

## NEW VIEWS AND VARIANTS OF THE GRÖNWALL INTEGRAL INEQUALITIES

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### Abstract

This article advances the understanding and development of the Grönwall integral inequalities. The first part examines the ability of the standard Grönwall integral inequalities to extend themselves. In such an extended setting, we also analyze the degree of importance of a usual non-negativity assumption. The second part establishes a new theorem on flexible variants of the Grönwall integral inequalities. This flexibility is characterized by five adaptable functions. Complementary results provide more tractable bounds under certain technical assumptions. Detailed proofs of all results ensure that the article remains self-contained.

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### 1. Introduction

The Grönwall integral inequalities are important tools in the study of differential and integral equations, particularly in establishing upper bounds on solutions. Various applications include stability analysis in dynamical systems and numerical approximations. A possible statement of the Grönwall integral inequalities is given below.

**Theorem 1.1** (Grönwall integral inequalities). *Let  $a, b \in \mathbb{R} \cup \{\pm\infty\}$  with  $a < b$ , and  $g, h : [a, b] \rightarrow \mathbb{R}$  and  $k : [a, b] \rightarrow (0, +\infty)$  be continuous functions. We suppose that, for any  $x \in [a, b]$ ,*

$$g(x) \leq h(x) + \int_a^x k(t)g(t)dt. \quad (1)$$

(1) *The first Grönwall integral inequality states that, for any  $x \in [a, b]$ ,*

$$g(x) \leq h(x) + \int_a^x h(r)k(r) \exp\left(\int_r^x k(t)dt\right)dr. \quad (2)$$

(2) *Furthermore, if  $h$  is non-decreasing, the second Grönwall integral inequality states that, for any  $x \in [a, b]$ ,*

$$g(x) \leq h(x) \exp\left(\int_a^x k(t)dt\right). \quad (3)$$

*Note: The interval  $[a, b]$  can be replaced by  $(a, b)$ ,  $[a, b)$ , or  $(a, b]$  (in this theorem and throughout the article).*

These inequalities have been the subject of much research and development. The main references are [1, 2, 4-21]. We also refer to the complete survey in [3], which provides a detailed historical overview of their applications and extensions.

The contributions of this article can be divided into two complementary parts. The first part provides a deeper understanding of the Grönwall integral inequalities. In particular, we highlight their

ability to be extended by the inclusion of an adaptable function. We also evaluate the importance of a common non-negativity assumption on this function, and show that it can be relaxed to obtain intermediate integral inequalities of potential interest. This provides a new perspective on ([20], Theorem 1.3.3). Several related results are established, including an alternative inequality to that in ([10], Lemma 2.1). Some of these results consider original assumptions on the functions involved, giving an original framework for integral inequalities. The second part introduces a new and general theorem on variants of the Grönwall integral inequalities characterized by five adaptable functions related by sophisticated integral inequality assumptions. Notably, to highlight its originality, one of the derived inequalities takes the following form:

$$\begin{aligned} g(x) \int_a^x k(t)h(t)dt + h(x) \int_a^x k(t)g(t)dt \\ \leq \ell(x) + f(x) \int_a^x \ell(r)k(r) \exp\left(\int_r^x k(t)f(t)dt\right)dr. \end{aligned}$$

The theorem is complemented by additional results. Some of these results include more tractable upper bounds, focusing on bounded  $g$ , and original assumptions. They are established in full generality, enhancing the applicability of the new inequalities in various analytical settings.

The other three sections of the article are as follows: Section 2 presents direct extensions to the Grönwall integral inequalities, focusing on some assumptions on the functions involved. Section 3 contains the main theorem and related results. The article concludes in Section 4 with a summary and some perspectives.

## 2. Direct Extensions

### 2.1. First approach

The proposition below is an extension of the Grönwall integral inequalities, using an additional function  $f$ . Interestingly, despite this

extension, the result follows directly from the Grönwall integral inequalities. In this sense, the Grönwall integral inequalities have the ability to extend themselves.

As a technical note, in all our statements and proofs, we omit the case of zero functions, since the results typically follow immediately. Furthermore, it is implicitly assumed that each division has a non-zero denominator.

**Proposition 2.1.** *Let  $a, b \in \mathbb{R} \cup \{\pm\infty\}$  with  $a < b$ , and  $g, h : [a, b] \rightarrow \mathbb{R}$  and  $f, k : [a, b] \rightarrow [0, +\infty)$  be continuous functions. We assume that, for any  $x \in [a, b]$ ,*

$$g(x) \leq h(x) + f(x) \int_a^x k(t)g(t)dt. \quad (4)$$

- *Then the first Grönwall integral inequality can be used to establish that, for any  $x \in [a, b]$ ,*

$$g(x) \leq h(x) + f(x) \int_a^x h(r)k(r) \exp\left(\int_r^x k(t)f(t)dt\right)dr.$$

- *Furthermore, if  $h/f$  is non-decreasing, the second Grönwall integral inequality can be used to establish that, for any  $x \in [a, b]$ ,*

$$g(x) \leq h(x) \exp\left(\int_a^x k(t)f(t)dt\right).$$

**Proof of Proposition 2.1.** We will prove the two points in turn by considering the Grönwall integral inequalities as stated in Theorem 1.1.

- Since  $f$  is non-negative, dividing by  $f(x)$ , the inequality in Equation (4) is equivalent to, for any  $x \in [a, b]$ ,

$$\frac{g(x)}{f(x)} \leq \frac{h(x)}{f(x)} + \int_a^x k(t)g(t)dt,$$

which can also be rewritten as

$$g_{\star}(x) \leq h_{\star}(x) + \int_a^x k_{\star}(t)g_{\star}(t)dt, \quad (5)$$

where  $g_{\star}(x)=g(x)/f(x)$ ,  $h_{\star}(x)=h(x)/f(x)$ , and  $k_{\star}(x)=k(x)f(x)$ . The assumption in Equation (1) is thus satisfied with these transformed functions. Note also that  $k_{\star}$  is non-negative. We can therefore apply the first Grönwall integral inequality in Equation (2), which gives

$$g_{\star}(x) \leq h_{\star}(x) + \int_a^x h_{\star}(r)k_{\star}(r) \exp\left(\int_r^x k_{\star}(t)dt\right)dr.$$

It can also be written as

$$\frac{g(x)}{f(x)} \leq \frac{h(x)}{f(x)} + \int_a^x \frac{h(r)}{f(r)} k(r)f(r) \exp\left(\int_r^x k(t)f(t)dt\right)dr,$$

and, after simplification and multiplication by  $f(x)$  (still using its non-negativity),

$$g(x) \leq h(x) + f(x) \int_a^x h(r)k(r) \exp\left(\int_r^x k(t)f(t)dt\right)dr.$$

The first desired result is proved.

- Now, adopting the above framework, if  $h/f$  is non-decreasing, i.e.,  $h_{\star}$  is non-decreasing, the second Grönwall integral inequality in Equation (3) gives, for any  $x \in [a, b]$

$$g_{\star}(x) \leq h_{\star}(x) \exp\left(\int_a^x k_{\star}(t)dt\right),$$

which can be expressed as

$$\frac{g(x)}{f(x)} \leq \frac{h(x)}{f(x)} \exp\left(\int_a^x k(t)f(t)dt\right).$$

Multiplying by  $f(x)$ , we get

$$g(x) \leq h(x) \exp\left(\int_a^x k(t)f(t)dt\right).$$

The second desired result is established.

This completes the proof.  $\square$

Obviously, if we take  $f = 1$ , i.e., the function equal to 1, Proposition 2.1 becomes Theorem 1.1. We have thus used the Grönwall integral inequality to extend its scope under an appropriate functional configuration. The non-negative assumption on  $f$  was a central point. This assumption will be discussed in more detail in the next subsection.

In a sense, this proposition also completes ([20], Theorem 1.3.3), which considers  $h$  non-decreasing and a different approach. See also ([3], Theorem 4).

The result below provides an alternative to Proposition 2.1 under different assumptions on the functions involved.

**Proposition 2.2.** *Let  $a, b \in \mathbb{R} \cup \{\pm\infty\}$  with  $a < b$ ,  $f, h: [a, b] \rightarrow [0, +\infty)$  be differentiable non-decreasing functions and  $g, k: [a, b] \rightarrow [0, +\infty)$  be continuous functions. We assume that, for any  $x \in [a, b]$ ,*

$$g(x) \leq h(x) + f(x) \int_a^x k(t)g(t)dt. \quad (6)$$

Then we have

$$g(x) \leq \frac{1}{f(a)} h(x)f(x) \exp\left(\int_a^x k(t)f(t)dt\right).$$

**Proof of Proposition 2.2.** Let us introduce, for any  $r \in [a, b]$ ,

$$U(r) = h(r) + f(r) \int_a^r k(t)g(t)dt.$$

Since all the functions involved are differentiable, a differentiation with respect to  $r$  gives, for any  $r \in [a, b]$ ,

$$U'(r) = h'(r) + f'(r) \int_a^r k(t)g(t)dt + f(r)k(r)g(r).$$

Note that, since  $f$  and  $h$  are differentiable and non-decreasing functions, we have  $f'(r) \geq 0$  and  $h'(r) \geq 0$  for any  $r \in [a, b]$ . Furthermore, since all the functions involved are non-negative, we have  $\int_a^r k(t)g(t)dt \geq 0$ ,  $f(r)k(r)g(r) \geq 0$ ,  $U(r) \geq 0$ ,  $U(r) \geq h(r)$ , and  $U(r) \geq f(r) \int_a^r k(t)g(t)dt$  for any  $r \in [a, b]$ , and, by Equation (6),  $U(r) \geq g(r)$  for any  $r \in [a, b]$ . As a result, choosing appropriate lower bounds for  $U(r)$  in each ratio term leads to

$$\begin{aligned} \frac{U'(r)}{U(r)} &= \frac{h'(r)}{U(r)} + \frac{f'(r)}{U(r)} \int_a^r k(t)g(t)dt + \frac{f(r)k(r)g(r)}{U(r)} \\ &\leq \frac{h'(r)}{h(r)} + \frac{f'(r)}{f(r) \int_a^r k(t)g(t)dt} \int_a^r k(t)g(t)dt + \frac{f(r)k(r)g(r)}{g(r)} \\ &= \frac{h'(r)}{h(r)} + \frac{f'(r)}{f(r)} + k(r)f(r). \end{aligned}$$

Integrating with respect to  $r$  for  $r \in [a, x]$  with  $x \in [a, b]$ , we get

$$\int_a^x \frac{U'(r)}{U(r)} dr \leq \int_a^x \frac{h'(r)}{h(r)} dr + \int_a^x \frac{f'(r)}{f(r)} dr + \int_a^x k(r)f(r)dr,$$

so that

$$\log\left(\frac{U(x)}{U(a)}\right) \leq \log\left(\frac{h(x)}{h(a)}\right) + \log\left(\frac{f(x)}{f(a)}\right) + \int_a^x k(r)f(r)dr,$$

which is equivalent to

$$\log\left(\frac{U(x)}{U(a)}\right) \leq \log\left(\frac{h(x)}{h(a)} \frac{f(x)}{f(a)} \exp\left(\int_a^x k(r)f(r)dr\right)\right).$$

Transforming by the (increasing) exponential function and noting that  $U(a) = h(a)$ , we get

$$\frac{U(x)}{h(a)} \leq \frac{h(x)}{h(a)} \frac{f(x)}{f(a)} \exp\left(\int_a^x k(r)f(r)dr\right),$$

and, multiplying by  $h(a)$ ,

$$U(x) \leq \frac{1}{f(a)} h(x)f(x) \exp\left(\int_a^x k(r)f(r)dr\right).$$

Using this,  $U(x) \geq g(x)$  for any  $x \in [a, b]$  by Equation (6), and a minor notation change, we establish that

$$g(x) \leq \frac{1}{f(a)} h(x)f(x) \exp\left(\int_a^x k(t)f(t)dt\right).$$

This ends the proof.  $\square$

Compared to the second result in Proposition 2.1, Proposition 2.2 is more restrictive on the sign of the functions involved; they must be non-negative and impose some differentiability conditions. However, it holds for  $f$  and  $h$  which are non-decreasing, whereas the second result of Proposition 2.1 requires  $h/f$  to be non-decreasing, which is a crucial difference. We also note that the results are slightly different with the significant role of  $f$  in the upper bound in Proposition 2.2. In particular, if  $f$  is non-decreasing, which means that  $f(x) \geq f(a)$  for any  $x \in [a, b]$ , and  $h$  is non-negative, we have

$$h(x) \exp\left(\int_a^x k(t)f(t)dt\right) \leq \frac{1}{f(a)} h(x)f(x) \exp\left(\int_a^x k(t)f(t)dt\right).$$

In this special case, if  $h/f$  is also non-decreasing, the second result in Proposition 2.1 must be privileged.



From another point of view, in a comparable setting, Proposition 2.2 gives an alternative bound to the result in ([10], Lemma 2.1), without the assumption  $h(x) > 1$  for any  $x \in [a, b]$ .

## 2.2. Extended approach

This subsection is devoted to contributions to Proposition 2.1. First, given the assumptions made, it is natural to ask the following question: What is the deep role of the non-negativity assumption on  $f$  in establishing these inequalities? The answer is given in the theorem below.

**Theorem 2.3.** *Let  $a, b \in \mathbb{R} \cup \{\pm\infty\}$  with  $a < b$ , and  $f, g, h : [a, b] \rightarrow \mathbb{R}$  and  $k : [a, b] \rightarrow [0, +\infty)$  be continuous functions. We assume that, for any  $x \in [a, b]$ ,*

$$g(x) \leq h(x) + f(x) \int_a^x k(t)g(t)dt. \quad (7)$$

(1) *Then we have, for any  $x \in [a, b]$ ,*

$$\int_a^x k(t)g(t)dt \leq \int_a^x h(r)k(r) \exp\left(\int_r^x k(t)f(t)dt\right)dr.$$

(2) *Under the additional assumption that  $f$  is non-negative, we have, for any  $x \in [a, b]$ .*

$$g(x) \leq h(x) + f(x) \int_a^x h(r)k(r) \exp\left(\int_r^x k(t)f(t)dt\right)dr.$$

*If the inequality in Equation (7) is reversed then the final inequalities are reversed.*

**Proof of Theorem 2.3.** First, note that, since  $f$  is not assumed to be non-negative, dividing by  $f(x)$  does not necessarily preserve the inequality in Equation (7), as was exploited in the proof of Proposition 2.1. For this reason, we propose a different approach, which is described

in detail below.

(1) Let us set, for any  $r \in [a, b]$ ,

$$V(r) = \left( \int_a^r k(t)g(t)dt \right) \exp \left( - \int_a^r k(t)f(t)dt \right). \quad (8)$$

Differentiating with respect to  $r$  and using standard differentiation rules, we obtain

$$\begin{aligned} V'(r) &= k(r)g(r) \exp \left( - \int_a^r k(t)f(t)dt \right) \\ &\quad + \left( \int_a^r k(t)g(t)dt \right) (-k(r)f(r)) \exp \left( - \int_a^r k(t)f(t)dt \right) \\ &= \left( g(r) - f(r) \int_a^r k(t)g(t)dt \right) k(r) \exp \left( - \int_a^r k(t)f(t)dt \right). \end{aligned}$$

It follows from this, Equation (7) and the non-negativity of  $k$ , that

$$V'(r) \leq h(r)k(r) \exp \left( - \int_a^r k(t)f(t)dt \right). \quad (9)$$

Integrating with respect to  $r$  for  $r \in [a, x]$  with  $x \in [a, b]$  and noting that  $V(a) = 0$ , we get

$$V(x) = V(x) - V(a) = \int_a^x V'(r)dr \leq \int_a^x h(r)k(r) \exp \left( - \int_a^r k(t)f(t)dt \right) dr.$$

Using this and the definition of  $V(x)$  in Equation (8), we obtain

$$\left( \int_a^x k(t)g(t)dt \right) \exp \left( - \int_a^x k(t)f(t)dt \right) \leq \int_a^x h(r)k(r) \exp \left( - \int_a^r k(t)f(t)dt \right) dr,$$

which can be expressed as follows:

$$\int_a^x k(t)g(t)dt \leq \int_a^x h(r)k(r) \exp \left( \int_a^x k(t)f(t)dt - \int_a^r k(t)f(t)dt \right) dr$$

$$= \int_a^x h(r)k(r) \exp\left(\int_r^x k(t)f(t)dt\right)dr. \quad (10)$$

This is the first desired inequality.

(2) Under the additional assumption that  $f$  is non-negative, the combination of Equations (7) and (10) directly gives

$$g(x) \leq h(x) + f(x) \int_a^x h(r)k(r) \exp\left(\int_r^x k(t)f(t)dt\right)dr. \quad (11)$$

This is the second desired inequality.

If the inequality in Equation (7) is reversed, then the inequality in Equation (9) is also reversed and the same in Equations (10) and (11).

This ends the proof.  $\square$

The first point of Theorem 2.3 presents a new integral inequality which does not require the non-negativity assumption on  $f$ . This inequality is clearly the main part of the proof. The non-negativity assumption is only needed for the second point, in the final step of the proof. This is a general version of the second Grönwall integral inequality. With these results, we complete the understanding of ([20], Theorem 1.3.3) and ([3], Theorem 4) by showing where the non-negativity assumption on  $f$  is crucial, and that no monotonicity assumption on  $f$  is necessary. The reversed integral inequalities, which are described at the end of the theorem, are also new contributions.

Some original assumptions under which a tractable upper bound for  $g$  is obtained are presented in the proposition below.

**Proposition 2.4.** *Let  $a, b \in \mathbb{R} \cup \{\pm\infty\}$  with  $a < b$ , and  $g, h, \ell : [a, b] \rightarrow \mathbb{R}$  and  $f, k : [a, b] \rightarrow [0, +\infty)$  be continuous functions.*

*We assume that*

- *for any  $x \in [a, b]$ ,*

$$g(x) \leq h(x) + f(x) \int_a^x k(t)g(t)dt, \quad (12)$$

- there exist  $\alpha, \beta \in \mathbb{R}$  such that, for any  $x \in [a, b]$  and  $r \in [a, x]$ ,

$$h(r)f(x) \leq \alpha f(r)\ell(x) \exp\left(\beta \int_r^x k(t)f(t)dt\right). \quad (13)$$

Then we have

$$g(x) \leq h(x) - \frac{\alpha}{\beta+1} \ell(x) + \frac{\alpha}{\beta+1} \ell(x) \exp\left((\beta+1) \int_a^x k(t)f(t)dt\right).$$

**Proof of Proposition 2.4.** We are in the framework of Theorem 2.3, whose main assumption is formulated in Equation (12). The second point of this theorem gives, for any  $x \in [a, b]$ ,

$$\begin{aligned} g(x) &\leq h(x) + f(x) \int_a^x h(r)k(r) \exp\left(\int_r^x k(t)f(t)dt\right)dr \\ &= h(x) + \int_a^x (h(r)f(x))k(r) \exp\left(\int_r^x k(t)f(t)dt\right)dr. \end{aligned} \quad (14)$$

Let us now majorize this last term. Applying the inequality in Equation (13), we get

$$\begin{aligned} &h(x) + \int_a^x (h(r)f(x))k(r) \exp\left(\int_r^x k(t)f(t)dt\right)dr \\ &\leq h(x) + \alpha \int_a^x \left(f(r)\ell(x) \exp\left(\beta \int_r^x k(t)f(t)dt\right)\right)k(r) \exp\left(\int_r^x k(t)f(t)dt\right)dr \\ &= h(x) + \alpha \ell(x) \int_a^x f(r)k(r) \exp\left((\beta+1) \int_r^x k(t)f(t)dt\right)dr \\ &= h(x) + \alpha \ell(x) \left[ -\frac{1}{\beta+1} \exp\left((\beta+1) \int_r^x k(t)f(t)dt\right) \right]_{r=a}^{r=x} \end{aligned}$$

$$\begin{aligned}
&= h(x) + \alpha \ell(x) \left( -\frac{1}{\beta+1} + \frac{1}{\beta+1} \exp \left( (\beta+1) \int_a^x k(t) f(t) dt \right) \right) \\
&= h(x) - \frac{\alpha}{\beta+1} \ell(x) + \frac{\alpha}{\beta+1} \ell(x) \exp \left( (\beta+1) \int_a^x k(t) f(t) dt \right).
\end{aligned}$$

Putting this inequality in Equation (14) gives

$$g(x) \leq h(x) - \frac{\alpha}{\beta+1} \ell(x) + \frac{\alpha}{\beta+1} \ell(x) \exp \left( (\beta+1) \int_a^x k(t) f(t) dt \right).$$

This concludes the proof.  $\square$

An alternative to this proposition is given below, under a monotonicity assumption on  $f$ .

**Proposition 2.5.** *Let  $a, b \in \mathbb{R} \cup \{\pm\infty\}$  with  $a < b$ , and  $g, h : [a, b] \rightarrow \mathbb{R}$  and  $f, k : [a, b] \rightarrow [0, +\infty)$  be continuous functions. We assume that*

- *for any  $x \in [a, b]$ ,*

$$g(x) \leq h(x) + f(x) \int_a^x k(t) g(t) dt, \quad (15)$$

- *$f$  is non-decreasing.*

Then we have

$$g(x) \leq h(x) - \sup_{r \in [a, x]} |h(r)| + \sup_{r \in [a, x]} |h(r)| \exp \left( f(x) \int_a^x k(t) dt \right).$$

**Proof of Proposition 2.5.** We are in the framework of Theorem 2.3, whose main assumption is formulated in Equation (15). Applying this theorem and the triangular inequality, taking into account that  $f$  and  $k$  are non-negative, we have, for any  $x \in [a, b]$ ,

$$g(x) \leq h(x) + f(x) \int_a^x h(r) k(r) \exp \left( \int_r^x k(t) f(t) dt \right) dr$$

$$\begin{aligned}
&\leq h(x) + f(x) \left| \int_a^x h(r)k(r) \exp\left(\int_r^x k(t)f(t)dt\right) dr \right| \\
&\leq h(x) + f(x) \int_a^x |h(r)|k(r) \exp\left(\int_r^x k(t)f(t)dt\right) dr. \tag{16}
\end{aligned}$$

Let us now majorize last term. Since  $|h|$  is continuous on  $[a, b]$ ,  $f$  is non-negative and non-decreasing, and  $k$  is non-negative, we get

$$\begin{aligned}
&h(x) + f(x) \int_a^x |h(r)|k(r) \exp\left(\int_r^x k(t)f(t)dt\right) dr \\
&\leq h(x) + f(x) \sup_{r \in [a, x]} |h(r)| \int_a^x k(r) \exp\left(f(x) \int_r^x k(t)dt\right) dr \\
&= h(x) + f(x) \sup_{r \in [a, x]} |h(r)| \left[ -\frac{1}{f(x)} \exp\left(f(x) \int_r^x k(t)dt\right) \right]_{r=a}^{r=x} \\
&= h(x) + f(x) \sup_{r \in [a, x]} |h(r)| \left( -\frac{1}{f(x)} + \frac{1}{f(x)} + \exp\left(f(x) \int_a^x k(t)dt\right) \right) \\
&= h(x) - \sup_{r \in [a, x]} |h(r)| + \sup_{r \in [a, x]} |h(r)| \exp\left(f(x) \int_a^x k(t)dt\right).
\end{aligned}$$

Putting this inequality in Equation (16) gives

$$g(x) \leq h(x) - \sup_{r \in [a, x]} |h(r)| + \sup_{r \in [a, x]} |h(r)| \exp\left(f(x) \int_a^x k(t)dt\right).$$

This concludes the proof.  $\square$

Note that, in the case where  $h$  is non-negative and non-decreasing, we have  $\sup_{r \in [a, x]} |h(r)| = h(x)$ , and Proposition 2.5 gives

$$g(x) \leq h(x) \exp\left(f(x) \int_a^x k(t)dt\right).$$

To the best of our knowledge, the upper bounds obtained in this section are new in the field of the Grönwall integral inequalities, under such general assumptions.

The rest of the article proposes the study of new variants, more technical on the mathematical aspects.

### 3. Technical Variants

This section begins with the main theorem, followed by some related propositions.

#### 3.1. Main theorem

The theorem below is a new variant of the Grönwall integral inequalities. It has the property of involving five adaptable functions.

**Theorem 3.1.** *Let  $a, b \in \mathbb{R} \cup \{\pm\infty\}$  with  $a < b$ , and  $f, g, h, \ell : [a, b] \rightarrow \mathbb{R}$  and  $k : [a, b] \rightarrow [0, +\infty)$  be continuous functions. We assume that, for any  $x \in [a, b]$ ,*

$$\begin{aligned} & g(x) \int_a^x k(t)h(t)dt + h(x) \int_a^x k(t)g(t)dt \\ & \leq \ell(x) + f(x) \left( \int_a^x k(t)g(t)dt \right) \left( \int_a^x k(t)h(t)dt \right). \end{aligned} \quad (17)$$

(1) *Then we have, for any  $x \in [a, b]$ ,*

$$\left( \int_a^x k(t)g(t)dt \right) \left( \int_a^x k(t)h(t)dt \right) \leq \int_a^x \ell(r)k(r) \exp \left( \int_r^x k(t)f(t)dt \right) dr.$$

(2) *Under the additional assumption that  $f$  is non-negative, we have, for any  $x \in [a, b]$ ,*

$$\begin{aligned} & g(x) \int_a^x k(t)h(t)dt + h(x) \int_a^x k(t)g(t)dt \\ & \leq \ell(x) + f(x) \int_a^x \ell(r)k(r) \exp \left( \int_r^x k(t)f(t)dt \right) dr. \end{aligned}$$

*If the inequality in Equation (17) is reversed, the final inequalities are reversed.*

**Proof of Theorem 3.1.**

(1) Let us set, for any  $r \in [a, b]$ ,

$$W(r) = \left( \int_a^r k(t)g(t)dt \right) \left( \int_a^r k(t)h(t)dt \right) \exp \left( - \int_a^r k(t)f(t)dt \right). \quad (18)$$

Differentiating with respect to  $r$  and using standard differentiation rules, we obtain

$$\begin{aligned} W'(r) &= k(r)g(r) \left( \int_a^r k(t)h(t)dt \right) \exp \left( - \int_a^r k(t)f(t)dt \right) \\ &+ \left( \int_a^r k(t)g(t)dt \right) k(r)h(r) \exp \left( - \int_a^r k(t)f(t)dt \right) \\ &+ \left( \int_a^r k(t)g(t)dt \right) \left( \int_a^r k(t)h(t)dt \right) (-k(r)f(r)) \exp \left( - \int_a^r k(t)f(t)dt \right) \\ &= \left( g(r) \int_a^r k(t)h(t)dt + h(r) \int_a^r k(t)g(t)dt \right. \\ &\quad \left. - f(r) \left( \int_a^r k(t)g(t)dt \right) \left( \int_a^r k(t)h(t)dt \right) \right) k(r) \exp \left( - \int_a^r k(t)f(t)dt \right). \end{aligned}$$

It follows from this, Equation (17) and the non-negativity of  $k$ , that

$$W'(r) \leq \ell(r)k(r) \exp \left( - \int_a^r k(t)f(t)dt \right). \quad (19)$$

Therefore, integrating with respect to  $r$  for  $r \in [a, x]$  with  $x \in [a, b]$  and noting that  $W(a) = 0$ , we have

$$W(x) = W(x) - W(a) = \int_a^x W'(r)dr \leq \int_a^x \ell(r)k(r) \exp \left( - \int_a^r k(t)f(t)dt \right) dr.$$

Using this and the definition of  $W(x)$  in Equation (18), we find that

$$\left( \int_a^x k(t)g(t)dt \right) \left( \int_a^x k(t)h(t)dt \right) \exp \left( - \int_a^x k(t)f(t)dt \right)$$



$$\leq \int_a^x \ell(r)k(r) \exp\left(-\int_a^r k(t)f(t)dt\right)dr,$$

which can be expressed as follows:

$$\begin{aligned} & \left(\int_a^x k(t)g(t)dt\right)\left(\int_a^x k(t)h(t)dt\right) \\ & \leq \int_a^r \ell(r)k(r) \exp\left(\int_a^x k(t)f(t)dt - \int_a^r k(t)f(t)dt\right)dr, \end{aligned}$$

so that

$$\left(\int_a^x k(t)g(t)dt\right)\left(\int_a^x k(t)h(t)dt\right) \leq \int_a^x \ell(r)k(r) \exp\left(\int_r^x k(t)f(t)dt\right)dr. \quad (20)$$

This is the first desired result.

(2) Under the additional assumption that  $f$  is non-negative, the combination of Equations (17) and (20) directly gives

$$\begin{aligned} & g(x)\int_a^x k(t)h(t)dt + h(x)\int_a^x k(t)g(t)dt \\ & \leq \ell(x) + f(x)\int_a^x \ell(r)k(r) \exp\left(\int_r^x k(t)f(t)dt\right)dr. \end{aligned} \quad (21)$$

This is the second desired result.

If the inequality in Equation (17) is reversed, the inequality in Equation (19) is reversed, and the same in Equations (20) and (21).

This concludes the proof.  $\square$

The first point of Theorem 3.1 presents a new integral inequality without the non-negativity assumption on  $f$ . However, this assumption is necessary for the second point. To the best of our knowledge, this variant of the Grönwall integral inequalities is new in the literature, and we show how five functions can be articulated under sophisticated integral inequality assumptions. The reversed integral inequalities described at

the end of the theorem also open up some interesting perspectives for applications.

The rest of this section is devoted to results related to this theorem.

### 3.2. Related results

In Theorem 3.1,  $g$  is not isolated in the left term. However, under some assumptions, it can be, as developed in the proposition below.

**Proposition 3.2.** *Let  $a, b \in \mathbb{R} \cup \{\pm\infty\}$  with  $a < b$ , and  $g, \ell : [a, b] \rightarrow \mathbb{R}$  and  $f, h, k : [a, b] \rightarrow [0, +\infty)$  be continuous functions. We assume that*

- for any  $x \in [a, b]$ ,

$$\begin{aligned} g(x) \int_a^x k(t)h(t)dt + h(x) \int_a^x k(t)g(t)dt \\ \leq \ell(x) + f(x) \left( \int_a^x k(t)g(t)dt \right) \left( \int_a^x k(t)h(t)dt \right), \end{aligned} \quad (22)$$

- there exists  $\lambda > -1$  such that, for any  $x \in [a, b]$ ,

$$\lambda g(x) \int_a^x k(t)h(t)dt \leq h(x) \int_a^x k(t)g(t)dt. \quad (23)$$

Then we have, for any  $x \in [a, b]$ ,

$$g(x) \leq \frac{1}{(\lambda + 1) \int_a^x k(t)h(t)dt} \left( \ell(x) + f(x) \int_a^x \ell(r)k(r) \exp \left( \int_r^x k(t)f(t)dt \right) dr \right).$$

**Proof of Proposition 3.2.** Using the inequality in Equation (23), we have, for any  $x \in [a, b]$

$$\begin{aligned} (\lambda + 1)g(x) \int_a^x k(t)h(t)dt &= g(x) \int_a^x k(t)h(t)dt + \lambda g(x) \int_a^x k(t)h(t)dt \\ &\leq g(x) \int_a^x k(t)h(t)dt + h(x) \int_a^x k(t)g(t)dt. \end{aligned}$$

Since we are in the framework of the second point of Theorem 3.1, whose main assumption is formulated in Equation (22), the last term can be bounded as

$$\begin{aligned} & g(x) \int_a^x k(t)h(t)dt + h(x) \int_a^x k(t)g(t)dt \\ & \leq \ell(x) + f(x) \int_a^x \ell(r)k(r) \exp\left(\int_r^x k(t)f(t)dt\right)dr. \end{aligned}$$

We therefore have

$$(\lambda + 1)g(x) \int_a^x k(t)h(t)dt \leq \ell(x) + f(x) \int_a^x \ell(r)k(r) \exp\left(\int_r^x k(t)f(t)dt\right)dr.$$

Since  $\lambda > -1$ , we have  $\lambda + 1 > 0$ , and the fact that  $h$  and  $k$  are non-negative implies that  $\int_a^x k(t)h(t)dt \geq 0$  for any  $x \in [a, b]$ . As a result, dividing by the appropriate term, we get

$$g(x) \leq \frac{1}{(\lambda + 1) \int_a^x k(t)h(t)dt} \left( \ell(x) + f(x) \int_a^x \ell(r)k(r) \exp\left(\int_r^x k(t)f(t)dt\right)dr \right).$$

This ends the proof.  $\square$

In particular, if  $g$  and  $h$  are non-negative, the inequality in Equation (23) is obviously satisfied with  $\lambda = 0$ , and we have, for any  $x \in [a, b]$ ,

$$g(x) \leq \frac{1}{\int_a^x k(t)h(t)dt} \left( \ell(x) + f(x) \int_a^x \ell(r)k(r) \exp\left(\int_r^x k(t)f(t)dt\right)dr \right).$$

Therefore, a valuable upper bound on  $g$  can be obtained under reasonable assumptions on the functions involved.

The proposition below gives a simpler upper bound than that obtained in Theorem 3.1 under different assumptions on the functions involved.

**Proposition 3.3.** *Let  $a, b \in \mathbb{R} \cup \{\pm\infty\}$  with  $a < b$ , and  $g, h, \ell : [a, b] \rightarrow \mathbb{R}$  and  $f, k : [a, b] \rightarrow [0, +\infty)$  be continuous functions. We assume that*

- *for any  $x \in [a, b]$ ,*

$$\begin{aligned} & g(x) \int_a^x k(t)h(t)dt + h(x) \int_a^x k(t)g(t)dt \\ & \leq \ell(x) + f(x) \left( \int_a^x k(t)g(t)dt \right) \left( \int_a^x k(t)h(t)dt \right), \end{aligned} \quad (24)$$

- *$\ell/f$  is non-decreasing.*

Then we have

$$g(x) \int_a^x k(t)h(t)dt + h(x) \int_a^x k(t)g(t)dt \leq \ell(x) \exp \left( \int_a^x k(t)f(t)dt \right).$$

**Proof of Proposition 3.3.** We are in the framework of Theorem 3.1, whose main assumption is formulated in Equation (24). The second point of this theorem ensures that, for any  $x \in [a, b]$ ,

$$\begin{aligned} & g(x) \int_a^x k(t)h(t)dt + h(x) \int_a^x k(t)g(t)dt \\ & \leq \ell(x) + f(x) \int_a^x \ell(r)k(r) \exp \left( \int_r^x k(t)f(t)dt \right) dr. \end{aligned}$$

Let us now majorize this last term. Since  $f$  is non-negative and  $\ell/f$  is non-decreasing, we get

$$\begin{aligned} & \ell(x) + f(x) \int_a^x \ell(r)k(r) \exp \left( \int_r^x k(t)f(t)dt \right) dr \\ & = \ell(x) + f(x) \int_a^x \frac{\ell(r)}{f(r)} k(r)f(r) \exp \left( \int_r^x k(t)f(t)dt \right) dr \\ & \leq \ell(x) + f(x) \frac{\ell(x)}{f(x)} \int_a^x k(r)f(r) \exp \left( \int_r^x k(t)f(t)dt \right) dr \end{aligned}$$

$$\begin{aligned}
&= \ell(x) + \ell(x) \left[ - \exp \left( \int_r^x k(t)f(t)dt \right) \right]_{r=a}^{r=x} \\
&= \ell(x) + \ell(x) \left( -1 + \exp \left( \int_a^x k(t)f(t)dt \right) \right) \\
&= \ell(x) \exp \left( \int_a^x k(t)f(t)dt \right).
\end{aligned}$$

We thus obtain

$$g(x) \int_a^x k(t)h(t)dt + h(x) \int_a^x k(t)g(t)dt \leq \ell(x) \exp \left( \int_a^x k(t)f(t)dt \right).$$

This ends the proof.  $\square$

Proceeding as in Proposition 3.2, we can isolate some bounds for  $g$  only under non-negativity assumptions on  $g$ ,  $h$ , and  $k$ .

The proposition below examines a more technical upper bound for  $g$  under some integral inequality conditions.

**Proposition 3.4.** *Let  $a, b \in \mathbb{R} \cup \{\pm\infty\}$  with  $a < b$ , and  $g, h, \ell, m : [a, b] \rightarrow \mathbb{R}$  and  $f, k : [a, b] \rightarrow [0, +\infty)$  be continuous functions. We assume that*

- *for any  $x \in [a, b]$ ,*

$$\begin{aligned}
&g(x) \int_a^x k(t)h(t)dt + h(x) \int_a^x k(t)g(t)dt \\
&\leq \ell(x) + f(x) \left( \int_a^x k(t)g(t)dt \right) \left( \int_a^x k(t)h(t)dt \right), \tag{25}
\end{aligned}$$

- *there exist  $\alpha, \beta \in \mathbb{R}$  such that, for any  $x \in [a, b]$  and  $r \in [a, x]$ ,*

$$\ell(r)f(x) \leq \alpha f(r)m(x) \exp \left( \beta \int_r^x k(t)f(t)dt \right). \tag{26}$$

Then we have

$$\begin{aligned}
& g(x) \int_a^x k(t)h(t)dt + h(x) \int_a^x k(t)g(t)dt \\
& \leq \ell(x) - \frac{\alpha}{\beta+1} m(x) + \frac{\alpha}{\beta+1} m(x) \exp\left((\beta+1) \int_a^x k(t)f(t)dt\right).
\end{aligned}$$

**Proof of Proposition 3.4.** We are in the framework of Theorem 3.1, whose main assumption is formulated in Equation (25). The second point of this theorem gives, for any  $x \in [a, b]$ ,

$$\begin{aligned}
& g(x) \int_a^x k(t)h(t)dt + h(x) \int_a^x k(t)g(t)dt \\
& \leq \ell(x) + f(x) \int_a^x \ell(r)k(r) \exp\left(\int_r^x k(t)f(t)dt\right)dr \\
& = \ell(x) + \int_a^x (\ell(r)f(x))k(r) \exp\left(\int_r^x k(t)f(t)dt\right)dr. \tag{27}
\end{aligned}$$

Let us now majorize this last term. It follows from the inequality in Equation (26) that

$$\begin{aligned}
& \ell(x) + \int_a^x (\ell(r)f(x))k(r) \exp\left(\int_r^x k(t)f(t)dt\right)dr \\
& \leq \ell(x) + \alpha \int_a^x \left(f(r)m(x) \exp\left(\beta \int_r^x k(t)f(t)dt\right)\right)k(r) \exp\left(\int_r^x k(t)f(t)dt\right)dr \\
& = \ell(x) + \alpha m(x) \int_a^x f(r)k(r) \exp\left((\beta+1) \int_r^x k(t)f(t)dt\right)dr \\
& = \ell(x) + \alpha m(x) \left[ -\frac{1}{\beta+1} \exp\left((\beta+1) \int_r^x k(t)f(t)dt\right) \right]_{r=a}^{r=x} \\
& = \ell(x) + \alpha m(x) \left( -\frac{1}{\beta+1} + \frac{1}{\beta+1} \exp\left((\beta+1) \int_a^x k(t)f(t)dt\right) \right) \\
& = \ell(x) - \frac{\alpha}{\beta+1} m(x) + \frac{\alpha}{\beta+1} m(x) \exp\left((\beta+1) \int_a^x k(t)f(t)dt\right).
\end{aligned}$$

Putting this inequality in Equation (27) gives

$$\begin{aligned} & g(x) \int_a^x k(t)h(t)dt + h(x) \int_a^x k(t)g(t)dt \\ & \leq \ell(x) - \frac{\alpha}{\beta+1} m(x) + \frac{\alpha}{\beta+1} m(x) \exp\left((\beta+1) \int_a^x k(t)f(t)dt\right). \end{aligned}$$

This concludes the proof.  $\square$

#### 4. Conclusion and Perspectives

In this article, we have studied two complementary aspects of the Grönwall integral inequalities: their natural extensions under reasonable assumptions and the establishment of new and general variants. In particular, we have discussed a key non-negativity assumption and shown that it can be relaxed to obtain flexible integral inequalities. We have also introduced and rigorously proved a new theorem on variants of the Grönwall integral inequalities involving five related functions. It is complemented by several secondary results which may be of independent interest. Some reversed versions of our main inequalities are also discussed. These contributions improve the theoretical understanding of Grönwall integral inequalities and provide more explicit upper bounds under technical conditions.

Future research directions include extending these results to stochastic settings, exploring applications in the stability analysis of differential equations, and investigating further generalizations in functional spaces.

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