = 0$.

Next is to determine the functions $c_{k\gamma}$ and ξ_k by employing the last equation of (3.21).

Now, put (3.23) in the equation and differentiate with respect to v_{γ} given that $c_{x\gamma}$ are constants. Hence we have

$$\frac{\partial \xi_k}{\partial u_j} + \frac{2}{4} \sum_{\lambda} \left(\sum_j \mathbb{F}_{ij}^{\lambda} u_i \right) c_{k\gamma} + \sum_{t,i} \mathbb{F}_{jt}^k b_{ti} u_i = 0.$$

The integrability conditions of this structure are given by

$$\sum_{\gamma} \mathbb{F}_{jt}^{\gamma} c_{k\gamma} - \sum_t \mathbb{F}_{jt}^k b_{ti} + \sum_t \mathbb{F}_{it}^k b_{tj} = 0, \quad (3.24)$$

considering (3.16) and (3.4) we see that (3.24) is the same as (3.17).

Conversely, assuming that (3.17) is given, then

$$\xi_k = -\frac{2}{4} \sum_{j,i,t} \mathbb{F}_{jt}^k b_{ti} u_j u_i + \zeta_k, \quad (3.25)$$

ζ_k being constants as before.

Hence the expression (3.18) follows immediately from (3.25), (3.23) and (3.22) \square

4. Conclusion

Homogeneous groups of Heisenberg type, which are generally reductive and generated with a heat kernels was considered. The generating kernel was then used to construct the Lie algebra associated with the Heisenberg group. Using the Lie algebra, we defined a homogeneous space associated with the Heisenberg group, which is a Riemannian manifold of Euclidian type. Detailed analysis of the geometry of this homogeneous space was provided and its relationship to the Lie algebra completely determined.

References

- [1] W. Ambrose and I. M. Singer, On homogeneous Riemannian manifolds, *Duke Mathematical Journal* 25(4) (1958), 647-669.
DOI: <https://doi.org/10.1215/S0012-7094-58-02560-2>
- [2] R. B. Brown and A. Gray, Vector cross products, *Commentarii Mathematici Helvetici* 42 (1967), 222-236.
DOI: <https://doi.org/10.1007/BF02564418>
- [3] G. Carron, T. Coulhon and E. M. Ouhabaz, Gaussian estimates and L^p -boundedness of Riesz means, *Journal of Evolution Equations* 2 (2002), 299-317.
DOI: <https://doi.org/10.1007/s00028-002-8090-1>
- [4] T. Coulhon and A. Sikora, Gaussian heat kernel upper bounds via the Phragmén-Lindelöf theorem, *The Proceedings of the London Mathematical Society* 96(2) (2008), 507-544.
DOI: <https://doi.org/10.1112/plms/pdm050>
- [5] J. E. D'Atri, Geodesic sphere and symmetries in naturally reductive homogeneous spaces, *Michigan Mathematical Journal* 22(1) (1975), 71-76.
DOI: <https://doi.org/10.1307/mmj/1029001423>
- [6] J. E. D'Atri and H. K. Nickerson, Geodesic symmetries in spaces with special curvature tensor, *Journal of Differential Geometry* 9(2) (1974), 251-262.
DOI: <https://doi.org/10.4310/jdg/1214432291>
- [7] J. E. D'Atri and W. Ziller, Naturally reductive metrics and Einstein metrics on compact Lie Groups, *Memoirs of the American Mathematical Society* 18(215) (1979), 215-227.
DOI: <https://doi.org/10.1090/memo/0215>

- [8] A. Gray, Riemannian manifolds with geodesic symmetries of order 3, *Journal of Differential Geometry* 7(3-4) (1972), 343-369.
DOI: <https://doi.org/10.4310/jdg/1214431159>
- [9] A. Kaplan, Riemannian manifolds attached to Clifford modules, *Geometriae Dedicata* 11 (1981), 127-136.
DOI: <https://doi.org/10.1007/BF00147615>
- [10] A. Kaplan, On the geometry of groups of Heisenberg type, *Bulletin of the London Mathematical Society* 15(1) (1983), 35-42.
DOI: <https://doi.org/10.1112/blms/15.1.35>
- [11] E. M. Ouhabaz, *Analysis of Heat Equations on Domains*, London Mathematical Society Monographs Series 31, Princeton University Press, 2005.
- [12] P. Petrushev and Y. Xu, Localized polynomial frames on the ball, *Constr. Approx.* 27 (2008), 121-148.
DOI: <https://doi.org/10.1007/s00365-007-0678-9>
- [13] F. Tricerri and L. Vanhecke, Homogeneous structures on Riemannian manifolds, *Lecture Note Series of the London Mathematical Society* 83 (1983), Cambridge University Press.
DOI: <https://doi.org/10.1017/CBO9781107325531>
- [14] F. Tricerri and L. Vanhecke, Naturally reductive homogeneous spaces and generalized Heisenberg groups, *Compositio Mathematica* 52(3) (1984), 389-408.
- [15] N. T. Varopoulos, L. Saloff-Coste and T. Coulhon, *Analysis and Geometry on Groups*, Cambridge Tracts in Mathematics 100, Cambridge University Press, Cambridge, 1992.
DOI: <https://doi.org/10.1017/CBO9780511662485>
- [16] J. A. Wolf, The geometry and structure of isotopy irreducible homogeneous spaces, *Acta Mathematica* 120 (1968), 59-148.
DOI: <https://doi.org/10.1007/BF02394607>

