Research and Communications in Mathematics and Mathematical Sciences
Vol. 16, Issue 2, 2024, Pages 135-158
ISSN 2319-6939
Published Online on November 14, 2024

http://jyotiacademicpress.org

ON VARIATIONAL ITERATION METHODS FOR THE FRACTIONAL MODEL OF BLOOD ETHANOL CONCENTRATION SYSTEM

G. MONZÓN

Instituto de Ciencias
Universidad Nacional de General Sarmiento
Buenos Aires
Argentina
e-mail: gmonzon@campus.ungs.edu.ar

Abstract

In this paper, we numerically approximate the solutions of the first-order fractional differential equation system that models blood ethanol concentration using a fractional variational iteration method (FVIM) developed for high-order differential equations. Comparative analysis with two different variational iteration methods, including one specifically tailored for this problem, shows that the FVIM achieves lower approximation errors.

1. Introduction

The use of fractional differential equations to model diverse phenomena has seen significant growth in recent years across numerous fields including finance, biology, mechanics and engineering [3, 4, 8-10, 17, 28, 36]. In particular, in chemical engineering, one such phenomenon is the

2020 Mathematics Subject Classification: 26A33, 41A10, 65D99.
Keywords and phrases: fractional blood ethanol concentration system, variational iteration method, Caputo fractional derivative, higher-order fractional differential equations.
Communicated by Cemil Tunc.
Received July 27, 2024; Revised September 6, 2024

concentration of ethanol in human blood. In [30], this process is analyzed using the following system of first-order fractional differential equations:

$$\begin{cases} D^{\alpha}v(t) = -\kappa^{\alpha}v(t), \\ D^{\alpha}u(t) = \kappa^{\alpha}v(t) - \eta^{\alpha}u(t), \\ v(0) = v_0, \quad u(0) = 0, \end{cases} \tag{1}$$

where $v(t)$ denotes the concentration of alcohol in the stomach at time t (mg/l), $u(t)$ represents the concentration of alcohol in the blood at time t (mg/l), κ and η are rate constants $(\min^{-1})$, v_0 is the initial concentration of alcohol in the stomach (mg/l) and D^{α} denotes the Caputo fractional derivative of order $\alpha \in (0, 1)$. The study in [30] demonstrates that this fractional model offers a more accurate estimate of real data compared to models that use integer-order derivatives.

The exact solution of (1) can be obtained using the Laplace transform technique. Indeed, from [30], we know that

$$v(t) = v_0 E_{\alpha}(-\kappa^{\alpha}t^{\alpha}) \text{ and } u(t) = v_0\kappa^{\alpha} \sum_{i\geq 0} \sum_{j\geq 0} \frac{(-1)^{i+j}\kappa^{\alpha i}\eta^{\alpha j}}{\Gamma((i+j+1)\alpha + 1)} t^{(i+j+1)\alpha},$$

where E_{α} denotes the Mittag-Leffler function defined as

$$E_{\alpha}(z) = \sum_{i\geq 0} \frac{z^i}{\Gamma(i\alpha + 1)}.$$

However, having numerical methods that provide accurate and efficient approximations is highly useful and advantageous. Many numerical approximation techniques for fractional differential equations have been explored [2, 5, 7, 11-13, 15, 19-22, 26, 31, 33], among which the variational iteration method (VIM) stands out as one of the most popular and effective [16, 18, 25, 29, 32, 34, 35]. In particular, in [1], a VIM is developed exclusively for the problem (1). This method essentially relies on constructing each correction term using the Riemann-Liouville operator and a Lagrange multiplier, which is determined through the stationary conditions. Moreover, the approximating sequences $(\phi_n)_n$ and $(\psi_n)_n$ for v and u, respectively, are given by

$$\phi_n(t) = \phi_{n-1}(t) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(D^\alpha \phi_{n-1}(\tau) + \kappa^\alpha \phi_{n-1}(\tau))d\tau, \quad (2a)$$

$$\psi_n(t) = \psi_{n-1}(t) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(D^\alpha \psi_{n-1}(\tau) - \kappa^\alpha \phi_{n-1}(\tau) + \eta^\alpha \psi_{n-1}(\tau))d\tau, \quad (2b)$$

with the initial terms

$$\phi_0(t) = v_0 \text{ and } \psi_0(t) = 0.$$

On the other hand, in [24], a VIM was proposed to obtain numerical approximations for arbitrary linear (and non-linear) systems of fractional differential equations. For the problem (1), the approximating sequences $(\varphi_n)_n$ and $(\chi_n)_n$ for v and u, respectively, are given by

$$\varphi_n(t) = \varphi_{n-1}(t) - \int_0^t (D^\alpha \varphi_{n-1}(\tau) + \kappa^\alpha \varphi_{n-1}(\tau))d\tau, \quad (3a)$$

$$\chi_n(t) = \chi_{n-1}(t) - \int_0^t (D^\alpha \chi_{n-1}(\tau) - \kappa^\alpha \varphi_{n-1}(\tau) + \eta^\alpha \chi_{n-1}(\tau))d\tau, \quad (3b)$$

with the initial terms

$$\varphi_0(t) = v_0 \text{ and } \chi_0(t) = 0.$$

In this work, we reformulate the partially coupled system (1) into a set of second-order homogeneous fractional differential equations and then, in order to obtain an alternative numerical method to (2) and (3), we use the fractional variational iteration method (FVIM) given in [27]. Through concrete numerical examples, we demonstrate that the fractional variational iteration method provides more accurate approximations.

Additionally, we derive alternative explicit expressions for the functions ϕ_n and ψ_n given in (2a) and (2b), which eliminate cross-term usage and unnecessary recurrences, relying solely on n (see Lemma 3.1 below). Clearly, this leads to a more straightforward and simplified implementation of the method. Nevertheless, the formulation we provide for the FVIM also offers this advantage and, as previously noted, yields improved results.

The organization of the paper is as follows: In Section 2, we introduce definitions, notations and basic properties related to the Riemann-Liouville integral operator, the Caputo fractional derivative and the Laplace transform, which are crucial for the comprehensive development of our work. Sections 3 and 4 are dedicated to presenting the approximations given by the variational iteration methods in [1] and [24], respectively, in a clear and concise manner. In Section 5, we use the FVIM developed in [27] to derive new approximating sequences for the real solutions of (1). Finally, in Section 6, numerical experiments are presented to compare the performance of the three methods considered.

2. Preliminaries

In this section, we briefly introduce definitions and well-known results that are useful for our purposes. For a more comprehensive and detailed discussion, we refer the reader to [6, 14, 23].

2.1. On the Riemann-Liouville integral operator and the Caputo fractional derivative

The Riemann-Liouville integral operator of w of order $\alpha \in (0, 1)$, $I^{\alpha}w$, is defined as

$$I^{\alpha}w(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}w(\tau)d\tau \quad (t \geq 0), \tag{4}$$

where $\Gamma : (0, +\infty) \to \mathbb{R}$ denotes the Gamma function given by

$$\Gamma(x) = \int_0^{+\infty} \tau^{x-1} e^{-\tau} d\tau.$$

A well-known fact about this operator is that, for any $r \geq 0$,

$$I^{\alpha} t^{r} = \frac{\Gamma(r+1)}{\Gamma(r+\alpha+1)} t^{\alpha+r}. \tag{5}$$

On the other hand, the Caputo fractional derivative of w of order $\alpha \in (0, 1)$, $D^{\alpha} w$, can be defined by $I^{1-\alpha}$ as follows

$$D^{\alpha} w = I^{1-\alpha} w',$$

where w' denotes the ordinary derivative of w, i.e.,

$$D^{\alpha} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{w'(\tau)}{(t-\tau)^{\alpha}} d\tau \quad (t \geq 0). \tag{6}$$

Taking into account that Gamma function Γ satisfies

$$x\Gamma(x) = \Gamma(x+1),$$

from (5) it follows that, for any $r \geq 0$,

$$D^{\alpha} t^{r} = \frac{\Gamma(r+1)}{\Gamma(r+1-\alpha)} t^{r-\alpha}. \tag{7}$$

The operators I^{α} and D^{α} are also related in the following ways:

$$D^{\alpha} I^{\alpha} w = w, \tag{8}$$

and

$$I^{\alpha} D^{\alpha} w(t) = w(t) - w(0). \tag{9}$$

2.2. On the Laplace transform

The Laplace transform of v, Lv, is given by

$$Lv(s) = \int_0^\infty e^{-s\tau} v(\tau) d\tau \quad (s > 0). \tag{10}$$

Among the many properties of this linear operator, we emphasize the following:

$$Lt^r(s) = \frac{\Gamma(r+1)}{s^{r+1}}, \quad r \geq 0, \tag{11}$$

$$LI^\alpha v(s) = \frac{1}{s^\alpha} Lv(s), \tag{12}$$

and

$$LD^\alpha v(s) = s^\alpha Lv(s) - s^{\alpha-1} v(0). \tag{13}$$

Finally, as usual, we use L^{-1} to denote the inverse Laplace transform. In particular, from (11), L^{-1} is a linear operator which satisfies

$$L^{-1} \frac{1}{s^{r+1}}(t) = \frac{t^r}{\Gamma(r+1)}, \quad r \geq 0. \tag{14}$$

3. On the VIM developed in [1]

A variational iteration method for the problem (1) is exclusively developed in [1] where the correction terms are built from the Riemann-Liouville operator (4) and the Lagrange multipliers are obtained from the stationary conditions via variational theory. In fact, the proposed approximating sequences $(\phi_n)_n$ of v and $(\psi_n)_n$ of u (previously introduced here in (2a) and (2b)) can be actually written as

$$\phi_n = \phi_{n-1} - I^\alpha(D^\alpha \phi_{n-1} + \kappa^\alpha \phi_{n-1}), \quad \phi_0 = v_0,$$

and

$$\psi_n = \psi_{n-1} - I^\alpha(D^\alpha \psi_{n-1} - \kappa^\alpha \phi_{n-1} + \eta^\alpha \psi_{n-1}), \quad \psi_0 = 0.$$

Now, thanks to the linearity of the operator I^{α} and the property (9) it follows that

$$\phi_n = \phi_{n-1} - I^{\alpha}D^{\alpha}\phi_{n-1} - \kappa^{\alpha}I^{\alpha}\phi_{n-1} = v_0 - \kappa^{\alpha}I^{\alpha}\phi_{n-1}, \tag{15}$$

and then

$$\psi_n = \psi_{n-1} - I^{\alpha}D^{\alpha}\psi_{n-1} + \kappa^{\alpha}I^{\alpha}\phi_{n-1} - \eta^{\alpha}\mathrm{I}^{\alpha}\psi_{n-1} = v_0 - \phi_n - \eta^{\alpha}I^{\alpha}\psi_{n-1}. \tag{16}$$

From these identities, we can derive alternative expressions for ϕ_n and ψ_n which are highly useful from a computational point of view since they depend solely on n, avoiding the cross-use of terms and unnecessary recurrences.

Lemma 3.1. *Let* ϕ_n *and* ψ_n, *with* $n \geq 1$, *be as in* (15) *and* (16) *respectively*, *then*

$$\phi_n(t) = v_0 \sum_{i=0}^{n} \frac{(-1)^i(\kappa t)^{\alpha i}}{\Gamma(i\alpha + 1)}, \tag{17}$$

and

$$\psi_n(t) = v_0\kappa^{\alpha} \sum_{i=1}^{n} \frac{(-1)^{i+1}\left(\sum_{j=0}^{i-1} \kappa^{(i-1-j)\alpha}\eta^{j\alpha}\right)t^{i\alpha}}{\Gamma(i\alpha + 1)}. \tag{18}$$

Proof. We argue by induction on n. For $n = 1$, from (15) and (5) we have

$$\phi_1(t) = v_0 - \kappa^{\alpha}I^{\alpha}v_0 = v_0(1 - \kappa^{\alpha}I^{\alpha}1) = v_0\left(1 - \kappa^{\alpha}\frac{t^{\alpha}}{\Gamma(\alpha + 1)}\right)$$

$$= v_0 \sum_{i=0}^{1} \frac{(-1)^i(\kappa^{\alpha}t^{\alpha})^i}{\Gamma(i\alpha + 1)};$$

on the other hand, from (16) we have

$$\psi_1(t) = v_0 - \phi_1(t) - \eta^{\alpha} I^{\alpha} 0 = v_0 - \phi_1(t) = v_0 \kappa^{\alpha} \frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

which proves the claim.

Now suppose that (17) holds for $n \geq 1$. From (15) and (5), it follows that

$$\begin{aligned}\phi_{n+1}(t) &= v_0 - \kappa^{\alpha} I^{\alpha} \phi_n(t) \\ &= v_0 - \kappa^{\alpha} I^{\alpha}\left(v_0 \sum_{i=0}^{n} \frac{(-1)^i \kappa^{\alpha i}}{\Gamma(i\alpha+1)} t^{\alpha i}\right) \\ &= v_0 - \kappa^{\alpha} v_0 \sum_{i=0}^{n} \frac{(-1)^i \kappa^{\alpha i}}{\Gamma(i\alpha+1)} I^{\alpha} t^{\alpha i} \\ &= v_0 + v_0 \sum_{i=0}^{n} \frac{(-1)^{i+1} \kappa^{\alpha i+\alpha}}{\Gamma(i\alpha+1)} \frac{\Gamma(i\alpha+1) t^{\alpha(i+1)}}{\Gamma((i+1)\alpha+1)} \\ &= v_0\left(1 + \sum_{i=0}^{n} \frac{(-1)^{i+1} \kappa^{\alpha(i+1)} t^{\alpha(i+1)}}{\Gamma((i+1)\alpha+1)}\right) = v_0 \sum_{i=0}^{n+1} \frac{(-1)^i (\kappa t)^{\alpha i}}{\Gamma(i\alpha+1)},\end{aligned}$$

which completes the argument regarding ϕ_n.

On the other hand, suppose that (18) holds for $n \geq 1$. From (16) and (5), it follows that

$$\psi_{n+1}(t) = v_0 - \phi_{n+1}(t) - \eta^{\alpha} I^{\alpha} \psi_n(t)$$

$$= v_0 - v_0 \sum_{i=0}^{n+1} \frac{(-1)^i (\kappa t)^{\alpha i}}{\Gamma(i\alpha+1)} - \eta^{\alpha} I^{\alpha}\left(v_0 \kappa^{\alpha} \sum_{i=1}^{n} \frac{(-1)^{i+1}\left(\sum_{j=0}^{i-1} \kappa^{(i-1-j)\alpha} \eta^{j\alpha}\right) t^{\alpha i}}{\Gamma(i\alpha+1)}\right)$$

$$= v_0 \sum_{i=1}^{n+1} \frac{(-1)^{i+1} (\kappa t)^{\alpha i}}{\Gamma(i\alpha+1)} - v_0 \eta^{\alpha} \kappa^{\alpha} \sum_{i=1}^{n} \frac{(-1)^{i+1}\left(\sum_{j=0}^{i-1} \kappa^{(i-1-j)\alpha} \eta^{j\alpha}\right)}{\Gamma(i\alpha+1)} I^{\alpha} t^{\alpha i}$$

$$= v_0 \sum_{i=1}^{n+1} \frac{(-1)^{i+1}(\kappa t)^{\alpha i}}{\Gamma(i\alpha+1)} - v_0 \eta^{\alpha}\kappa^{\alpha} \sum_{i=1}^{n} \frac{(-1)^{i+1}\left(\sum_{j=0}^{i-1} \kappa^{(i-1-j)\alpha}\eta^{j\alpha}\right)t^{\alpha i}}{\Gamma(i\alpha+1)} \times \frac{\Gamma(i\alpha+1)t^{\alpha(i+1)}}{\Gamma((i+1)\alpha+1)}$$

$$= v_0 \sum_{i=1}^{n+1} \frac{(-1)^{i+1}(\kappa t)^{\alpha i}}{\Gamma(i\alpha+1)} + v_0 \eta^{\alpha}\kappa^{\alpha} \sum_{i=2}^{n+1} \frac{(-1)^{i+1}\left(\sum_{j=0}^{i-2} \kappa^{(i-2-j)\alpha}\eta^{j\alpha}\right)t^{\alpha i}}{\Gamma(i\alpha+1)}$$

$$= v_0\kappa^{\alpha}\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \sum_{i=2}^{n+1} \frac{(-1)^{i+1}\kappa^{\alpha(i-1)}t^{\alpha i}}{\Gamma(i\alpha+1)} + \sum_{i=2}^{n+1} \frac{(-1)^{i+1}\left(\sum_{j=0}^{i-2} \kappa^{(i-2-j)\alpha}\eta^{(j+1)\alpha}\right)t^{\alpha i}}{\Gamma(i\alpha+1)}\right]$$

$$= v_0\kappa^{\alpha}\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \sum_{i=2}^{n+1} \frac{(-1)^{i+1}t^{\alpha i}}{\Gamma(i\alpha+1)}\left(\kappa^{\alpha(i-1)} + \sum_{j=0}^{i-2} \kappa^{(i-2-j)\alpha}\eta^{(j+1)\alpha}\right)\right]$$

$$= v_0\kappa^{\alpha}\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \sum_{i=2}^{n+1} \frac{(-1)^{i+1}t^{\alpha i}}{\Gamma(i\alpha+1)}\left(\sum_{j=0}^{i-1} \kappa^{(i-1-j)\alpha}\eta^{j\alpha}\right)\right]$$

$$= v_0\kappa^{\alpha} \sum_{i=1}^{n-1} \frac{(-1)^{i+1}t^{\alpha i}}{\Gamma(i\alpha+1)}\left(\sum_{j=0}^{i-1} \kappa^{(i-1-j)\alpha}\eta^{j\alpha}\right).$$

This proves the lemma. □

Remark 3.1. The explicit expression for ψ_3 given in [1] is

$$\psi_3(t) = v_0\kappa^{\alpha}\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{4^{-\alpha}(\kappa^{\alpha}+\eta^{\alpha})\sqrt{\pi}t^{2\alpha}}{\Gamma(0.5+\alpha)\Gamma(1+\alpha)} + \frac{4^{-\alpha}(\kappa^{2\alpha}+(\eta\kappa)^{\alpha}+\eta^{2\alpha})\sqrt{\pi}\Gamma(1+2\alpha)t^{3\alpha}}{\Gamma(0.5+\alpha)\Gamma(1+\alpha)\Gamma(1+3\alpha)}\right],$$

whereas the expression derived from (18) is

$$\psi_3(t) = v_0\kappa^{\alpha}\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{(\kappa^{\alpha}+\eta^{\alpha})t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{(\kappa^{2\alpha}+(\eta\kappa)^{\alpha}+\eta^{2\alpha})t^{3\alpha}}{\Gamma(3\alpha+1)}\right].$$

The differences between both expressions can be explained using the Legendre's duplication formula:

$$\Gamma(\beta)\Gamma(0.5+\beta) = 2^{1-2\beta}\sqrt{\pi}\Gamma(2\beta),$$

in the particular case $\beta = 0.5 + \alpha$. A similar situation arises with the expressions for ψ_2, ϕ_2 and ϕ_3 provided in [1] compared to those derived here.

4. On the VIM developed in [24]

Unlike the VIM proposed in [1], it is neither simple nor straightforward to find expressions for the approximating sequences φ_n and χ_n provided in [24] (given here in (3a)-(3b)) that depend solely on n. We present only the first five terms of each sequence, which will suffice for our purposes:

$$\varphi_1(t) = v_0(1-\kappa^{\alpha}t),$$

$$\varphi_2(t) = v_0\left(1 - 2\kappa^{\alpha}t + \frac{\kappa^{2\alpha}}{2}t^2 + \frac{\kappa^{\alpha}}{\Gamma(3-\alpha)}t^{2-\alpha}\right),$$

$$\varphi_3(t) = v_0\left(1 - 3\kappa^{\alpha}t + \frac{3\kappa^{2\alpha}}{2}t^2 - \frac{\kappa^{3\alpha}}{6}t^3 + \frac{3\kappa^{\alpha}}{\Gamma(3-\alpha)}t^{2-\alpha}\right.$$
$$\left. - \frac{2\kappa^{2\alpha}}{\Gamma(4-\alpha)}t^{3-\alpha} - \frac{\kappa^{\alpha}}{\Gamma(4-2\alpha)}t^{3-2\alpha}\right),$$

$$\varphi_4(t) = v_0\left(1 - 4\kappa^{\alpha}t + 3\kappa^{2\alpha}t^2 - \frac{2\kappa^{3\alpha}}{3}t^3 + \frac{\kappa^{4\alpha}}{24}t^4 + \frac{6\kappa^{\alpha}}{\Gamma(3-\alpha)}t^{2-\alpha} - \frac{8\kappa^{2\alpha}}{\Gamma(4-\alpha)}t^{3-\alpha}\right.$$
$$\left. - \frac{4\kappa^{\alpha}}{\Gamma(4-2\alpha)}t^{3-2\alpha} + \frac{3\kappa^{3\alpha}}{\Gamma(5-\alpha)}t^{4-\alpha} + \frac{3\kappa^{2\alpha}}{\Gamma(5-2\alpha)}t^{4-2\alpha} + \frac{\kappa^{\alpha}}{\Gamma(5-3\alpha)}t^{4-3\alpha}\right),$$

$$\varphi_5(t) = v_0\Bigg(1 - 5\kappa^{\alpha}t + 5\kappa^{2\alpha}t^2 - \frac{5\kappa^{3\alpha}}{3}t^3 + \frac{5\kappa^{4\alpha}}{24}t^4 - \frac{\kappa^{5\alpha}}{120}t^5$$

$$+ \frac{10\kappa^{\alpha}}{\Gamma(3-\alpha)}t^{2-\alpha} - \frac{20\kappa^{2\alpha}}{\Gamma(4-\alpha)}t^{3-\alpha} - \frac{10\kappa^{\alpha}}{\Gamma(4-2\alpha)}t^{3-2\alpha} + \frac{15\kappa^{3\alpha}}{\Gamma(5-\alpha)}t^{4-\alpha}$$

$$+ \frac{15\kappa^{2\alpha}}{\Gamma(5-2\alpha)}t^{4-2\alpha} + \frac{5\kappa^{\alpha}}{\Gamma(5-3\alpha)}t^{4-3\alpha} - \frac{4\kappa^{4\alpha}}{\Gamma(6-\alpha)}t^{5-\alpha}$$

$$- \frac{6\kappa^{3\alpha}}{\Gamma(6-2\alpha)}t^{5-2\alpha} - \frac{4\kappa^{2\alpha}}{\Gamma(6-3\alpha)}t^{5-3\alpha} - \frac{\kappa^{\alpha}}{\Gamma(6-4\alpha)}t^{5-4\alpha}\Bigg),$$

and, calling $\sigma_i = \sum_{j=0}^{i} \kappa^{(i-j)\alpha}\eta^{j\alpha}$ for $i = 1, \ldots, 4,$ we have

$$\chi_1(t) = v_0\kappa^{\alpha}t,$$

$$\chi_2(t) = v_0\kappa^{\alpha}\left(2t - \frac{\sigma_1}{2}t^2 - \frac{1}{\Gamma(3-\alpha)}t^{2-\alpha}\right),$$

$$\chi_3(t) = v_0\kappa^{\alpha}\Bigg(3t - \frac{3\sigma_1}{2}t^2 + \frac{\sigma_2}{6}t^3 - \frac{3}{\Gamma(3-\alpha)}t^{2-\alpha}$$

$$+ \frac{2\sigma_1}{\Gamma(4-\alpha)}t^{3-\alpha} + \frac{1}{\Gamma(4-2\alpha)}t^{3-2\alpha}\Bigg),$$

$$\chi_4(t) = v_0\kappa^{\alpha}\Bigg(4t - 3\sigma_1t^2 + \frac{2\sigma_2}{3}t^3 - \frac{\sigma_3}{24}t^4 - \frac{6}{\Gamma(3-\alpha)}t^{2-\alpha} + \frac{8\sigma_1}{\Gamma(4-\alpha)}t^{3-\alpha}$$

$$+ \frac{4}{\Gamma(4-2\alpha)}t^{3-2\alpha} - \frac{3\sigma_2}{\Gamma(5-\alpha)}t^{4-\alpha} - \frac{3\sigma_1}{\Gamma(5-2\alpha)}t^{4-2\alpha} - \frac{1}{\Gamma(5-3\alpha)}t^{4-3\alpha}\Bigg),$$

$$\chi_5(t) = v_0\kappa^{\alpha}\bigg(5t - 5\sigma_1 t^2 + \frac{5\sigma_2}{3}t^3 - \frac{5\sigma_3}{24}t^4 + \frac{\sigma_4}{120}t^5 - \frac{10}{\Gamma(3-\alpha)}t^{2-\alpha}$$

$$+ \frac{20\sigma_1}{\Gamma(4-\alpha)}t^{3-\alpha} + \frac{10}{\Gamma(4-2\alpha)}t^{3-2\alpha} - \frac{15\sigma_2}{\Gamma(5-\alpha)}t^{4-\alpha} - \frac{15\sigma_1}{\Gamma(5-2\alpha)}t^{4-2\alpha}$$

$$- \frac{5}{\Gamma(5-3\alpha)}t^{4-3\alpha} + \frac{4\sigma_3}{\Gamma(6-\alpha)}t^{5-\alpha} + \frac{6\sigma_2}{\Gamma(6-2\alpha)}t^{5-2\alpha}$$

$$+ \frac{4\sigma_1}{\Gamma(6-3\alpha)}t^{5-3\alpha} + \frac{1}{\Gamma(6-4\alpha)}t^{5-4\alpha}\bigg).$$

5. Application of the FVIM [27]

From the partially coupled system (1), we can derive the following second-order fractional differential equation for the function u:

$$\begin{cases} D^{2\alpha}u + (\kappa^{\alpha} + \eta^{\alpha})D^{\alpha}u + (\kappa\eta)^{\alpha}u = 0, \\ u(0) = 0, \quad D^{\alpha}u(0) = v_0\kappa^{\alpha}, \end{cases}$$

where $D^{2\alpha}$ denotes the iterated derivative $D^{\alpha} \circ D^{\alpha}$. Clearly, this equation can be rewritten as

$$\begin{cases} D^{2\alpha}u = pD^{\alpha}u + qu, \\ u(0) = 0, \quad D^{\alpha}u(0) = v_0\kappa^{\alpha}, \end{cases} \tag{19}$$

where

$$p = -(\kappa^{\alpha} + \eta^{\alpha}) \text{ and } q = -(\kappa\eta)^{\alpha}.$$

Applying the fractional variational iteration method proposed in [27], we obtain the approximating sequence $(u_n)_n$ given by

$$u_n(t) = \frac{v_0\kappa^{\alpha}}{\Gamma(\alpha+1)}t^{\alpha} - L^{-1}\left[\frac{((\kappa^{\alpha} + \eta^{\alpha})s^{\alpha} + (\kappa\eta)^{\alpha})Lu_{n-1}}{s^{2\alpha}}\right] \; (n \geq 1), \tag{20}$$

with the initial data

$$u_0(t) = \frac{v_0 \kappa^{\alpha}}{\Gamma(\alpha + 1)} t^{\alpha}.$$

On the other hand, in order to obtain numerical approximations of the function v, it may be natural to consider the following single-term differential equation:

$$\begin{cases} D^{\alpha} v(t) = -\kappa^{\alpha} v(t), \\ u(0) = v_0. \end{cases}$$

However, we show that the FVIM proposed in [27] can also be applied in this case (and more generally to any first-order single-term fractional differential equation) with highly favorable results. Since a high-order fractional differential equation is necessary to apply such a method, we observe that the following second-order fractional differential equation can be derived from the system (1):

$$\begin{cases} D^{2\alpha} v(t) = \kappa^{2\alpha} v(t), \\ u(0) = v_0, \qquad D^{\alpha} v(0) = -\kappa^{\alpha} v_0. \end{cases} \tag{21}$$

Thus, the approximating sequence provided by the FVIM in [27] is given by

$$v_n(t) = v_0 \left(1 - \frac{\kappa^{\alpha} t^{\alpha}}{\Gamma(\alpha + 1)} \right) + \kappa^{2\alpha} L^{-1} \left[\frac{1}{s^{2\alpha}} L v_{n-1} \right] \quad (n \geq 1), \tag{22}$$

with the initial data

$$v_0(t) = v_0 \left(1 - \frac{\kappa^{\alpha} t^{\alpha}}{\Gamma(\alpha + 1)} \right).$$

Lemma 5.1. *Let* u_n *and* v_n, *with* $n \geq 1$, *be as in* (20) *and* (22) *respectively*, *then*

$$u_n(t) = v_0 \kappa^{\alpha} \sum_{j=0}^{n} \sum_{i=0}^{j} \binom{j}{i} \frac{(-1)^j (\kappa^{\alpha} + \eta^{\alpha})^{j-i} (\kappa\eta)^{i\alpha} t^{(i+j+1)\alpha}}{\Gamma((i + j + 1)\alpha + 1)} \tag{23}$$

and

$$v_n(t) = v_0 \sum_{i=0}^{2n+1} \frac{(-\kappa^\alpha t^\alpha)^i}{\Gamma(i\alpha + 1)}. \tag{24}$$

Proof. We argue by induction on n. For $n = 1$, from (20), (11) and (14) we have

$$u_1(t) = \frac{v_0 \kappa^\alpha}{\Gamma(\alpha+1)} t^\alpha - L^{-1}\left[\frac{((\kappa^\alpha + \eta^\alpha)s^\alpha + (\kappa\eta)^\alpha)Lu_0(s)}{s^{2\alpha}}\right]$$

$$= \frac{v_0 \kappa^\alpha}{\Gamma(\alpha+1)} t^\alpha - L^{-1}\left[\frac{((\kappa^\alpha + \eta^\alpha)s^\alpha + (\kappa\eta)^\alpha)L\left[\frac{v_0\kappa^\alpha}{\Gamma(\alpha+1)} t^\alpha\right]}{s^{2\alpha}}\right]$$

$$= \frac{v_0 \kappa^\alpha}{\Gamma(\alpha+1)} t^\alpha - L^{-1}\left[\frac{((\kappa^\alpha + \eta^\alpha)s^\alpha + (\kappa\eta)^\alpha)\frac{v_0\kappa^\alpha}{s^{\alpha+1}}}{s^{2\alpha}}\right]$$

$$= v_0\kappa^\alpha\left(\frac{t^\alpha}{\Gamma(\alpha+1)} - (\kappa^\alpha + \eta^\alpha)L^{-1}\left[\frac{1}{s^{2\alpha+1}}\right] - (\kappa\eta)^\alpha L^{-1}\left[\frac{1}{s^{3\alpha+1}}\right]\right)$$

$$= v_0\kappa^\alpha\left(\frac{t^\alpha}{\Gamma(\alpha+1)} - (\kappa^\alpha + \eta^\alpha)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - (\kappa\eta)^\alpha \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right)$$

$$= v_0\kappa^\alpha \sum_{j=0}^{1}\sum_{i=0}^{j}\binom{j}{i}\frac{(-1)^j(\kappa^\alpha + \eta^\alpha)^{j-i}(\kappa\eta)^{i\alpha}t^{(i+j+1)\alpha}}{\Gamma((i+j+1)\alpha+1)};$$

and, from (22), (11) and (14) it follows that

$$v_1(t) = v_0\left(1 - \frac{\kappa^\alpha t^\alpha}{\Gamma(\alpha+1)}\right) + \kappa^{2\alpha}L^{-1}\left[\frac{1}{s^{2\alpha}} Lv_0(s)\right]$$

$$= v_0\left(1 - \frac{\kappa^\alpha t^\alpha}{\Gamma(\alpha+1)}\right) + \kappa^{2\alpha}L^{-1}\left[\frac{1}{s^{2\alpha}} L\left[v_0\left(1 - \frac{\kappa^\alpha t^\alpha}{\Gamma(\alpha+1)}\right)\right]\right]$$

$$= v_0\left(1 - \frac{\kappa^{\alpha}t^{\alpha}}{\Gamma(\alpha+1)}\right) + \kappa^{2\alpha}L^{-1}\left[\frac{v_0}{s^{2\alpha}}\left(\frac{1}{s} - \frac{\kappa^{\alpha}}{s^{\alpha+1}}\right)\right]$$

$$= v_0\left(1 - \frac{\kappa^{\alpha}t^{\alpha}}{\Gamma(\alpha+1)} + \kappa^{2\alpha}\left(L^{-1}\left[\frac{1}{s^{2\alpha+1}}\right] - \kappa^{\alpha}L^{-1}\left[\frac{1}{s^{3\alpha+1}}\right]\right)\right)$$

$$= v_0\left(1 - \frac{\kappa^{\alpha}t^{\alpha}}{\Gamma(\alpha+1)} + \kappa^{2\alpha}\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{\kappa^{\alpha}t^{3\alpha}}{\Gamma(3\alpha+1)}\right)\right)$$

$$= v_0\sum_{i=0}^{3}\frac{(-\kappa^{\alpha}t^{\alpha})^i}{\Gamma(i\alpha+1)},$$

which proves the claim.

Now, suppose that (23) holds for $n \geq 1$. Then, thanks to (11),

$$Lu_n(s) = L\left[v_0\kappa^{\alpha}\sum_{j=0}^{n}\sum_{i=0}^{j}\binom{j}{i}\frac{(-1)^j(\kappa^{\alpha}+\eta^{\alpha})^{j-i}(\kappa\eta)^{i\alpha}t^{(i+j+1)\alpha}}{\Gamma((i+j+1)\alpha+1)}\right]$$

$$= v_0\kappa^{\alpha}\sum_{j=0}^{n}\sum_{i=0}^{j}\binom{j}{i}(-1)^j(\kappa^{\alpha}+\eta^{\alpha})^{j-i}(\kappa\eta)^{i\alpha}L\left[\frac{t^{(i+j+1)\alpha}}{\Gamma((i+j+1)\alpha+1)}\right]$$

$$= v_0\kappa^{\alpha}\sum_{j=0}^{n}\sum_{i=0}^{j}\binom{j}{i}(-1)^j(\kappa^{\alpha}+\eta^{\alpha})^{j-i}(\kappa\eta)^{i\alpha}\frac{1}{s^{(i+j+1)\alpha+1}}.$$

From (20), the newly derived identity and (14), it follows that

$$u_{n+1}(t) = \frac{v_0\kappa^{\alpha}}{\Gamma(\alpha+1)}t^{\alpha} - L^{-1}\left[\frac{((\kappa^{\alpha}+\eta^{\alpha})s^{\alpha}+(\kappa\eta)^{\alpha})Lu_n(s)}{s^{2\alpha}}\right]$$

$$= \frac{v_0\kappa^{\alpha}}{\Gamma(\alpha+1)}t^{\alpha} - v_0\kappa^{\alpha}L^{-1}\left[\sum_{j=0}^{n}\sum_{i=0}^{j}\binom{j}{i}\frac{(-1)^j(\kappa^{\alpha}+\eta^{\alpha})^{j-i+1}(\kappa\eta)^{i\alpha}}{s^{(i+j+2)\alpha+1}}\right.$$

$$\left. + \sum_{j=0}^{n}\sum_{i=0}^{j}\binom{j}{i}\frac{(-1)^j(\kappa^{\alpha}+\eta^{\alpha})^{j-i}(\kappa\eta)^{(i+1)\alpha}}{s^{(i+j+3)\alpha+1}}\right]$$

$$= v_0\kappa^\alpha\left(\frac{t^\alpha}{\Gamma(\alpha+1)} - \sum_{j=0}^{n}\sum_{i=0}^{j}\binom{j}{i}\frac{(-1)^j(\kappa^\alpha+\eta^\alpha)^{j-i+1}(\kappa\eta)^{i\alpha}t^{(i+j+2)\alpha}}{\Gamma((i+j+2)\alpha+1)}\right.$$

$$\left. - \sum_{j=0}^{n}\sum_{i=0}^{j}\binom{j}{i}\frac{(-1)^j(\kappa^\alpha+\eta^\alpha)^{j-i}(\kappa\eta)^{(i+1)\alpha}t^{(i+j+3)\alpha}}{\Gamma((i+j+3)\alpha+1)}\right)$$

$$= v_0\kappa^\alpha\left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \sum_{j=1}^{n+1}\sum_{i=0}^{j-1}\binom{j-1}{i}\frac{(-1)^j(\kappa^\alpha+\eta^\alpha)^{j-i}(\kappa\eta)^{i\alpha}t^{(i+j+1)\alpha}}{\Gamma((i+j+1)\alpha+1)}\right.$$

$$\left. + \sum_{j=1}^{n+1}\sum_{i=1}^{j}\binom{j-1}{i-1}\frac{(-1)^j(\kappa^\alpha+\eta^\alpha)^{j-i}(\kappa\eta)^{i\alpha}t^{(i+j+1)\alpha}}{\Gamma((i+j+1)\alpha+1)}\right)$$

$$= v_0\kappa^\alpha\left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \sum_{j=1}^{n+1}\left(\sum_{i=1}^{j-1}\left(\binom{j-1}{i}+\binom{j-1}{i-1}\right) \times \frac{(-1)^j(\kappa^\alpha+\eta^\alpha)^{j-i}(\kappa\eta)^{i\alpha}t^{(i+j+1)\alpha}}{\Gamma((i+j+1)\alpha+1)}\right.\right.$$

$$\left.\left. + \frac{(-1)^j(\kappa^\alpha+\eta^\alpha)^{j}t^{(j+1)\alpha}}{\Gamma((j+1)\alpha+1)} + \frac{(-1)^j(\kappa\eta)^{j\alpha}t^{(2j+1)\alpha}}{\Gamma((2j+1)\alpha+1)}\right)\right)$$

$$= v_0\kappa^\alpha\left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \sum_{j=1}^{n+1}\left(\sum_{i=1}^{j-1}\binom{j}{i}\frac{(-1)^j(\kappa^\alpha+\eta^\alpha)^{j-i}(\kappa\eta)^{i\alpha}t^{(i+j+1)\alpha}}{\Gamma((i+j+1)\alpha+1)}\right.\right.$$

$$\left.\left. + \frac{(-1)^j(\kappa^\alpha+\eta^\alpha)^{j}t^{(j+1)\alpha}}{\Gamma((j+1)\alpha+1)} + \frac{(-1)^j(\kappa\eta)^{j\alpha}t^{(2j+1)\alpha}}{\Gamma((2j+1)\alpha+1)}\right)\right)$$

$$= v_0\kappa^\alpha\left(\frac{t^\alpha}{\Gamma(\alpha+1)} + \sum_{j=1}^{n+1}\sum_{i=0}^{j}\binom{j}{i}\frac{(-1)^j(\kappa^\alpha+\eta^\alpha)^{j-i}(\kappa\eta)^{i\alpha}t^{(i+j+1)\alpha}}{\Gamma((i+j+1)\alpha+1)}\right)$$

$$= v_0\kappa^\alpha\sum_{j=0}^{n+1}\sum_{i=0}^{j}\binom{j}{i}\frac{(-1)^j(\kappa^\alpha+\eta^\alpha)^{j-i}(\kappa\eta)^{i\alpha}t^{(i+j+1)\alpha}}{\Gamma((i+j+1)\alpha+1)},$$

which completes the argument regarding u_n.

Finally, suppose that (24) holds for $n \geq 1$. From (22), (11) and (14), it follows that

$$v_{n+1}(t) = v_0\left(1 - \frac{\kappa^{\alpha}t^{\alpha}}{\Gamma(\alpha+1)}\right) + \kappa^{2\alpha}L^{-1}\left[\frac{1}{s^{2\alpha}}\,Lv_n(s)\right]$$

$$= v_0\left(1 - \frac{\kappa^{\alpha}t^{\alpha}}{\Gamma(\alpha+1)}\right) + \kappa^{2\alpha}L^{-1}\left[\frac{1}{s^{2\alpha}}\,L\left[v_0\sum_{i=0}^{2n+1}\frac{(-\kappa^{\alpha}t^{\alpha})^i}{\Gamma(i\alpha+1)}\right]\right]$$

$$= v_0\left(1 - \frac{\kappa^{\alpha}t^{\alpha}}{\Gamma(\alpha+1)}\right) + \kappa^{2\alpha}L^{-1}\left[\frac{1}{s^{2\alpha}}\,v_0\sum_{i=0}^{2n+1}(-\kappa^{\alpha})^i L\left[\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}\right]\right]$$

$$= v_0\left(1 - \frac{\kappa^{\alpha}t^{\alpha}}{\Gamma(\alpha+1)}\right) + \kappa^{2\alpha}L^{-1}\left[\frac{1}{s^{2\alpha}}\,v_0\sum_{i=0}^{2n+1}(-\kappa^{\alpha})^i\,\frac{1}{s^{i\alpha+1}}\right]$$

$$= v_0\left(1 - \frac{\kappa^{\alpha}t^{\alpha}}{\Gamma(\alpha+1)}\right) + v_0\sum_{i=0}^{2n+1}(-1)^i\kappa^{(i+2)\alpha}L^{-1}\left[\frac{1}{s^{(i+2)\alpha+1}}\right]$$

$$= v_0\left(1 - \frac{\kappa^{\alpha}t^{\alpha}}{\Gamma(\alpha+1)} + \sum_{i=0}^{2n+1}(-1)^i\kappa^{(i+2)\alpha}\,\frac{t^{(i+2)\alpha}}{\Gamma((i+2)\alpha+1)}\right)$$

$$= v_0\left(1 - \frac{\kappa^{\alpha}t^{\alpha}}{\Gamma(\alpha+1)} + \sum_{i=0}^{2n+1}\frac{(-\kappa^{\alpha}t^{\alpha})^{(i+2)}}{\Gamma((i+2)\alpha+1)}\right)$$

$$= v_0\left(\sum_{i=0}^{1}\frac{(-\kappa^{\alpha}t^{\alpha})^i}{\Gamma(i\alpha+1)} + \sum_{i=2}^{2(n+1)+1}\frac{(-\kappa^{\alpha}t^{\alpha})^i}{\Gamma(i\alpha+1)}\right) = v_0\sum_{i=0}^{2(n+1)+1}\frac{(-\kappa^{\alpha}t^{\alpha})^i}{\Gamma(i\alpha+1)},$$

and the proof is complete. □

6. Numerical Experiments

With the aim to compare our results with some of those obtained in [1] we will consider

$$\kappa = 0.02873, \quad \eta = 0.08442, \quad v_0 = 4, \quad \text{and} \quad t \in [0, 7].$$

For these values, from [27, Theorem 4.1], we know that u_n converges to the solution u of (19) for any $\alpha \geq 0.88632$ due that the condition

$$\frac{(\kappa^{\alpha} + \eta^{\alpha})7^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\kappa\eta)^{\alpha} 7^{2\alpha}}{\Gamma(2\alpha+1)} < 1$$

is satisfied. On the other hand, v_n converges to the solution v of (21) for any α since the requirement in [27, Theorem 4.1] is always verified.

Some of the α values within the interval [0.88632, 1) are 0.97 and 0.89; for these values, the approximation errors are tabulated in Tables 1 and 2, respectively. It can be observed that the results obtained by the FVIM are more accurate than those yielded by VIM [1] and VIM [24] in all cases, and that, according to [27, Theorem 4.1], the error improves as the number of iterations increases and as α approaches 1.

Table 1. Approximation errors for the FVIM, VIM [1] and VIM [24] for $\alpha = 0.97$

n	$\alpha = 0.97$					
	$\|u - u_n\|_\infty$	$\|u - \psi_n\|_\infty$	$\|u - \chi_n\|_\infty$	$\|v - v_n\|_\infty$	$\|v - \phi_n\|_\infty$	$\|v - \varphi_n\|_\infty$
2	2.0990e-02	7.1449e-02	9.5971e-02	6.6364e-07	6.6284e-03	1.4004e-02
3	3.8142e-03	1.2195e-02	1.8357e-02	6.3351e-10	3.7802e-04	1.0353e-03
4	5.7492e-04	1.6267e-03	2.4934e-03	3.8058e-13	1.7316e-05	4.7095e-05
5	7.4152e-05	1.7963e-04	2.4163e-04	null	6.6364e-07	7.2062e-07
6	8.3667e-06	1.6987e-05	– –	null	2.1882e-08	– –
7	8.3964e-07	1.4072e-06	– –	null	6.3351e-10	– –

Table 2. Approximation errors for the FVIM, VIM [1] and VIM [24] for $\alpha = 0.89$

n	$\alpha = 0.89$					
	$\|u - u_n\|_\infty$	$\|u - \psi_n\|_\infty$	$\|u - \chi_n\|_\infty$	$\|v - v_n\|_\infty$	$\|v - \phi_n\|_\infty$	$\|v - \varphi_n\|_\infty$
2	4.5438e-02	1.17671e-01	2.75256e-01	3.3643e-06	1.2729e-02	7.3797e-02
3	1.0826e-02	2.5248e-02	8.1245e-02	6.5858e-09	9.8529e-04	1.0465e-02
4	2.1697e-03	4.2964e-03	1.4022e-02	6.5858e-09	6.2456e-05	4.8578e-04
5	3.7654e-04	6.1335e-04	7.6709e-04	7.9936e-15	3.3643e-06	1.0588e-04
6	5.7760e-05	7.5888e-05	– –	null	1.5797e-07	– –
7	7.9533e-06	8.3130e-06	– –	null	6.5858e-09	– –

In Tables 3 and 4, we compile the error approximations obtained for $\alpha = 0.72$ and $\alpha = 0.62$, both values considered in [1] but outside the interval [0.88632, 1). In these cases, we observe that the FVIM remains the most efficient method for approximating v, while the VIM [1] performs slightly better than the FVIM for approximating u. The lower precision of the VIM [24] in all these cases is the main reason we have only considered the first five terms provided by this method. Nevertheless, it should be noted that the computational cost of obtaining each iteration with this method is higher than that of the other methods.

Table 3. Approximation errors for the FVIM, VIM [1] and VIM [24] for $\alpha = 0.72$

n	$\alpha = 0.72$					
	$\|u - u_n\|_\infty$	$\|u - \psi_n\|_\infty$	$\|u - \chi_n\|_\infty$	$\|v - v_n\|_\infty$	$\|v - \phi_n\|_\infty$	$\|v - \varphi_n\|_\infty$
2	2.177e-01	3.159e-01	1.4866	9.0844e-05	4.6935e-02	6.1106e-01
3	9.1415e-02	1.08192e-01	8.55439e-01	7.7956e-07	6.7892e-03	1.9295e-01
4	3.3241e-02	3.0269e-02	2.29284e-01	4.7707e-09	8.3796e-04	1.8086e-02
5	1.0729e-02	7.2970e-03	3.9745e-02	2.2218e-11	9.0844e-05	1.5215e-02
6	3.1276e-03	1.5621e-03	– –	8.2601e-14	8.8251e-06	– –
7	8.3422e-04	3.0253e-04	– –	4.4409e-16	7.7955e-07	– –

Table 4. Approximation errors for the FVIM, VIM [1] and VIM [24] for $\alpha = 0.62$

n	$\alpha = 0.62$					
	$\|u - u_n\|_\infty$	$\|u - \psi_n\|_\infty$	$\|u - \chi_n\|_\infty$	$\|v - v_n\|_\infty$	$\|v - \phi_n\|_\infty$	$\|v - \varphi_n\|_\infty$
2	5.1944e-01	5.3679e-01	3.3947	5.6313e-04	9.5226e-02	1.5580
3	3.0242e-01	2.3864e-01	2.6857	1.1120e-05	1.9538e-02	6.9515e-01
4	1.5517e-01	8.8148e-02	9.3978e-01	1.6453e-07	3.5049e-03	8.6431e-02
5	7.1717e-02	2.8468e-02	2.3613e-01	1.9274e-09	5.6313e-04	1.0027e-01
6	3.0307e-02	8.2782e-03	– –	1.8556e-11	8.2406e-05	– –
7	1.1851e-02	2.2015e-03	– –	1.5143e-13	1.1119e-05	– –

7. Conclusions

Some systems of first-order differential equations, such as the one used to model ethanol concentration in blood and stomach (cf. (1)), can be rewritten as a set of higher-order differential equations (cf. (19), (21)). Consequently, any method designed to handle these higher-order equations can be used to obtain numerical approximations for the original system. In this work, we explore this approach for the system (1) and the fractional variational iteration method proposed in [27]. We find that the FVIM is effective for obtaining numerical approximations of (1), particularly demonstrating notable performance in addressing the single-term differential equation involved (see the first line in (1)).

We compare the results obtained using the FVIM with those derived from the variational iteration methods VIM [1] and VIM [24]. In this comparison, the FVIM consistently yields lower approximation errors.

Although the FVIM is originally defined by a recurrence relation, we provide explicit expressions for the approximating terms that depend solely on the iteration order, which simplifies and streamlines its implementation. Furthermore, we enhance the implementation of VIM [1] by providing explicit expressions for the approximating terms that avoid the use of cross-terms.

References

[1] M. Adel, M. M. Khader, H. Ahmad and T. A. Assiri, Approximate analytical solutions for the blood ethanol concentration system and predator-prey equations by using variational iteration method, AIMS Mathematics 8(8) (2023), 19083-19096.

DOI: https://doi.org/10.3934/math.2023974

[2] A. A. M. Arafa and A. M. SH. Hagaga, A new analytic solution of fractional coupled Ramani equation, Chinese Journal of Physics 60 (2019), 388-406.

DOI: https://doi.org/10.1016/j.cjph.2019.05.011

[3] A. A. M. Arafa, M. Khalil and A. Sayed, A Non-Integer Variable Order Mathematical Model of Human Immunodeficiency Virus and Malaria Coinfection with Time Delay, Hindawi, Complexity (2019), 13 pp.

DOI: https://doi.org/10.1155/2019/4291017

[4] D. Bravo, M. Barrios and G. Reyero, Analysis of a fractional-order predator-prey model with harvest incorporating an Allee effect, Journal of Fractional Calculus and Applications 14(2) (2023), 14 pp.

DOI: https://doi.org/10.21608/jfca.2023.204624.1010

[5] M. Bohner, O. Tunç and C. Tunç, Qualitative analysis of caputo fractional integro-differential equations with constant delays, Computational and Applied Mathematics 40(6) (2021); Article 214.

DOI: https://doi.org/10.1007/s40314-021-01595-3

[6] M. A. Cohen, Numerical Methods for Laplace Transform Inversion, Numerical Methods and Algorithms, Springer, Boston, 2007.

DOI: https://doi.org/10.1007/978-0-387-68855-8

[7] K. Diethelm, J. Ford, N. Ford and M. Weilbeer, Pitfalls in fast numerical solvers for fractional differential equations, Journal of Computational and Applied Mathematics 186(2) (2006), 482-503.

DOI: https://doi.org/10.1016/j.cam.2005.03.023

[8] A. M. A. El-Sayed, A. A. M. Arafa, M. Khalil and A. Hassan, A mathematical model with memory for propagation of computer virus under human intervention, Progress in Fractional Differentiation and Applications 2(2) (2016), 105-113.

DOI: https://doi.org/10.18576/pfda/020203

[9] S. El-Wakil, E. Abulwafa, E. El-Shewy and A. Mahmoud, Ion-acoustic waves in unmagnetized collisionless weakly relativistic plasma of warm-ion and isothermal-electron using time-fractional KdV equation, Advances in Space Research 49(12) (2012), 1721-1727.

DOI: https://doi.org/10.1016/j.asr.2012.02.018

[10] V. Ertürk, Z. Odibat and S. Momani, An approximate solution of a fractional order differential equation model of human T-cell lymphotropic virus I (HTLV-I) infection of $CD4^{+}$ T-cells, Computers & Mathematics with Applications 62(3) (2011), 996-1002.

DOI: https://doi.org/10.1016/j.camwa.2011.03.091

[11] S. Esmaeili, M. Shamsi and Y. Luchko, Numerical solution of fractional differential equations with a collocation method based on Müntz polynomials, Computers & Mathematics with Applications 62(3) (2011), 918-929.

DOI: https://doi.org/10.1016/j.camwa.2011.04.023

[12] N. Ford and J. Connolly, Comparison of numerical methods for fractional differential equations, Communications on Pure and Applied Analysis 5(2) (2006), 289-307.

DOI: https://doi.org/10.3934/cpaa.2006.5.289

[13] F. Ghoreishi and S. Yazdani, An extension of the spectral Tau method for numerical solution of multi-order fractional differential equations with convergence analysis, Computers & Mathematics with Applications 61(1) (2011), 30-43.

DOI: https://doi.org/10.1016/j.camwa.2010.10.027

[14] R. Gorenflo and F. Mainardi, Fractional Calculus: Integral and Differential Equations of Fractional Order, Fractals and Fractional Calculus, Carpinteri and Mainardi, New York, 1997.

[15] H. Jafari, S. Das and H. Tajadodi, Solving a multi-order fractional differential equation using homotopy analysis method, Journal of King Saud University - Science 23(2) (2011), 151-155.

DOI: https://doi.org/10.1016/j.jksus.2010.06.023

[16] H. Jafari, H. Tajadodi and D. Baleanu, A modified variational iteration method for solving fractional Riccatti differential equation by Adomian polynomials, Fractional Calculus and Applied Analysis 16(1) (2013), 109-122.

DOI: https://doi.org/10.2478/s13540-013-0008-9

[17] M. Johansyah, A. Supriatna, E. Rusyaman and J. Saputra, Application of fractional differential equation in economic growth model: A systematic review approach, AIMS Mathematics 6(9) (2021), 10266-10280.

DOI: https://doi.org/10.3934/math.2021594

[18] Y. Khan, N. Faraz, A. Yildirim and Q. Wu, Fractional variational iteration method for fractional initial-boundary value problems arising in the application of nonlinear science, Computers & Mathematics with Applications 62(5) (2011), 2273-2278.

DOI: https://doi.org/10.1016/j.camwa.2011.07.014

[19] V. Krishnasamy, S. Mashayekhi and M. Razzaghi, Numerical solutions of fractional differential equations by using fractional Taylor basis, IEEE/CAA Journal of Automatica Sinica 4(1) (2017), 98-106.

DOI: https://doi.org/10.1109/JAS.2017.7510337

[20] C. Li and Y. Wang, Numerical algorithm based on Adomian decomposition for fractional differential equations, Computers & Mathematics with Applications 57(10) (2009), 1672-1681.

DOI: https://doi.org/10.1016/j.camwa.2009.03.079

[21] C. Li and F. Zeng, Finite difference methods for fractional differential equations, International Journal of Bifurcation and Chaos 22(4) (2012), 1230014.

DOI: https://doi.org/10.1142/S0218127412300145

[22] Y. Luchko and R. Gorenflo, An operational method for solving fractional differential equations with the Caputo derivatives, Acta Mathematica Vietnamica 24(2) (1999), 207-233.

[23] S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley and Sons, USA, 1993.

[24] S. Momani and Z. Odibat, Numerical approach to differential equations of fractional order, Journal of Computational and Applied Mathematics 207(1) (2007), 96-110.

DOI: https://doi.org/10.1016/j.cam.2006.07.015

[25] S. Momani and Z. Odibat, The variational iteration method: An efficient scheme for handling fractional partial differential equations in fluid mechanics, Computers & Mathematics with Applications 58(11-12) (2009), 2199-2208.

DOI: https://doi.org/10.1016/j.camwa.2009.03.009

[26] G. Monzón, On numerical approximation of second-order fractional differential equations in the frame of the Caputo fractional derivative, Journal of Fractional Calculus and Applications 14(1) (2023), 200-213.

[27] G. Monzón, Fractional variational iteration method for higher-order fractional differential equations, Journal of Fractional Calculus and Applications 15(1) (2024), 15 pp.

DOI: https://doi.org/10.21608/jfca.2023.229366.1027

[28] K. Oldham, Fractional differential equations in electrochemistry, Advances in Engineering Software 41(1) (2010), 9-12.

DOI: https://doi.org/10.1016/j.advengsoft.2008.12.012

[29] A. Prakash, M. Goyal and S. Gupta, Fractional variational iteration method for solving time-fractional Newell-Whitehead-Segel equation, Nonlinear Engineering 8(1) (2019), 164-171.

DOI: https://doi.org/10.1515/nleng-2018-0001

[30] S. Qureshi, A. Yusuf, A. A. Shaikh, M. Inc and D. Baleanu, Fractional modeling of blood ethanol concentration system with real data application, Chaos 29(1) (2019); Article 013143.

DOI: https://doi.org/10.1063/1.5082907

[31] A. Saadatmandi and M. Dehghan, A new operational matrix for solving fractional-order differential equations, Computers & Mathematics with Applications 59(3) (2010), 1326-1336.

DOI: https://doi.org/10.1016/j.camwa.2009.07.006

[32] B. Si-yuan, Fractional variational iteration method for fractional Cauchy problems, Mathematical Problems in Engineering 4 (2014), 1-5.

DOI: https://doi.org/10.1155/2014/925016

[33] O. Tunç and C. Tunç, Solution estimates to Caputo proportional fractional derivative delay integro-differential equations, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales: Serie A. Matemáticas 117 (2023), 12.

DOI: https://doi.org/10.1007/s13398-022-01345-y

[34] G.-C. Wu, A fractional variational iteration method for solving fractional nonlinear differential equations, Computers & Mathematics with Applications 61(8) (2011), 2186-2190.

DOI: https://doi.org/10.1016/j.camwa.2010.09.010

[35] G.-C. Wu and D. Baleanu, Variational iteration method for fractional calculus - A universal approach by Laplace transform, Advances in Difference Equations (2013); Article 18.

DOI: https://doi.org/10.1186/1687-1847-2013-18

[36] Z. U. A. Zafar, S. Zaib, M. T. Hussain, C. Tunç and S. Javeed, Analysis and numerical simulation of tuberculosis model using different fractional derivatives, Chaos, Solitons & Fractals 160 (2022), 12 pp; Article 112202.

DOI: https://doi.org/10.1016/j.chaos.2022.112202

■