Research and Communications in Mathematics and Mathematical Sciences
Vol. 16, Issue 2, 2024, Pages 159-185
ISSN 2319-6939
Published Online on December 16, 2024

http://jyotiacademicpress.org

KERNELS OF COVERED GROUPS WITH OPERATORS FROM AN INVERSE SEMIGROUP

PETER R. FUCHS

Institute for Algebra
Johannes Kepler University
Austria
e-mail: peter.fuchs@jku.at

Abstract

In a previous paper, [3], we determined the $J_\nu(N)$-radical, $\nu \in \{0, 1, 2\}$, and semisimplicity of the kernel of a covered group with operators from a Clifford semigroup. In this paper we generalize these results to operators from an arbitrary inverse semigroup.

1. Introduction

Let $(G, +)$ be a group with identity 0, written additively, but not necessarily commutative. A set $\mathcal{C} = \{C_i \mid i \in I\}$ of subgroups is called a cover of G, if $G = \bigcup_{i \in I} C_i$. The pair $(G, \mathcal{C})$ is called a covered group with cells C_i. A semigroup S of endomorphisms of G is called a semigroup of operators for (G, C), if for all $\sigma \in S$, $C_i \in \mathcal{C}$, there exists a

2020 Mathematics Subject Classification: Primary 16Y30; Secondary 20M20, 51J20.

Keywords and phrases: covered group, semigroup of operators, generalized translation structure, inverse semigroup, near-ring, radical, semisimple.

Communicated by Suayip Yuzbasi.

Received September 3, 2024

cell $C_j \in \mathcal{C}$ such that $\sigma(C_i) \subseteq C_j$. To each triple $(G, \mathcal{C}, S)$, we can associate the near-ring $M_S(G, \mathcal{C}) := \{f : G \to G \mid f(0) = 0, f(C_i) \subseteq C_i$ for all $i \in I$ and $f \circ \sigma = \sigma \circ f$ for all $\sigma \in S\}$. In fact, under the operations of function addition and composition, $M_S(G, \mathcal{C})$ is a zero-symmetric, right near-ring with identity, called the kernel of $(G, \mathcal{C}, S)$. In [1], [6] and [7], a connection between covered groups and generalized translation spaces is established, the kernel has been introduced in [7]. For general information on near-rings consult [2], [9], and [12]. Also recall that for a nearring N and $\nu \in \{0, 1, 2\}$, the radical $J_\nu(N)$ is defined as

$$J_\nu(N) := \bigcap (0 : \Gamma),$$

where Γ is an N-group of type ν and $(0 : \Gamma) := \{n \in N | \, n\Gamma = \{0\}\}$, see [12], Section 5. N is called ν-semisimple, if $J_\nu(N) = 0$.

In [3] we determined for a Clifford semigroup S the J_ν-radicals, $\nu \in \{0, 1, 2\}$ of the kernel and characterized when it is 2-semisimple. The aim of the present paper is to generalize these results to arbitrary inverse semigroups S of operators of a finite covered group $(G, \mathcal{C})$. Recall that a semigroup S is inverse, if S is regular and its idempotents commute. Alternatively, S is inverse, if every $s \in S$ has a unique inverse s^{-1}, that is there is a unique element s^{-1} such that $ss^{-1}s = s$ and $s^{-1}ss^{-1} = s^{-1}$. Without loss of generality we may assume that $\{0, id\} \subseteq S$.

If $\mathcal{C} = \{G\}$, then $M_S(G, \mathcal{C})$ is equal to the centralizer near-ring $M_S(G) = \{f : G \to G \mid f(0) = 0, f \circ \sigma = \sigma \circ f$ for all $\sigma \in S\}$, see, for example, [5], [8] and [12], Chapter 9h for an overview. Thus the results in this paper also apply to centralizer near-rings. In [5], Kabza laid some of the groundwork for centralizer near-rings over inverse semigroups, to which we often refer.

We now give a brief summary of the paper. In [4], we studied the kernel $M_A(G, \mathcal{C})$ of a covered group $(G, \mathcal{C}, A)$, where A is a group of operators. In Theorem II.2 of that paper 2-semisimplicity has been characterized. This work has been continued in [3], where we determined the J_ν-radicals of $M_A(G, \mathcal{C})$ for $\nu \in \{0, 1, 2\}$. In the present paper, we show that for an arbitrary inverse semigroup S and a covered group $(G, \mathcal{C}, S)$ such that $\{0, id\} \subseteq S$ and $e(C) \subseteq C$ for every idempotent $e \in S$ and $C \in \mathcal{C}$, some quotient of the kernel $M_S(G, \mathcal{C})$ is a direct product of kernels of the form $M_{A_i}(G_i, \mathcal{C}_i)$, where the A_i are groups of operators. This will finally enable us to determine $J_\nu(M_S(G, \mathcal{C}))$ for $\nu \in \{0, 1, 2\}$. We also characterize the 2-semisimplicity of $M_S(G, \mathcal{C})$ using Theorem II.2 in [4] for the components.

2. Radical of $M_S(G, \mathcal{C})$

For all of the following, we let $(G, \mathcal{C})$ be a finite covered group with operators from an inverse semigroup S such that $\{0, id\} \subseteq S$ and $e(C) \subseteq C$ for every idempotent $e \in S$ and $C \in \mathcal{C}$. In [3], page 1557, a decomposition of the group G has been derived. The same procedure is possible for arbitrary inverse semigroups. For $A \subseteq G$ let $A^* := A - \{0\}$. We denote by E the set of all idempotents of S. If $e \in E$, then for $g \in G$, $g = e(g) + (-e(g) + g)$, hence $G = e(G) \oplus \ker(e)$. Since different primitive idempotents are orthogonal, we obtain as in [8] for the set $\{e_{01}, \dots, e_{0n_0}\}$ of all primitive idempotents

$$G = e_{01}(G) \oplus \cdots \oplus e_{0n_0}(G) \oplus \bigcap_{t=1}^{n_0} \ker(e_{0t}).$$

We now decompose $\bigcap_{t=1}^{n_0} \ker(e_{0t})$ further to arrive at a complete decomposition of G.

Let $K_{01} := G$ and

$$K_{0j} := \bigcap_{t=1}^{j-1} \ker(e_{0t}) \text{ for } 2 \leq j \leq n_0.$$

Now suppose $i \geq 1$ and we have already defined idempotents e_{kj}, for all $k \leq i-1$ and $1 \leq j \leq n_k$. If

$$K_{i1} := \bigcap_{s=0}^{i-1} K_{s1} \cap \bigcap_{t=1}^{n_{i-1}} \ker(e_{i-1t}) \neq 0,$$

we let $\{e_{i1}, \dots, e_{in_i}\} \subseteq E$ be the set of all idempotents, which are minimal (in the natural partial ordering for idempotents of S) with respect to the property that $e_{ij}(K_{i1}) \neq 0$ for $j \in \{1, \dots, n_i\}$. Note that such idempotents exist, since we have assumed that $id \in S$ and G is finite. For $2 \leq j \leq n_i$, we define

$$K_{ij} := K_{i1} \cap \bigcap_{t=1}^{j-1} \ker(e_{it}).$$

Since $id \in S$ we have $\bigcap_{e \in E} \ker(e) = 0$, hence there exists an integer l such that $K_{l+11} = 0$.

Kabza has shown in [5] that if the centralizer $M_S(G)$, for a finite inverse semigroup S, is simple, then $K_{11} = 0$. In Theorem 15 we shall generalize this result to the case where $M_S(G, \mathcal{C})$ is 2-semisimple.

The next two results can be proved like in [3].

Theorem 1 ([3], Theorem 1)**.**

(1) *Every* $g \in G$ *has a unique representation* $g = \sum_{i=0}^{l} \sum_{j=1}^{n_i} g_{ij}$, *for some* $g_{ij} \in e_{ij}(K_{ij})$.

(2) *For all* $i \in \{0, \dots, l\}$, $j \in \{1, \dots, n_i\}$, $e_{ij}(K_{ij}) = e_{ij}(K_{i1})$ *and* $e_{ij}(K_{ij}) \neq 0$.

Theorem 2 ([3], Theorem 3)**.** *Let* $g = \sum_{i=0}^{l} \sum_{j=1}^{n_i} g_{ij} \in G$, *where* $g_{ij} \in e_{ij}(K_{ij})$ *for all* i, j. *Then for every cell* $C \in \mathcal{C}$:

$$g \in C \Leftrightarrow \forall i, j : g_{ij} \in C.$$

We also need the following result from previous papers. For simplicity of notation, we make the abbreviation $K_i := K_{i1}$, for $i \in \{0, \dots, l+1\}$.

Theorem 3. *Let* $\{e_1, \dots, e_n\}$ *be the set of primitive idempotents of* S. *Then*

(1) ([8], *page* 48) $G = e_1(G) \oplus \cdots \oplus e_n(G) \oplus K_1$, *where* $K_1 = \bigcap_{i=1}^{n} \ker(e_i)$.

(2) ([8], *page* 53) *If* $f \in M_S(G, \mathcal{C})$ *and* $x = x_1 + \cdots x_n + k \in G$, *where* $x_i \in e_i(G)$, $k \in K_1$, *then* $f(x) = f(x_1) + \cdots + f(x_n) + k'$ *for some* $k' \in K_1$.

(3) ([5], *Lemma* 5) $\forall \alpha \in S : \alpha(K_1) \subseteq K_1$.

Theorem 4. *For all* $i \in \{0, \dots, l\}$, *let* $T_i := S|K_i$ (*restriction of* S *to* K_i).

(1) T_i *is an inverse semigroup of endomorphisms of the group* K_i *and* $\{e_{i1}|K_i, \dots, e_{in_i}|K_i\}$ *is the set of primitive idempotents of* T_i.

(2) $K_i = e_{i1}(K_i) \oplus \cdots \oplus e_{in_i}(K_i) \oplus K_{i+1}$.

(3) $\forall \alpha \in S : \alpha(K_{i+1}) \subseteq K_{i+1}$.

Proof. We proceed by induction on $i \in \{0, \dots, l\}$. Since $K_0 = G$, the result follows from Theorem 3 for $i = 0$. Now let $i \geq 1$ and suppose the result has been shown for all $j \leq i - 1$. By (3), $\alpha(K_i) \subseteq K_i$ for all $\alpha \in S$, hence $T_i = S|K_i$ is a semigroup of endomorphisms of the group K_i. Since the restriction map $S \to S|K_i$ is a semigroup epimorphism and S is inverse, it follows from [11], Lemma II.1.10 that T_i is also an inverse semigroup. Let $0 \neq f \in T_i$ be an idempotent. By [11], Lemma I.7.10, $f = e|K_i$ for some idempotent $e \in S$. Since $e(K_i) \neq 0$ and $\{e_{i1}, \dots, e_{in_i}\}$ is the set of all idempotents of S, which are minimal with respect to the property that $e_{ij}(K_i) \neq 0$ for $j \in \{1, \dots, n_i\}$ (in the sequel we shall refer to this saying "by the minimality of the e_{ij}"), it follows that $e_{ij} \leq e$ for some $j \in \{1, \dots, n_i\}$. But then, $e_{ij}|K_i \leq e|K_i = f$ and (1) follows.

(2) By Theorem 3, we have for $f_j := e_{ij}|K_i$, $j \in \{1, \dots, n_i\}$ that $K_i = f_1(K_i) \oplus \dots \oplus f_{n_i}(K_i) \oplus K$, where $K = \bigcap_{j=1}^{n_i} \ker(f_j)$. But $K = K_i \cap \bigcap_{j=1}^{n_i} \ker(e_{ij}) = K_{i+1}$, from which we obtain (2). Finally, (3) follows from (2) and Theorem 3. □

The semigroup T_i will be used in Theorems 14 and 15. To obtain a decomposition of the kernel $M_S(G, \mathcal{C})$, we need another semigroup constructed from T_i.

Since K_i is a normal subgroup of G for $i \in \{0, \dots, l+1\}$, the inverse semigroup $T_i = S|K_i$ acts on the quotient $G_i := K_i/K_{i+1}$ by $\bar{s}(k/K_{i+1}) := s(k)/K_{i+1}$ for $s \in T_i$ and a coset $k/K_{i+1} \in G_i$. Note that by Theorem 4, $s(K_i) \subseteq K_i$ for all $s \in S$.

Theorem 5. (1) $S_i := \{\bar{s} \mid s \in T_i\}$ *is an inverse semigroup of endomorphisms of the group* G_i.

(2) *For* $f_j := e_{ij}|K_i$, $j \in \{1, \dots, n_i\}$, $\{\bar{f}_j \mid, j \in \{1, \dots, n_i\}\}$ *is the set of all primitive idempotents of* S_i.

(3) $\bigcap_{j=1}^{n_i} \ker \bar{f}_j = 0$.

(4) $G_i = \bar{f}_1(G_i) \oplus \cdots \oplus \bar{f}_{n_i}(G_i)$.

(5) *If* $x \in \bar{f}_j(G_i)$ *and* $y \in \bar{f}_k(G_i)$ *such that* $k \neq j$, *then* $x + y = y + x$.

(6) $\mathcal{C}_i := \{(C \cap K_i)/K_{i+1} \,|\, C \in \mathcal{C}\}$ *is a cover of* G_i *and* S_i *is a semigroup of operators for* $(G_i, \mathcal{C}_i)$.

(7) *If* $f \in M_{S_i}(G_i, \mathcal{C}_i)$ *and* $x = x_1 + \cdots + x_{n_i} \in G_i$, $x_j \in \bar{f}_j(G_i)$, *then* $f(x) = f(x_1) + \cdots + f(x_{n_i})$.

Proof. (1) Let $s \in T_i$ and $k/K_{i+1} = k'/K_{i+1}$, then $-k' + k \in K_{i+1}$ and by Theorem 4, $-s(k') + s(k) \in K_{i+1}$, hence $s(k)/K_{i+1} = s(k')/K_{i+1}$, which shows that $\bar{s}$ is well defined. Clearly S_i is a semigroup of endomorphisms of the group G_i. Since T_i is inverse and the map $h : T_i \to S_i$, $s \mapsto \bar{s}$ is a semigroup epimorphism, S_i is an inverse semigroup by [11], Lemma II.1.10.

(2) Let $0 \neq f$ be an idempotent of S_i. Since $h : T_i \to S_i$, $s \mapsto \bar{s}$ is an epimorphism, we have by [11], Lemma I.7.10, that $f = \bar{e}$ for some idempotent $0 \neq e \in T_i$. By Theorem 4, $\{e_{i1}|K_i, \ldots, e_{in_i}|K_i\}$ is the set of all primitive idempotents of T_i, hence there exists $j \in \{1, \ldots, n_i\}$, such that $e_{ij}|K_i \leq e$, which implies $\bar{f}_j \leq f$.

(3) Let $x \in G_i$. From Theorem 4 (2), $x = e_{i1}(k_1)/K_{i+1} + \cdots + e_{in_i}(k_{n_i})/K_{i+1}$ for some elements $k_j \in K_i$, $j \in \{1, \ldots, n_i\}$. If $x \in \bigcap_{j=1}^{n_i} \ker \bar{f}_j$, then $0 = \bar{f}_j(x) = e_{ij}e_{i1}(k_1)/K_{i+1} + \cdots + e_{ij}e_{in_i}(k_{n_i})/K_{i+1} = e_{ij}e_{ij}(k_j)/K_{i+1} = e_{ij}(k_j)/K_{i+1}$, for all $j \in \{1, \ldots, n_i\}$, since $e_{ij}e_{ik} = 0$ for $k \neq j$, from which we conclude $x = 0$.

(4) This follows from (2), (3) and (1) of Theorem 3.

(5) Let $u = x + y - x - y$. Since primitive idempotents are pairwise orthogonal, $u \in \bigcap_{j=1}^{n_i} \ker \bar{f}_j = 0$, hence $u = 0$.

(6) Since $\bigcup_{C \in \mathcal{C}} C \cap K_i = (\bigcup_{C \in \mathcal{C}} C) \cap K_i = K_i$, we have $\bigcup_{C \in \mathcal{C}} (C \cap K_i)/K_{i+1} = K_i / K_{i+1}$, hence $\mathcal{C}_i$ is a cover. Now let $(C \cap K_i)/K_{i+1} \in \mathcal{C}_i$, for some $C \in \mathcal{C}$, and let $\beta \in S_i$. Then $\beta = \overline{\alpha}$, for some $\alpha = \alpha_1 |K_i$, $\alpha_1 \in S$. Since $\alpha_1(C) \subseteq C'$ for some $C' \in \mathcal{C}$ and $\alpha_1(K_i) \subseteq K_i$ by Theorem 4, $\beta((C \cap K_i)/K_{i+1}) = \alpha(C \cap K_i)/K_{i+1} \subseteq (C' \cap K_i)/K_{i+1} \in \mathcal{C}_i$.

(7) This follows from (2), (3) and Theorem 3, (2). □

In our next result we collect a few tools for subsequent use.

Theorem 6. *Let* $i \in \{0, \ldots, l\}$, $f \in M_S(G, \mathcal{C})$. *Then*

(1) $f(K_i) \subseteq K_i$.

(2) *If* $x = x_{i1} + \cdots + x_{in_i} + k \in K_i$, *where* $x_{ij} \in e_{ij}(K_{ij}) = e_{ij}(K_i)$ *and* $k \in K_{i+1}$, *then* $f(x) = f(x_{i1}) + \cdots + f(x_{in_i}) + k'$ *for some* $k' \in K_{i+1}$.

(3) *For all* $x/K_{i+1} \in G_i$, *there exist unique elements* $x_{ij} \in e_{ij}(K_i)$ *such that* $x/K_{i+1} = (x_{i1} + \cdots + x_{in_i})/K_{i+1}$.

(4) *Let* $x/K_{i+1} = x_{i1}/K_{i+1} + \cdots + x_{in_i}/K_{i+1} \in G_i$, *where* $x_{ij} \in e_{ij}(K_i)$. *Then* $x/K_{i+1} \in (C \cap K_i)/K_{i+1}$ *for some* $C \in \mathcal{C}$, *if and only if* $x_{ij}/K_{i+1} \in (C \cap K_i)/K_{i+1}$ *for all* $j \in \{1, \ldots, n_i\}$.

Proof. (1) This holds for $i = 0$, since $K_0 = G$. Let $i \geq 1$, $x \in K_i = \bigcap_{s=0}^{i-1} K_s \cap \bigcap_{t=1}^{n_{i-1}} \ker e_{i-1t}$, and let $e \in \bigcup_{s=0}^{i-1} \{e_{st} \mid 1 \leq t \leq n_s\}$. Then $ef(x) = f(ex) = f(0) = 0$, thus $f(x) \in K_i$.

(2) In the proof of Theorem 4 (2) we have shown for $f_j = e_{ij}|K_i$, $j \in \{1, \ldots, n_i\}$ that $\bigcap_{j=1}^{n_i} \ker f_j = K_{i+1}$. The assertion now follows from (1), Theorem 4 (1) and Theorem 3 (2).

(3) Suppose $x = x_{i1} + \cdots + x_{in_i} + k$, $x' = x'_{i1} + \cdots + x'_{in_i} + k'$, where x_{ij}, $x'_{ij} \in e_{ij}(K_i)$, $k, k' \in K_{i+1}$, are such that $x/K_{i+1} = x'/K_{i+1}$. Then for some $k'' \in K_{i+1}$, $k'' = -x' + x = -k' - x'_{in_i} - \cdots - x'_{i1} + x_{i1} + \cdots + x_{in_i} + k$. Therefore, $0 = e_{ij}(-x' + x) = -e_{ij}(x') + e_{ij}(x) = -x'_{ij} + x_{ij}$, hence $x_{ij} = x'_{ij}$ for all $j \in \{1, \ldots, n_i\}$,

(4) If $x = x_{i1} + \cdots + x_{in_i} + k \in K_i$ is such that $x/K_{i+1} \in (C \cap K_i)/K_{i+1}$ for some $C \in \mathcal{C}$, then there exists an element $x' \in C \cap K_i$ and $k' \in K_{i+1}$ such that $x = x' + k'$. Therefore, $x_{ij} = e_{ij}(x) = e_{ij}(x')$ for $j \in \{1, \ldots, n_i\}$. Since $x' \in C \cap K_i$, also $x_{ij} = e_{ij}(x') \in C \cap K_i$, hence $x_{ij}/K_{i+1} \in (C \cap K_i)/K_{i+1}$. □

For $i \in \{0, \ldots, l\}$, let $N_i := M_{S_i}(G_i, \mathcal{C}_i)$ and $N := M_S(G, \mathcal{C})$. We are now ready to decompose a quotient of N into the components N_i.

Theorem 7. *Let* $\phi : N \to \bigoplus_{i=0}^{l} N_i$, $\phi(f) := (f_0, \cdots, f_l)$, *where* $f_i : G_i \to G_i$, $f_i(x/K_{i+1}) := f(x)/K_{i+1}$. *Then* ϕ *is a near-ring epimorphism, hence* $N/\ker \phi \cong \bigoplus_{i=0}^{l} N_i$.

Proof. Let $f \in N = M_S(G, \mathcal{C})$. We show first that each f_i is well defined. If $x/K_{i+1} \in G_i$, $x \in K_i$, then there exist by Theorem 6 unique elements $x_{ij} \in e_{ij}(K_i)$, $j \in \{1, \ldots, n_i\}$, such that $x/K_{i+1} = x_{i1}/K_{i+1} + \cdots + x_{in_i}/K_{i+1}$ and there exists an element $k \in K_{i+1}$ with $x = x_{i1} + \cdots + x_{in_i} + k$. If $x' \in K_i$ is such that $x/K_{i+1} = x'/K_{i+1}$, then, by the uniqueness of the elements x_{ij}, $x' = x_{i1} + \cdots + x_{in_i} + k'$ for some $k' \in K_{i+1}$. By Theorem 6 (2) $f(x) = f(x_{i1}) + \cdots + f(x_{in_i}) + z$ and $f(x') = f(x_{i1}) + \cdots + f(x_{in_i}) + z'$ for some $z, z' \in K_{i+1}$, thus $f(x)/K_{i+1} = f(x')/K_{i+1}$, which shows f_i is well defined. By Theorem 6 (1), $f(K_i) \subseteq K_i$, hence f_i is a map from G_i to G_i. We proceed to show that $f_i \in N_i, i \in \{0, \dots, l\}$. If $\beta \in S_i$, then $\beta = \overline{\alpha}$ for some $\alpha = \alpha_1 | K_i, \alpha_1 \in S$. For $x/K_{i+1} \in G_i, f_i(\beta(x/K_{i+1})) = f_i(\alpha_1(x)/K_{i+1}) = f(\alpha_1(x))/K_{i+1} = \alpha_1(f(x))/K_{i+1} = \beta(f(x)/K_{i+1}) = \beta f_i(x/K_{i+1})$. Moreover, if $(C \cap K_i)/K_{i+1} \in \mathcal{C}_i$, then $f_i((C \cap K_i)/K_{i+1}) = f(C \cap K_i)/K_{i+1} \subseteq (C \cap K_i)/K_{i+1}$, since $f \in M_S(G, \mathcal{C})$ and $f(K_i) \subseteq K_i$. It now follows that $f_i \in N_i$ for all $i \in \{0, \dots, l\}$.

Clearly ϕ is a near-ring homomorphism. It remains to show that ϕ is surjective. For this let $(f_0, \dots, f_l) \in \bigoplus_{i=0}^{l} N_i$. For each $s \in \{0, \dots, l\}$ define a function $\bar{f}_s : G \to G$ by

$$\bar{f}_s\Big(\sum_{i=0}^{l} \sum_{j=1}^{n_i} g_{ij}\Big) := \sum_{j=1}^{n_s} h_{sj},$$

where $h_{sj} \in e_{sj}(K_s)$, $j \in \{1, \dots, n_s\}$ is such that $f_s((\sum_{j=1}^{n_s} g_{sj})/K_{s+1}) = \sum_{j=1}^{n_s} h_{sj}/K_{s+1}$. By Theorem 6 (3), the elements h_{sj} are uniquely determined. Note that $\bar{f}_s$ is well defined on G by Theorem 1. By Theorem 5 (7), $\sum_{j=1}^{n_s} h_{sj}/K_{s+1} = \sum_{j=1}^{n_s} f_s(g_{sj}/K_{s+1})$. We show that $\bar{f}_s \in N$ for all $s \in \{0, \dots, l\}$.

If $C \in \mathcal{C}$ and $\sum_{i=0}^{l}\sum_{j=1}^{n_i} g_{ij} \in C$, then by Theorem 2, $g_{sj} \in C$ for all $j \in \{1, \dots, n_s\}$. Since $f_s \in N_s$, $f_s(\sum_{j=1}^{n_s} g_{sj} / K_{s+1}) = \sum_{j=1}^{n_s} h_{sj} / K_{s+1}$ $= \sum_{j=1}^{n_s} f_s(g_{sj}/K_{s+1}) \in (C \cap K_s)/K_{s+1}$, thus there exists an element $x = x_{s1} + \cdots + x_{sn_s} + k \in K_s \cap C$, $k \in K_{s+1}$, such that $x / K_{s+1} = \sum_{j=1}^{n_s}$ h_{sj} / K_{s+1}. By Theorem 6 (3), we have $x_{sj} = h_{sj}$ for all $j \in \{1, \dots, n_s\}$. But since $x \in C$, $e_{sj}(x) = x_{sj} = h_{sj} \in C$ for all j. It now follows that $\bar{f}_s(C) \subseteq C$ for all $C \in \mathcal{C}$.

Now let $\beta \in S$. By Theorem 4, $\beta(K_i) \subseteq K_i$ and $\beta | K_i \in T_i$ for $i \in \{0, \dots, l\}$. Let $g = \sum_{i=0}^{l}\sum_{j=1}^{n_i} g_{ij} \in G$. Since $\{e_{ij} | K_i | \ j \in \{1, \dots, n_i\}\}$ is the set of primitive idempotents of T_i by Theorem 4, we obtain from [5], Lemma 7 that for each $j \in \{1, \dots, n_s\}$ with $\beta(g_{sj}) \neq 0$, there exists an element $k(j) \in \{1, \dots, n_s\}$ such that $\beta(g_{sj}) \in e_{jk(j)}(K_i)$ and that if $\beta(g_{sj'}) \neq 0$ for $j' \neq j$, then $k(j) \neq k(j')$. Consequently, we obtain for $g = \sum_{i=0}^{l}\sum_{j=1}^{n_i} g_{ij}$ from Theorem 5 (7), $f_s(\sum_{j=1}^{n_s} \beta(g_{sj})/K_{s+1}) =$ $\sum_{j=1}^{n_s} f_s(\beta(g_{sj})/K_{s+1}) = \sum_{j=1}^{n_s} \beta f_s(g_{sj} / K_{s+1}) = \beta\sum_{j=1}^{n_s} f_s(g_{sj} / K_{s+1}) =$ $\beta f_s(\sum_{j=1}^{n_s} g_{sj} / K_{s+1})$, from which we obtain $\bar{f}_s(\beta g) = \beta \bar{f}_s(g)$. It now follows that $\bar{f}_s \in N$ for all $s \in \{0, \dots, l\}$.

If $f = \sum_{i=0}^{l}\bar{f}_i \in N$ and $\phi(f) = (f_1', \dots, f_l')$, then for $s \in \{0, \dots, l\}$ and $x \in K_s$, $f_s'(x/K_{s+1}) = f(x)/K_{s+1} = \bar{f}_s(x)/K_{s+1} = f_s(x/K_{s+1})$. Therefore, $f_i' = f_i$ for all $i \in \{0, \dots, l\}$, which proves that ϕ is surjective and the result follows. $\square$

In the proof of Theorem 7, we have defined functions $\bar{f}_s \in N$ for all $f_s \in N_s$, $s \in \{0, \ldots, l\}$. For a subnear-ring T of $\bigoplus_{i=0}^{l} N_i$ let $\bar{T} := \{\sum_{i=0}^{l} \bar{f}_i \mid (f_0, \cdots, f_l) \in T\} \subseteq N$. We can now determine the radicals of N, depending on the radicals of the components N_i, $i \in \{0, \ldots, l\}$.

Theorem 8. *Let* $J := \ker \phi$, $\nu \in \{0, 1, 2\}$ *and let* $T := \bigoplus_{i=0}^{l} J_\nu(N_i)$. *Then*

(1) $J^{l+1} = 0$, $J \subseteq J_\nu(N)$,

(2) $J_\nu(N) = \bar{T} \oplus J$.

Proof. (1) If $f \in \ker \phi$, then $(f_0, \cdots, f_l) = (0, 0, \cdots, 0)$, hence for all $i \in \{0, \ldots, l\}$ and $x \in K_i$, $f_i(x/K_{i+1}) = f(x)/K_{i+1} = 0$. Therefore, $f(K_i) \subseteq K_{i+1}$ for all $i \in \{0, \ldots, l\}$. Since $K_{l+1} = 0$, we have $f^{l+1} = 0$, thus $J^{l+1} = 0$. By [12], Theorem 5.37, $J \subseteq J_0(N) \subseteq J_1(N) \subseteq J_2(N)$.

(2) The proof of this is almost identical to the proof of [3], Theorem 5, and shall be omitted. □

It remains to determine the structure of $N_i = M_{S_i}(G_i, \mathcal{C}_i), i \in \{0, \ldots, l\}$. For this we decompose N_i further.

The following result can be seen as a consequence of the proof of [5], Lemma 7.

Theorem 9 ([5], Lemma 7)**.** *Let G be a group, S an inverse semigroup of endomorphisms of G with primitive idempotents* $e_1, \ldots, e_n$, *and* $I_i := e_i(G)$ *for* $i \in \{1, \ldots, n\}$. *If* $\alpha \in S$ *and* $\alpha(I_i) \neq 0$ *for some* $i \in \{1, \ldots, n\}$, *then there exists an element* $j(i) \in \{1, \ldots, n\}$, *such that* $\alpha|I_i : I_i \to I_{j(i)}$ *is a group isomorphism with inverse map* $\alpha^{-1}|I_{j(i)} : I_{j(i)} \to I_i$. *Moreover, if* $k \neq i$ *and* $\alpha(I_k) \neq 0$, *then* $j(k) \neq j(i)$.

For $i \in \{0, \ldots, l\}$, $j \in \{1, \ldots, n_i\}$, *let* $I_{ij} := e_{ij}(K_{ij}) = e_{ij}(K_i)$, *where the second equation follows from Theorem* 1. *Define a relation* $\sim_i$ *on* $\{I_{i1}, \ldots, I_{in_i}\}$ *by*

$$I_{ij_1} \sim_i I_{ij_2} :\Leftrightarrow \exists \alpha \in T_i = S|K_i : (\alpha|I_{ij_1} : I_{ij_1} \to I_{ij_2} \text{ is an isomorphism}).$$

Theorem 10. *For all* $i \in \{0, \ldots, l\}$, $\sim_i$ *is an equivalence relation on* $\{I_{ij} | j \in \{1, \ldots, n_i\}\}$.

Proof. By our assumptions, $id \in S$, hence $I_{ij} \sim_i I_{ij}$ for $j \in \{1, \ldots, n_i\}$. If $I_{ij_1} \sim_i I_{ij_2}$, then there exists an isomorphism $\alpha|I_{ij_1} : I_{ij_1} \to I_{ij_2}$. By Theorem 9, the inverse map $\alpha^{-1}|I_{ij_2} : I_{ij_2} \to I_{ij_1}$ is also an isomorphism, hence $I_{ij_2} \sim_i I_{ij_1}$. If $I_{ij_1} \sim_i I_{ij_2} \sim_i I_{ij_3}$ and $\alpha|I_{ij_1} : I_{ij_1} \to I_{ij_2}, \beta|I_{ij_2} : I_{ij_2} \to I_{ij_3}$ are isomorphisms, then $(\beta \circ \alpha)|I_{ij_1} : I_{ij_1} \to I_{ij_3}$ is an isomorphism, hence $I_{ij_1} \sim_i I_{ij_3}$. □

Recall from Theorem 5 that $G_i = K_i / K_{i+1}$, $T_i = S|K_i$, $S_i = \{\bar{s} \mid s \in T_i\}$, $\mathcal{C}_i = \{(C \cap K_i)/K_{i+1} | C \in \mathcal{C}\}$ and $N_i = M_{S_i}(G_i, \mathcal{C}_i)$ for $i \in \{0, \ldots, l\}$. By Theorem 5 (4), $G_i = \bar{f}_1(G_i) \oplus \cdots \oplus \bar{f}_{n_i}(G_i)$, where $f_j = e_{ij}|K_i$. By Theorem 5 (2), $\{\bar{f}_j | j \in \{1, \ldots, n_i\}\}$ is the set of all primitive idempotents of S_i. For $j \in \{1, \ldots, n_i\}$ let $\bar{I}_{ij} := \bar{f}_j(G_i)$. Since $f_j = e_{ij}|K_i$ and $e_{ij}(K_i) = e_{ij}(K_{ij})$ by Theorem 1, we have $\bar{I}_{ij} := e_{ij}(K_{ij})/K_{i+1} = I_{ij}/K_{i+1}$. We can now extend the equivalence relation $\sim_i$, $i \in \{0, \ldots, l\}$ to a relation $\equiv_i$ on $\{\bar{I}_{ij} | j \in \{1, \ldots, n_i\}\}$ by defining

$$\bar{I}_{ij_1} \equiv_i \bar{I}_{ij_2} :\Leftrightarrow \exists \bar{\alpha} \in S_i : (\bar{\alpha}|\bar{I}_{ij_1} : \bar{I}_{ij_1} \to \bar{I}_{ij_2} \text{ is an isomorphism}).$$

Like in Theorem 10 one can prove that $\equiv_i$ is an equivalence relation.

If there are t_i equivalence classes with respect to $\equiv_i$, then we can renumerate the indices of the $\bar{I}_{ij}$ to find numbers

$$0 = l_0 < l_1 < l_2 < \cdots < l_{t_i} = n_i,$$

such that $\{\bar{I}_{ij} \mid l_{k-1} < j \leq l_k\}$ is the equivalence class of $\bar{I}_{il_k}$, $k \in \{1, \ldots, t_i\}$. Now let $G_{ik} = \bigoplus_{j=l_{k-1}+1}^{l_k} \bar{I}_{ij}$ for $k \in \{1, \ldots, t_i\}$. Since entries coming from different $\bar{I}_{ij}$ commute by Theorem 5 (5), it follows that G_{ik} is a group for all $k \in \{1, \ldots, t_i\}$. Also, $G_i = G_{i1} \oplus \cdots \oplus G_{it_i}$, since $G_i = \bar{I}_{i1} \oplus \cdots \oplus \bar{I}_{in_i}$. Now we can decompose $N_i = M_{S_i}(G_i, \mathcal{C}_i)$ as follows: For $k \in \{1, \ldots, t_i\}$, let $S_{ik} := S_i|G_{ik}$ and $\mathcal{C}_{ik} := \{C \cap G_{ik} \mid C \in \mathcal{C}_i\}$. Then

Theorem 11. (1) *$\mathcal{C}_{ik}$ is a cover of G_{ik} and S_{ik} is an inverse semigroup of operators for $(G_{ik}, \mathcal{C}_{ik})$ for all $k \in \{1, \ldots, t_i\}$.*

(2) $\psi : M_{S_i}(G_i, \mathcal{C}_i) \to \bigoplus_{k=1}^{t_i} M_{S_{ik}}(G_{ik}, \mathcal{C}_{ik})$, $\psi(f) := (f|G_{i1}, \ldots, f|G_{it_i})$ *is a near-ring isomorphism.*

Proof. (1) For $k \in \{1, \ldots, t_i\}$, $\bigcup_{C \in \mathcal{C}_i} C \cap G_{ik} = (\bigcup_{C \in \mathcal{C}_i} C) \cap G_{ik} = G_i \cap G_{ik} = G_{ik}$, hence $\mathcal{C}_{ik}$ is a cover of G_{ik}.

If $\alpha \in S_{ik}$ and $x = x_{l_{k-1}+1} + \cdots + x_{l_k} \in \bigoplus_{j=l_{k-1}+1}^{l_k} \bar{I}_{ij} = G_{ik}$, then $\alpha(x) = \alpha(x_{l_{k-1}+1}) + \cdots + \alpha(x_{l_k})$. If $\alpha(x_{ij}) \neq 0$ for some $l_{k-1} < j \leq l_k$, then by Theorem 9 there exists an element $s \in \{1, \ldots, n_i\}$ such that $\alpha|\bar{I}_{ij} : \bar{I}_{ij} \to \bar{I}_{is}$ is an isomorphism, hence $l_{k-1} < s \leq l_k$. Consequently $\alpha(x) \in G_{ik}$, which shows that S_{ik} is a semigroup of endomorphisms of the group G_{ik}. That S_{ik} is a semigroup of operators for $(G_{ik}, \mathcal{C}_{ik})$ now follows from the fact that S_i is a semigroup of operators for $(G_i, \mathcal{C}_i)$. By [11], Lemma II.1.10, S_{ik} is an inverse semigroup, since S_i is inverse and the restriction map $h : S_i \to S_{ik} = S_i | G_{ik}$ is a semigroup epimorphism.

(2) Let $f \in M_{S_i}(G_i, \mathcal{C}_i)$ and $x = x_{l_{k-1}+1} + \cdots + x_{l_k} \in G_{ik}$ for some $k \in \{1, \ldots, t_i\}$. By Theorem 5 (7), $f(x) = f(x_{l_{k-1}+1}) + \cdots + f(x_{l_k}) \in G_{ik}$, thus $f(G_{ik}) \subseteq G_{ik}$. Also, if $C \in \mathcal{C}_{ik}$, say $C = C_1 \cap G_{ik}$ for some $C_1 \in \mathcal{C}_i$, we have $f(C_1) \subseteq C_1$ since $f \in M_{S_i}(G_i, \mathcal{C}_i)$, hence $f | G_{ik} \in M_{S_{ik}}(G_{ik}, \mathcal{C}_{ik})$ for all $k \in \{1, \ldots, t_i\}$. Now it is clear that ψ is a near-ring homomorphism. If $f \in \ker \psi$, then $f | G_{ik} = 0$ for all $k \in \{1, \ldots, t_i\}$. For $x \in G_i$, there exist elements $x_j \in G_{ij}$, $j \in \{1, \ldots, t_i\}$, such that $x = x_1 + \cdots + x_{t_i}$. By Theorem 5 (7), $f(x) = f(x_1) + \cdots + f(x_{t_i}) = 0$, which shows that $f = 0$. It now follows that ψ is a monomorphism.

To prove that ψ is surjective, let $(f_1, \ldots, t_{t_i}) \in \bigoplus_{k=1}^{t_i} M_{S_{ik}}(G_{ik}, \mathcal{C}_{ik})$. Define $f : G_i \to G_i$ by

$$f\Big(\sum_{k=1}^{t_i} (x_{l_{k-1}+1} + \cdots + x_{l_k})\Big) := \sum_{k=1}^{t_i} f_k(x_{l_{k-1}+1} + \cdots + x_{l_k}).$$

Since $G_i = G_{i1} \oplus \cdots \oplus G_{it_i}$, f is well defined and since $f_k \in M_{S_{ik}}(G_{ik}, \mathcal{C}_{ik})$, we have again by Theorem 5 (7) that $\sum_{k=1}^{t_i} f_k(x_{l_{k-1}+1} + \cdots + x_{l_k}) = \sum_{k=1}^{t_i} f_k(x_{l_{k-1}+1}) + \cdots + f_k(x_{l_k})$. Also, $f | G_{ik} = f_k$ for $k \in \{1, \ldots, t_i\}$. If $x = \sum_{k=1}^{t_i} (x_{l_{k-1}+1} + \cdots + x_{l_k}) \in C$, for some $C \in \mathcal{C}_i$, then by Theorem 6 (4), $x_{l_{k-1}+1}, \ldots, x_{l_k} \in C \cap G_{ik}$ for all $k \in \{1, \ldots, t_i\}$, thus $f_k(x_{l_{k-1}+1} + \cdots + x_{l_k}) = f_k(x_{l_{k-1}+1}) + \cdots + f_k(x_{l_k}) \in C \cap G_{ik}$. From this we can conclude that $f(C) \subseteq C$, for all $C \in \mathcal{C}_i$.

Now let $\beta \in S_i$ and $x = \sum_{k=1}^{t_i}(x_{l_{k-1}+1} + \cdots + x_{l_k}) \in G_i$. Then $f(\beta x) = f(\sum_{k=1}^{t_i}(\beta x_{l_{k-1}+1} + \cdots + \beta x_{l_k}))$. But since $S_{ik} = S_i | G_{ik}$ is a semigroup of operators for $(G_{ik}, \mathcal{C}_{ik})$ for all $k = \{1, \ldots, t_i\}$, we have from Theorem 9 and Theorem 5 (7) that $f(\beta x) = \sum_{k=1}^{t_i} f_k(\beta x_{l_{k-1}+1} + \cdots + \beta x_{l_k})$ $= \sum_{k=1}^{t_i} f_k(\beta x_{l_{k-1}+1}) + \cdots + f_k(\beta x_{l_k}) = \sum_{k=1}^{t_i} \beta f_k(x_{l_{k-1}+1}) + \cdots + \beta f_k(x_{l_k}) = \beta f(x)$. It now follows that $f \in M_{S_i}(G_i, \mathcal{C}_i)$ and that $\psi(f) = (f_1, \ldots, f_{t_i})$. The proof is now complete. □

Following an idea of Kabza ([5], Lemmas 16-19 and Theorem 20), we can now show that every kernel $M_{S_{ik}}(G_{ik}, \mathcal{C}_{ik})$, $k = \{1, \ldots, t_i\}$ in Theorem 11 is isomorphic to a kernel with operators from an automorphism group. The J_ν-radicals, $\nu \in \{0, 1, 2\}$ of such kernels has been determined in [3], Theorem 15. Using the decomposition in Theorem 11, we can determine the radicals of $M_{S_i}(G_i, \mathcal{C}_i)$ to obtain the radicals of N from Theorem 8.

In fact, if $H_{ik} := \bar{I}_{il_{k-1}+1}$ and $A_{ik} := \{\alpha | \bar{I}_{il_{k-1}+1} | \alpha \in S_{ik}, \alpha(\bar{I}_{il_{k-1}+1}) \subseteq \bar{I}_{il_{k-1}+1}\}$, for $i \in \{0, \ldots, l\}$, $k \in \{1, \ldots, t_i\}$, then we have

Theorem 12. (1) *A_{ik} is a group with* 0 *of automorphisms of the group H_{ik}.*

(2) *There exists a cover $\mathcal{D}_{ik}$ of the group H_{ik}, such that A_{ik} is a group of operators for $(H_{ik}, \mathcal{D}_{ik})$ and*

$$M_{S_{ik}}(G_{ik}, \mathcal{C}_{ik}) \cong M_{A_{ik}}(H_{ik}, \mathcal{D}_{ik}).$$

Proof. (1) If $0 \neq \beta \in A_{ik}$, then $0 \neq \beta(\bar{I}_{il_{k-1}+1}) \subseteq \bar{I}_{il_{k-1}+1}$. By Theorem 9, $\beta : \bar{I}_{il_{k-1}+1} \to \bar{I}_{il_{k-1}+1}$ is an isomorphism, hence A_{ik} is a group of automorphisms with 0.

(2) In [5], Lemmas 16-19 and Theorem 20, Kabza has shown that $\psi : M_{S_{ik}}(G_{ik}) \to M_{A_{ik}}(H_{ik}),\ \psi(g) := g|H_{ik}$ is a near-ring isomorphism between the centralizers. Therefore, it suffices to show that there exists a suitable cover D_{ik} of H_{ik}, such that ψ is also an isomorphism from $M_{S_{ik}}(G_{ik}, \mathcal{C}_{ik})$ to $M_{A_{ik}}(H_{ik}, \mathcal{D}_{ik})$. In fact, let $D_{ik} := \{\alpha(C \cap \bar{I}_{ij}) \,|\, C \in \mathcal{C}_{ik}, l_{k-1} + 1 \leq j \leq l_k, \alpha \in S_{ik}, \alpha|\bar{I}_{ij} : \bar{I}_{ij} \to H_{ik}$ is an isomorphism$\}$. Note that there does exist an isomorphism $\alpha|\bar{I}_{ij} : \bar{I}_{ij} \to H_{ik} = \bar{I}_{il_{k-1}+1}$, since $\bar{I}_{ij} \equiv_i H_{ik}$ for all $l_{k-1} + 1 \leq j \leq l_k$.

By our assumptions $id \in S$ which implies that the identity map on G_{ik} is an element of S_{ik}. If we restrict to H_{ik}, we obtain $\bigcup_{D \in \mathcal{D}_{ik}} D \supseteq \bigcup_{C \in \mathcal{C}_{ik}} C \cap H_{ik} = (\bigcup_{C \in \mathcal{C}_{ik}} C) \cap H_{ik} = H_{ik}$, thus $\mathcal{D}_{ik}$ is a cover of H_{ik}. If $0 \neq \beta \in A_{ik}$ and $\alpha|\bar{I}_{ij} : \bar{I}_{ij} \to H_{ik}, l_{k-1} + 1 \leq j \leq l_k$, is an isomorphism, then for all $C \in \mathcal{C}_{ik}$, $\beta\alpha(C \cap \bar{I}_{ij}) \in \mathcal{D}_{ik}$, hence A_{ik} is a group of operators for $(H_{ik}, \mathcal{D}_{ik})$.

For $f \in M_{A_{ik}}(H_{ik}, \mathcal{D}_{ik})$ and $x = \sum_{j=l_{k-1}+1}^{l_k} x_j \in G_{ik} = \bigoplus_{j=l_{k-1}+1}^{l_k} \bar{I}_{ij}$, define $\bar{f}(x) = \sum_{j=l_{k-1}+1}^{l_k} f_j(x_j)$, where $f_1 := f$ and for $j \geq l_{k-1} + 2$, $f_j(x_j) := \alpha f(y)$, where $\alpha : \bar{I}_{il_{k-1}+1} \to \bar{I}_{ij}$ is an isomorphism such that $\alpha(y) = x_j$. It has been shown in ([5], Lemmas 16-19 and Theorem 20) that $\bar{f} \in M_{S_{ik}}(G_{ik})$ and $\psi(\bar{f}) = f$ and that the definition of f_j is independent of the special choice of α. Therefore, it remains to show $\bar{f} \in M_{S_{ik}}(G_{ik}, \mathcal{C}_{ik})$.

For this, let $C \in \mathcal{C}_{ik}$ and $x = \sum_{j=l_{k-1}+1}^{l_k} x_j \in C$. By Theorem 6 (4), all $x_j \in C$ for $j \in \{l_{k-1}+1, \ldots, l_k\}$. If $\alpha(y) = x_j$ for some $y \in \bar{I}_{il_{k-1}+1}$, then $y = \alpha^{-1}(x_j) \in \alpha^{-1}(C \cap \bar{I}_{ij})$, where $\alpha^{-1} : \bar{I}_{ij} \to \bar{I}_{ij_{k-1}+1}$ is the inverse isomorphism of α. By construction, $\alpha^{-1}(C \cap \bar{I}_{ij}) \in \mathcal{D}_{ik}$ and since $f \in M_{A_{ik}}(H_{ik}, \mathcal{D}_{ik}), f(y) \in \alpha^{-1}(C \cap \bar{I}_{ij})$, therefore $f_j(x_j) = \alpha f(y) \in C \cap \bar{I}_{ij}$. It follows that $\bar{f}(x) = \sum_{j=l_{k-1}+1}^{l_k} f_j(x_j) \in C$, hence $\bar{f}(C) \subseteq C$, which shows that $\bar{f} \in M_{S_{ik}}(G_{ik}, \mathcal{C}_{ik})$.

Finally, we have to show that for $g \in M_{S_{ik}}(G_{ik}, \mathcal{C}_{ik})$, $\psi(g) = g|\, H_{ik} \in M_{A_{ik}}(H_{ik}, \mathcal{D}_{ik})$. By Kabza, we have $\psi(g) \in M_{A_{ik}}(H_{ik})$. If $\alpha(C \cap \bar{I}_{ij}) \in \mathcal{D}_{ik}$ for some isomorphism $\alpha|\bar{I}_{ij} : \bar{I}_{ij} \to H_{ik} = \bar{I}_{ij_{k-1}+1}$ and $x \in C \cap \bar{I}_{ij}$, then $g|H_{ik}(\alpha x) = g(\alpha x) = \alpha g(x)$. Since $g \in M_{S_{ik}}(G_{ik}, \mathcal{C}_{ik})$, $g(C \cap \bar{I}_{ij}) \subseteq C \cap \bar{I}_{ij}$, from which we have $g|H_{ik}(\alpha(C \cap \bar{I}_{ij})) \subseteq \alpha(C \cap \bar{I}_{ij})$. It now follows that $g|H_{ik} \in M_{A_{ik}}(H_{ik}, \mathcal{D}_{ik})$, which completes the proof. $\square$

We are now able to determine the radicals of $M_S(G, \mathcal{C})$. Since each A_{ik} is a group of operators with 0 for $(H_{ik}, \mathcal{D}_{ik})$, $J_\nu(M_{A_{ik}}(H_{ik}, \mathcal{D}_{ik}))$ (and therefore by Theorem 12, $J_\nu(M_{S_{ik}}(G_{ik}, \mathcal{C}_{ik}))$, $\nu \in \{0, 1, 2\}$) has been determined in [3], Theorem 15. Note that $J_1(M_{A_{ik}}(H_{ik}, \mathcal{D}_{ik})) = J_2(M_{A_{ik}}(H_{ik}, \mathcal{D}_{ik}))$ by ([12], Proposition 5.3), since the near-ring $M_{A_{ik}}(H_{ik}, \mathcal{D}_{ik})$ has an identity. If $N_i = M_{S_i}(G_i, \mathcal{C}_i)$, $i \in \{0, \ldots, l\}$ is the near-ring defined in Theorem 5, then $N_i \cong \bigoplus_{k=1}^{t_i} M_{S_{ik}}(G_{ik}, \mathcal{C}_{ik})$ by Theorem 11 and by [12], Theorem 5.20,

$$J_\nu(\bigoplus_{k=1}^{t_i} M_{S_{ik}}(G_{ik}, \mathcal{C}_{ik})) = \bigoplus_{k=1}^{t_i} J_\nu(M_{S_{ik}}(G_{ik}, \mathcal{C}_{ik})).$$

This allows us to compute $J_\nu(N_i)$ for $i \in \{0, \dots, l\}$, from which we finally obtain $J_\nu(M_S(G, C))$ from Theorem 8, for all $\nu \in \{0, 1, 2\}$.

We now return to the case where $(G, \mathcal{C})$ is a finite covered group with operators from an inverse semigroup S such that $e(C) \subseteq C$ for every idempotent $e \in S$ and $C \in \mathcal{C}$. We want to characterize when $N := M_S(G, \mathcal{C})$ is 2-semisimple. It turns out that this problem is related to the 2-semisimplicity of the near-rings $M_{A_{ik}}(H_{ik}, \mathcal{D}_{ik})$, as defined prior to Theorem 12. Since each A_{ik} is a group of operators for the covered group $(H_{ik}, \mathcal{D}_{ik})$, we can apply [4], Theorem II.2. Also, it turns out that the equivalence relation $\sim_i$ as defined in Theorem 10 is useful.

Theorem 13 ([10], Lemma 1.3)**.** *Let S be an inverse semigroup of endomorphisms of a group G. Then for $\alpha \in S$.*

(1) $\mathrm{Im}(\alpha) = \mathrm{Im}(\alpha\alpha^{-1})$.

(2) $\ker(\alpha^{-1}\alpha) = \ker(\alpha)$.

Here $\alpha\alpha^{-1}$ and $\alpha^{-1}\alpha$ are the idempotents associated with α.

Let $i \in \{0, \dots, l\}$, $j \in \{1, \dots, n_i\}$ and $r \le i$. Since $K_i \subseteq K_r$, we have $e_{ij}(K_r) \neq 0$, hence by the minimality of the idempotents e_{rt}, there exists an element $t \in \{1, \dots, n_r\}$, such that $e_{rt} \le e_{ij}$. For $r \le i$ and $j \in \{1, \dots, n_i\}$, let $X_{rj} := \{t \in \{1, \dots, n_r\} \mid e_{rt} \le e_{ij}\}$.

Theorem 14. *Let $i \in \{0, \dots, l\}$ and let $j, k \in \{1, \dots, n_i\}$ be such that $\alpha|I_{ij} : I_{ij} \to I_{ik}$ is an isomorphism for some $\alpha \in S$. Then for all $r \in \{0, \dots, l\}$, $r \le i$, there exists a permutation $\pi : X_{rj} \to X_{rk}$ such that $\alpha|I_{rs} : I_{rs} \to I_{r\pi(s)}$ is an isomorphism for all $s \in X_{rj}$.*

Proof. Let $r \in \{0, \ldots, l\}$, $r \leq i$. If $r = i$, then the assertion follows from our assumption that $\alpha|I_{ij} : I_{ij} \to I_{ik}$ is an isomorphism, since $X_{ij} = \{j\}$. Now let $r < i$. Since $(\alpha e_{ij})(\alpha e_{ij})^{-1}|I_{ik} = e_{ik}|I_{ik} = id|I_{ik}$, we have $(\alpha e_{ij})(\alpha e_{ij})^{-1} \circ e_{ik} \neq 0$ and $(\alpha e_{ij})(\alpha e_{ij})^{-1} \circ e_{ik} \leq e_{ik}$. By the minimality of the e_{is}, $(\alpha e_{ij})(\alpha e_{ij})^{-1} \circ e_{ik} = e_{ik}$, hence $e_{ik} \leq (\alpha e_{ij})(\alpha e_{ij})^{-1}$. Therefore, $e_{rt} \leq e_{ik} \leq (\alpha e_{ij})(\alpha e_{ij})^{-1}$ for all $t \in X_{rk}$, from which we obtain $(\alpha e_{ij})(\alpha e_{ij})^{-1}|I_{rt} = e_{rt}|I_{rt} = id|I_{rt}$. By Theorem 13 (applied to $G = K_r$ and $S = T_r = S|K_r$), it follows that $I_{rt} \subseteq Im((\alpha e_{ij})(\alpha e_{ij})^{-1}|K_r)$ $= Im(\alpha e_{ij}|K_r)$. Therefore, we obtain for each $t \in X_{rk}$ an element $s \in \{1, \ldots, n_r\}$, such that $(\alpha e_{ij})(I_{rs}) = (\alpha e_{ij} e_{rs})(K_r) = I_{rt}$. In particular, $e_{ij}e_{rs} \neq 0$ and since $e_{ij}e_{rs} \leq e_{rs}$, we have $e_{ij}e_{rs} = e_{rs}$ by the minimality of the idempotents e_{rs}, hence $e_{rs} \leq e_{ij}$, which shows that $s \in X_{rj}$. Since $\alpha(I_{rs}) = \alpha e_{rs}(K_r) = \alpha e_{ij} e_{rs}(K_r) = I_{rt}$, $\alpha|I_{rs} : I_{rs} \to I_{rt}$ is an isomorphism by Theorem 9 and we obtain an injective map $\tau : X_{rk} \to X_{rj}$ such that $\alpha|I_{r\tau(t)} : I_{r\tau(t)} \to I_{rt}$ is an isomorphism for $t \in X_{rk}$. Applying the above arguments to the inverse isomorphism $\alpha^{-1}|I_{ik} : I_{ik} \to I_{ij}$, it follows that τ is bijective. $\square$

In Theorem 12, we have defined for $i \in \{0, \ldots, l\}$ and $k \in \{1, \ldots, t_i\}$ covered groups $(H_{ik}, \mathcal{D}_{ik})$ and groups with 0 of operators A_{ik} for $(H_{ik}, \mathcal{D}_{ik})$, where t_i is the number of equivalence classes with respect to $\equiv_i$. For simplicity of notation, we make for all of the following the abbreviations $H_{0k} =: H_k$, $\mathcal{D}_{0k} =: \mathcal{D}_k$ and $A_{0k} =: A_k$, $k \in \{1, \ldots, t_0\}$. We can now reduce the semisimplicity of $M_S(G, \mathcal{C})$ to the semisimplicity of the near-rings $M_{A_k}(H_k, \mathcal{D}_k)$, which has been determined in [4], Theorem II.2. For this we use sets $I(w)$, $S(w)$, $E(w)$, $s(w)$, $e(w)$, which have been defined in [4]. We refer here to the definition given there. Also, for $i \in \{0, \dots, l\}$, $j \in \{1, \dots, n_i\}$, let $E_{ij} := \{t \in \{1, \dots, n_i\} \mid I_{it} \sim_i I_{ij}\}$.

Theorem 15. *For a covered group $(G, \mathcal{C})$ and an inverse semigroup of operators S for $(G, \mathcal{C})$ such that $id \in S$ and $e(C) \subseteq C$ for all $C \in \mathcal{C}$, the following are equivalent*:

(1) *$M_S(G, \mathcal{C})$ is 2-semisimple.*

(2) (a) *$\bigcap_{i=1}^{n_0} \ker(e_{0i}) = \{0\}$, where $\{e_{01}, \dots, e_{0n_0}\}$ is the set of primitive idempotents of S (as defined on page 161).*

(b) *$\forall k \in \{1, \dots, t_0\} : M_{A_k}(H_k, \mathcal{D}_k)$ is 2-semisimple.*

(3) (a) *$\bigcap_{i=1}^{n_0} \ker(e_{0i}) = \{0\}$.*

(b) *$\forall k \in \{1, \dots, t_0\}\ \forall w \in H_k^* : (S(w) \cap s(w))^* = E(w) \cap e(w)$ (where S, E, s, e are taken with respect to H_k, A_k, and D_k).*

Proof. (1) $\Rightarrow$ (2) : We use the notation on page 162. Suppose $K_{11} = \bigcap_{i=1}^{n_0} \ker(e_{0i}) \neq 0$. Then there exists an integer $l \geq 1$ such that $K_{ln_l} \neq 0$ and $K_{l+11} = 0$. We have $e_{l1}(K_{l1}) \neq 0$. Note that if $j \in E_{l1} = \{j \in \{1, \dots, n_l \mid I_{lj} \sim_l I_{l1}\}$ and $\alpha|I_{lj} : I_{lj} \to I_{lk}$ is an isomorphism for some $k \in \{1, \dots, n_l\}$, then also $k \in E_{l1}$. For $g = \sum_{i=0}^{l} \sum_{j=1}^{n_i} g_{ij} \in G$, let $Y_g = \{j \in E_{l1} \mid \forall k \in X_{l-1j} : g_{l-1k} \neq 0\}$. Define a function $f : G \to G$ by

$$f(\sum_{i=0}^{l} \sum_{j=1}^{n_i} g_{ij}) := \sum_{j \in Y_g} g_{lj}.$$

Note that if $Y_g = \emptyset$, then $\sum_{j \in Y_g} g_{lj} := 0$. Since $K_{\ln_l} \neq 0$, f is not the zero function. We show that $f \in M_S(G, \mathcal{C})$. By Theorem 2, $f(C) \subseteq C$ for all $C \in \mathcal{C}$.

Let $g = \sum_{i=0}^{l} \sum_{j=1}^{n_i} g_{ij} \in G$. If $\beta \in S$, then $\beta g = \sum_{i=0}^{l} \sum_{j=1}^{n_i} \beta g_{ij}$ $=: \sum_{i=0}^{l} \sum_{j=1}^{n_i} \overline{g}_{ij}$, hence $f(\beta g) = \sum_{k \in Y_{\beta g}} \overline{g}_{lk}$. We need to prove that $f(\beta g) = \beta f(g)$. Suppose $k \in Y_{\beta g}$ is such that $\overline{g}_{lk} \neq 0$. Then there exists an element $j \in \{1, \dots, n_l\}$ such that $\overline{g}_{lk} = \beta(g_{lj})$. By Theorem 9, this means that $\beta | I_{lj} : I_{lj} \to I_{lk}$ is an isomorphism, hence $j \in E_{l1}$. By Theorem 14, there exists a permutation $\pi : X_{l-1j} \to X_{l-1k}$ such that $\beta | I_{l-1s} : I_{l-1s} \to I_{l-1\pi(s)}$ is an isomorphism for all $s \in X_{l-1j}$, in particular $|X_{l-1j}| = |X_{l-1k}|$. Suppose $j \notin Y_g$. Then there exists an element $t \in X_{l-1j}$ such that $g_{l-1t} = 0$, hence $\beta(g_{l-1t}) = 0$. Since $|X_{l-1j}| = |X_{l-1k}|$, we then cannot have that $k \in Y_{\beta g}$, which contradicts our assumption. Therefore, we conclude that $j \in Y_g$ and $\overline{g}_{lk} = \beta(g_{lj})$. Now let $j \in Y_g$ be arbitrary. Then either $\beta(g_{lj}) = 0$ or $\beta(g_{lj}) \neq 0$. If $\beta(g_{lj}) \neq 0$, then by Theorem 9, $\beta | I_{lj} : I_{lj} \to I_{lk}$ is an isomorphism for some $k \in \{1, \dots, n_l\}$, and since $j \in E_{l1}$. also $k \in E_{l1}$. By Theorem 14, there exists a permutation $\pi : X_{l-1j} \to X_{l-1k}$ such that $\beta | I_{l-1s} : I_{l-1s} \to I_{l-1\pi(s)}$ is an isomorphism for all $s \in X_{l-1j}$. Since $j \in Y_g$, $g_{l-1s} \neq 0$ for all $s \in X_{l-1j}$, hence $\beta(g_{l-1j}) \neq 0$ for all $s \in X_{l-1j}$, which means that $\overline{g}_{l-1t} \neq 0$ for all $t \in X_{l-1k}$. Therefore, $k \in Y_{\beta g}$ and $\beta(g_{lj}) = \overline{g}_{lk}$. It now follows that $f(\beta g) = f(\sum_{i=0}^{l} \sum_{j=1}^{n_i} \overline{g}_{ij}) =$ $\sum_{k \in Y_{\beta g}} \overline{g}_{lk} = \sum_{j \in Y_g} \beta(g_{lj}) = \beta(\sum_{j \in Y_g} g_{lj}) = \beta f(g)$. We have now shown that $f \in M_S(G, \mathcal{C})$. By our construction, $f(K_i) \subseteq K_l$ for all $0 \leq i \leq l-1$ and

$f(K_l) = \{0\}$. Therefore $0 \neq f \in \ker \phi$, as defined in Theorem 7. By Theorem 8, $\ker \phi \subseteq J_2(N)$, which contradicts our assumption that $M_S(G, \mathcal{C})$ is 2-semisimple. Therefore, we can conclude that $\bigcap_{i=1}^{n_0} \ker(e_{0i}) = \{0\}$, which shows (2)(a).

By Theorems 7, 11, and 12, $N \cong \bigoplus_{i=1}^{t_0} M_{A_i}(H_i, \mathcal{D}_i)$, hence $0 = J_2(\bigoplus_{i=1}^{t_0} M_{A_i}(H_i, \mathcal{D}_i)) = \bigoplus_{i=1}^{t_0} J_2(M_{A_i}(H_i, \mathcal{D}_i))$ by ([12], Theorem 5.20). Consequently, $J_2(M_{A_i}(H_i, \mathcal{D}_i))) = 0$ for all $i \in \{1, \dots, t_0\}$, which implies (2)(b).

(2) $\Rightarrow$ (1) : By Theorems 7, 11, and 12, $N \cong \bigoplus_{i=1}^{t_0} M_{A_i}(H_i, \mathcal{D}_i)$, and $J_2(\bigoplus_{i=1}^{t_0} M_{A_i}(H_i, \mathcal{D}_i)) = \bigoplus_{i=1}^{t_0} J_2(M_{A_i}(H_i, \mathcal{D}_i)) = \{0\}$.

(2) $\Leftrightarrow$ (3) : Follows from Theorem II.2 in [4]. □

Example 1. Let $G := \mathbb{Z}_2^5$, $B := \{b_1, b_2, b_3, b_4, b_5\}$ the canonical basis of G. Define linear maps e_1, e_2, e_3, α on G as follows: $e_1(b_1) = b_1$, $e_1(b_2) = \dots = e_1(b_5) = 0$, $e_2(b_1) = b_1$, $e_2(b_2) = b_2$, $e_2(b_3) = b_3$, $e_2(b_4) = e_2(b_5) = 0$, $e_3(b_1) = b_1$, $e_3(b_2) = e_3(b_3) = 0$, $e_3(b_4) = b_4$, $e_3(b_5) = b_5$, $\alpha(b_1) = b_1$, $\alpha(b_2) = b_4$, $\alpha(b_3) = b_5$, $\alpha(b_4) = b_2$, $\alpha(b_5) = b_3$. Let $\mathcal{C} := \{C_1 = \langle b_1, b_2 + b_3, b_4 + b_5 \rangle$, $C_2 = \langle b_1, b_2 + b_3, b_4 \rangle$, $C_3 = \langle b_1, b_2 + b_3, b_5 \rangle$, $C_4 = \langle b_1, b_2, b_4 + b_5 \rangle$, $C_5 = \langle b_1, b_2, b_4 \rangle$, $C_6 = \langle b_1, b_2, b_5 \rangle$, $C_7 = \langle b_1, b_3, b_4 \rangle$, $C_8 = \langle b_1, b_3, b_5 \rangle$, $C_9 = \langle b_1, b_3, b_4 + b_5 \rangle\}$. Then $\mathcal{C}$ is a cover of G and $S := \{e_1, e_2, e_3, \alpha, \alpha e_2, e_2\alpha, id, 0\}$ is an inverse semigroup with idempotents $e_1, e_2, e_3, id, 0$, $e_1 \leq e_2$, $e_1 \leq e_3$, $\alpha^2 = id$, $\alpha e_2 = e_3 \alpha$, $e_2 \alpha = \alpha e_3$. Further, one checks that $e(C) \subseteq C$ for every idempotent $e \in S$, $C \in \mathcal{C}$ and that S is a semigroup of operators for $(G, \mathcal{C})$.

Let $N := M_S(G, \mathcal{C})$. We want to determine the decomposition of N in Theorem 7 and the radicals $J_\nu(N)$, $\nu \in \{0, 1, 2\}$ of N in Theorem 8. We have $K_0 = G$, $K_1 = \ker e_1$, $K_2 = \ker e_1 \cap \ker e_2 \cap \ker e_3 = 0$. By Theorems 7, 8, $N/J \cong \bigoplus_{i=0}^{1} M_{S_i}(G_i, \mathcal{C}_i)$, where $G_i = K_i/K_{i+1}$, $S_i = \{\bar{s} \,|\, s \in S|K_i = T_i\}$, where $\bar{s}(k/K_{i+1}) = s(k)/K_{i+1}$ for $k \in K_i$ and $\mathcal{C}_i = \{(C \cap K_i)/K_{i+1} | C \in \mathcal{C}\}$.

$G_0 \cong \langle b_1 \rangle$, $S_0 = \{id, 0\}$, $\mathcal{C}_0 = \{\langle b_1 \rangle\}$. Therefore, $M_{S_0}(G_0, \mathcal{C}_0) \cong \mathbb{Z}_2$. $G_1 \cong K_1 = \ker(e_1)$, $S_1 = \{\bar{e}_2, \bar{e}_3, \bar{\alpha}, \overline{\alpha e_2}, \overline{e_2 \alpha}, \overline{id}, \bar{0}\}$, $\mathcal{C}_1 = \{C \cap \ker e_1 | C \in \mathcal{C}\}$, e_2 and e_3 are the idempotents which are minimal with respect to the property that $e(K_1) = e(\ker e_1) \neq 0$. Following the notation on page 171, if $\bar{I}_{11} := \bar{e}_2(G_1) \cong \langle b_2, b_3 \rangle$ and $\bar{I}_{12} := \bar{e}_3(G_1) \cong \langle b_4, b_5 \rangle$, then $\bar{\alpha} : \bar{I}_{11} \to \bar{I}_{12}$ is a group isomorphism. Since $G_1 = \bar{I}_{11} \oplus \bar{I}_{12}$, there is only one equivalence class with respect to the equivalence relation $\equiv_1$. By Theorem 12, $M_{S_1}(G_1, \mathcal{C}_1) \cong M_{A_{11}}(H_{11}, \mathcal{D}_{11})$, where $H_{11} = \bar{I}_{11}$, $A_{11} = \{\alpha|H_{11}| \alpha \in S_1$, $\alpha(H_{11}) \subseteq H_{11}\}$ and $\mathcal{D}_{11} = \{\alpha(C \cap \bar{I}_{1j}) | C \in \mathcal{C}_1, \alpha \in S_1, \alpha|\bar{I}_{1j} : \bar{I}_{1j} \to H_{11}$ is an isomorphism, $j \in \{1, 2\}\}$. It is easy to check that $H_{11} = \langle b_2, b_3 \rangle$, $A_{11} = \{id, 0\}$, and $\mathcal{D}_{11} = \{\langle b_2 + b_3 \rangle, \langle b_2 \rangle, \langle b_3 \rangle\}$. If, for a group F, $M_0(F)$ denotes the near-ring $\{f : F \to F \,|\, f$ is a function, $f(0) = 0\}$ with respect to function addition and composition, then $M_{S_1}(G_1, \mathcal{C}_1) \cong M_{A_{11}}(H_{11}, \mathcal{D}_{11})$ $= \{f : H_{11} \to H_{11} \,|\, f(0) = 0, f(b_2) \in \langle b_2 \rangle, f(b_3) \in \langle b_3 \rangle, f(b_2 + b_3) \in \langle b_2 + b_3 \rangle\}$ $\cong M_0(\langle b_2 \rangle) \times M_0(\langle b_3 \rangle) \times M_0(\langle b_2 + b_3 \rangle) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Since the field $\mathbb{Z}_2$ is semisimple, we have that $J_\nu(M_{S_0}(G_0, \mathcal{C}_0)) = J_\nu(M_{S_1}(G_1, \mathcal{C}_1)) = 0$ for $\nu \in \{0, 1, 2\}$, from which we conclude that in Theorem 8, $\bar{T} = 0$, hence $J_\nu(N) = J$ for $\nu \in \{0, 1, 2\}$. It remains to compute J.

By Theorem 8, $J = \{f \mid \forall i \in \{0, \dots, l\} : f(K_i) \subseteq K_{i+1}\} = \{f \mid f(G) \subseteq \ker e_1,$ $f(\ker e_1) \subseteq K_2 = \{0\}\}$. Since e_1 is the only primitive idempotent and $K_1 = \ker e_1 \neq 0$, we know from Theorem 15 that $J \neq 0$. e_2 and e_3 are the idempotents e minimal with respect to the property that $e(K_1) \neq 0$. Let $e_{01} := e_1, e_{11} := e_2, e_{12} := e_3$. Then $K_{12} = \ker e_1 \cap \ker e_{11} = \langle b_4, b_5 \rangle$, $K_2 = \ker e_{01} \cap \ker e_{11} \cap \ker e_{12} = 0$. By Theorem 1, every $g \in G$ has a unique representation $g = g_{01} + g_{11} + g_{12}$, where $g_{01} \in e_{01}(G) = \langle b_1 \rangle$, $g_{11} \in \langle b_2, b_3 \rangle, g_{12} \in \langle b_4, b_5 \rangle$. If $f \in J$, then since $f(G) \subseteq \ker e_1, f(\ker e_1) = 0$, we have

$$f(g_{01} + g_{11} + g_{12}) = \begin{cases} 0 & \text{if } g_{01} = 0, \\ h_{11} + h_{12} & \text{for some } h_{11} \in \langle b_2, b_3 \rangle, \\ & \quad h_{12} \in \langle b_4, b_5 \rangle, \text{ if } g_{01} \neq 0, \end{cases}$$

$f(b_1) = f(e_{01}b_1) = e_{01}f(b_1) \in e_{01}(G) = \langle b_1 \rangle$, thus $f(b_1) \in \{0, b_1\}$. But since $f(G) \subseteq \ker e_1$, it follows that $f(b_1) = 0$. Further, $g_{01} = b_1, f(e_2 g) = f(b_1 + g_{11})$ $= e_2(f(g)) = e_2(h_{11} + h_{12}) = h_{11}$ and similarly, $f(e_3 g) = h_{12}$. Therefore, for each $f \in J$, we get a function $h : \langle b_4, b_5 \rangle \to \langle b_4, b_5 \rangle$ such that $h(x) := f(b_1 + x)$. Since $f(b_1) = 0$, we have $h(0) = 0$ and since $f(C) \subseteq C$ for all $C \in \mathcal{C}$, one can check that $h(b) \in \{0, b\}$ for all $b \in \langle b_4, b_5 \rangle$. Also, $h(\alpha g_{11}) = f(b_1 + \alpha g_{11}) = \alpha f(b_1 + g_{11}) = \alpha h_{11}$. Combining our calculations, we get for $f \in J, f(g_{01} + g_{11} + g_{12}) = \alpha^{-1} h(\alpha g_{11}) + h(g_{12})$.

Conversely, if $h : \langle b_4, b_5 \rangle \to \langle b_4, b_5 \rangle$ is an arbitrary function such that $h(0) = 0$ and $h(b) \in \{0, b\}$ for $b \in \langle b_4, b_5 \rangle$, then f_h such that

$$f_h(g_{01} + g_{11} + g_{12}) := \begin{cases} 0, & \text{if } g_{01} = 0, \\ \alpha^{-1} h(\alpha g_{11}) + h(g_{12}), & \text{if } g_{01} \neq 0, \end{cases}$$

is an element of J. In fact, it is easy to check that for $g \in G$, $i \in \{1, 2, 3\}$, $f_h(e_i g) = e_i(f_h(g))$ and $f_h(\alpha g) = \alpha(f_h(g))$. Since e_1, e_2, e_3, α generate all of S, it follows that $f_h(\sigma g) = \sigma(f_h(g))$ for all $\sigma \in S$. Also, $f_h(C) \subseteq C$ for all $C \in \mathcal{C}$.

Combining our results we obtain $J_\nu(N) = J = \{f_h \,|\, h : \langle b_4, b_5 \rangle \to \langle b_4, b_5 \rangle$, $h(0) = 0$, $h(b) \in \{0, b\}$, for $b \in \langle b_4, b_5 \rangle$ is a function$\}$, for $\nu \in \{0, 1, 2\}$.

References

[1] J. André, Über Parallelstrukturen, II: Translationsstrukturen, Mathematische Zeitschrift 76(1) (1961), 155-163.

DOI: https://doi.org/10.1007/BF01210967

[2] C. Cotti Ferrero and G. Ferrero, Some Developments Linked to Semigroups and Groups, Advances in Mathematics, Volume 4, Kluwer Academic Publishers, Dordrecht, 2002.

[3] P. R. Fuchs, Kernels of covered groups over completely regular inverse semigroups, Communications in Algebra 51(4) (2023), 1556-1564.

DOI: https://doi.org/10.1080/00927872.2022.2138419

[4] P. R. Fuchs and C. J. Maxson, Kernels of covered groups with operators, Journal of Algebra 114(1) (1988), 68-80.

DOI: https://doi.org/10.1016/0021-8693(88)90213-X

[5] L. Kabza, The centralizer near-ring of an inverse semigroup of endomorphisms of a group, Communications in Algebra 23(14) (1995), 5419-5435.

DOI: https://doi.org/10.1080/00927879508825542

[6] C. J. Maxson, Near rings associated with Sperner Spaces, Journal of Geometry 20(1) (1983), 128-145.

DOI: https://doi.org/10.1007/BF01918003

[7] C. J. Maxson, Near-rings associated with generalized translation structures, Journal of Geometry 24 (1985), 175-193.

DOI: https://doi.org/10.1007/BF01220486

[8] C. J. Maxson and K. C. Smith, Centralizer near-rings determined by completely regular inverse semigroups, Semigroup Forum 22(1) (1981), 47-58.

DOI: https://doi.org/10.1007/BF02572783

[9] J. D. P. Meldrum, Near-rings and their Links with Groups, Research Notes in Mathematics, Volume 134, Pitman, London, 1985.

[10] S. J. Mahmood, J. D. P. Meldrum and L. O'Carrol, Inverse semigroups and near-rings, Journal of the London Mathematical Society (2), 23(1) (1981), 45-60.

DOI: https://doi.org/10.1112/jlms/s2-23.1.45

[11] M. Petrich, Inverse Semigroups, Pure and Applied Mathematics, John Wiley, New York, 1984.

[12] G. Pilz, Near-Rings, Rev. Ed., North-Holland, New York, 1983.

■