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VARIATIONAL ITERATION TECHNIQUE BASED ON FOURTH KIND CHEBYSHEV POLYNOMIALS FOR SOLVING BOUNDARY VALUE PROBLEMS OF TENTH-ORDER

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Abstract

In this study, the Variational Iteration Technique (VIT) was applied, utilizing fourth kind Chebyshev polynomials, to compute numerical solutions for tenth-order boundary value problems. The novel approach involved creating fourth order Chebyshev polynomials specifically tailored for the given boundary value problems. These polynomials were then employed as basis functions for approximation within the VIT framework. The effectiveness and dependability of this method were demonstrated through numerical examples provided in the study. The calculations were performed using Maple 18 software, which provided the implementation and execution of the proposed approach.

1. Introduction

Consider a generalized boundary value problem of the form:

$$\eta_m \frac{d^m}{dz^m} v + \eta_{m-1} \frac{d^{m-1}}{dz^{m-1}} v + \eta_{m-2} \frac{d^{m-2}}{dz^{m-2}} v \ldots \eta_1 \frac{d}{dz} v + \eta_0 v = f(z), \quad a < z < b, \tag{1}$$

with boundary conditions

$$v(a) = A_1,\ v'(a) = A_2,\ v''(a) = A_3,\ \ldots,\ v^m(a) = A_i,\ v(b) = B_1,\ v'(b) = B_2,$$

$$v''(b) = B_3,\ v^m(b) = B_i, \tag{2}$$

where $\eta_m, \eta_{m-1}, \ldots, \eta_0$ are constants, $f(z)$ continuous on $[\alpha, \beta]$ and $A_j,\ j = 1, 2, 3, \ldots, m$ and $B_j,\ j = 1, 2, 3, \ldots, m$. This category of problems finds application in the mathematical modeling of various real-world scenarios such as viscoelastic flow and heat transfer, among others, within engineering sciences. Over the years, numerous numerical techniques have been developed to tackle such problems. Abdulla and Mohammad [1] introduced the fundamentals of the variational iteration method to address seventh-order boundary value problems. Pandey [14] employed quartic spline methods to solve third-order boundary value problems. Kasi and Sreenivasulu [6] utilized the Galerkin Method with Septic B-splines for tackling tenth-order boundary value problems. Additionally, Ali et al. [2] applied the reproducing kernel Hilbert space method to address tenth-order boundary value problems.

Iqbal et al. [4] employed both polynomial and non-polynomial cubic spline techniques to address linear tenth-order boundary value problems. Kasi and Reddy [7] utilized the Petrov-Galerkin method for solving ninth-order boundary value problems, while Reddy [15] applied the collocation method to tackle similar problems of the same order. Ghazala and Maasoomah [3] utilized the homotopy analysis method to solve ninth-order boundary value problems. Noor and Mohyud-Din [11] utilized the variational iteration decomposition method for solving fifth-order boundary value problems. In their subsequent work, Noor and Mohyud-Din [13] combined the homotopy perturbation method and the variational iteration method (VIM) to address fifth-order boundary value problems. Furthermore, Mohyud-Din and Yildirim [9] applied a modified VIM to solve ninth and tenth-order boundary value problems. Additionally, Noor and Mohyud-Din [12] developed and utilized the adomian decomposition method along with the VIM to solve fifth-order and other higher-order boundary value problems. In Shahid and Iftikhar's [16] research, the VIM employing He's polynomials was utilized to address seventh-order boundary value problems. Recently, Njoseh and Mamadu [10] introduced a generalized approach named the Power Series Approximation Method (PSAM) for tackling similar problems. Additionally, Mamadu and Njoseh [8] extensively employed the method of tau and tau-collocation approximation for solving first and second-order ordinary differential equations. Similarly, Islam et al. [5] applied the Differential Transform Method (DTM) to successfully handle a twelfth-order boundary value problem. In this research, we adopt the VIT utilizing Chebyshev polynomials of the fourth kind to address a tenth-order boundary value problem. Our proposed method corrects the correction functional for the boundary value problem (BVP), and optimally computes the Lagrange multiplier using variational theory. The results obtained thus far demonstrate the effectiveness and consistency of our proposed approach. Finally, we present the solution in an infinite series format, typically yielding accurate results.

2. The Standard Variational Iteration Technique

Consider the following general differential equation to demonstrate the basic concept of the technique:

$$Lv + N_l v - g(z) = 0, \tag{3}$$

where L stands for a linear operator, N_l stands for a nonlinear operator, and $g(z)$ stands for an inhomogeneous term. We can construct a correction functional using the variational iteration method as follows:

$$v_{i+1} = v_i(z) + \int_0^z \lambda(t)\left(Lv_i(t) + N_l \widetilde{v_l(t)} - g(t)\right)dt, \tag{4}$$

where $\lambda(t)$ is a Lagrange multiplier that can be optimally identified using a variational iteration technique. The nth approximation is denoted by the subscripts n. $\widetilde{v_l}$ is classified as a restricted variation, i.e., $\widetilde{v_l} = 0$. The relation (4) is referred to as a correction functional. Because of the exact identification of the Lagrange multiplier, linear problems can be solved in a single iteration step. In this method, we need to determine the Langrange multiplier $\lambda(t)$ optimally, and thus the successive approximation of solution v will be easily obtained by employing the Langrange multiplier and our v_0, and the solution is given by

$$\lim_{i - \infty} v_i = v. \tag{5}$$

The Lagrange Multiplier also plays an important role in determining the solution of the problem, and can be defined as follows:

$$\lambda(t) = (-1)^m \frac{1}{(m-1)!}(t - z)^{m-1}. \tag{6}$$

2.1. Chebyshev polynomials of the fourth kind

The Chebyshev polynomials of the fourth kind are orthogonal polynomials with respect to the weight function $W(z) = \sqrt{\frac{1-z}{1+z}}, \ \forall z \in [-1, 1]$. The Chebyshev polynomials of the fourth kind is defined by $W_m(z) = \frac{\sin\left(m + \frac{1}{2}\right)\theta}{\sin\left(\frac{\theta}{2}\right)}$ with $W_0(z) = 1$ and $W_1(z) = 2z + 1$.

Hence, the first five Chebyshev polynomials of the fourth kind is given below:

$$\left.\begin{aligned} W_0(z) &= 1 \\ W_1(z) &= 2z + 1 \\ W_2(z) &= 4z^2 + 2z - 1 \\ W_3(z) &= 8z^3 + 4z^2 - 4z - 1 \\ W_4(z) &= 16z^4 + 8z^3 - 12z^2 - 4z + 1 \\ W_5(z) &= 32z^5 + 16z^4 - 32z^3 - 12z^2 + 6z + 1 \end{aligned}\right\}. \tag{7}$$

2.1.1. Shifted Chebyshev polynomials of the fourth kind

The shifted Chebyshev polynomials of the fourth kind are orthogonal polynomials with respect to the weight function $W^*(z) = \sqrt{\frac{1-z}{z}}, \ \forall z \in [0, 1]$ with starting values $W_0^*(z) = 1$ and $W_1^*(z) = 4z - 1$.

Hence, the first five shifted Chebyshev polynomials of the fourth kind is given below:

$$\left.\begin{aligned} W_0^*(z) &= 1 \\ W_1^*(z) &= 4z - 1 \\ W_2^*(z) &= 16z^2 - 12z + 1 \\ W_3^*(z) &= 64z^3 - 80z^2 + 24z - 1 \\ W_4^*(z) &= 256z^4 - 448z^3 + 2402z^2 - 40z + 1 \\ W_5^*(z) &= 1024z^5 - 2304z^4 + 1792z^3 - 560z^2 + 60z - 1 \end{aligned}\right\}. \tag{8}$$

2.2. Modified variational iteration technique using Chebyshev and shifted Chebyshev polynomials of the fourth kind (MVITCS-CP)

Using (3) and (4), we assume an approximate solution of the form

$$v_{i,N-1}(z) = \sum_{i=0}^{N-1} \eta_{i,N-1} W_{i,N-1}(z), \tag{9}$$

$$v^*_{i,N-1}(z) = \sum_{i=0}^{N-1} \eta_{i,N-1} W^*_{i,N-1}(z), \tag{10}$$

where $W_{i,N-1}(z)$ and $W^*_{i,N-1}(z)$ are fourth-order Chebyshev polynomials and shifted fourth-order Chebyshev polynomials, respectively, $\eta_{i,N-1}$ are unknown constants, and N is the degree of approximation. As a result, we get the iterative method shown below:

$$\begin{aligned} v_{i+1,N-1}(z) = {} & \sum_{i=0}^{N-1} \eta_{i,N-1} P_{i,N-1}(x) \\ & + \int_0^z \lambda(t)(L\sum_{i=0}^{N-1} \eta_{i,N-1} P_{i,N-1}(z) \\ & + N_l \sum_{i=0}^{N-1} \eta_{i,N-1} P_{i,N-1}(z))dt, \end{aligned} \tag{11}$$

$$\begin{aligned} v^*{}_{i+1,N-1}(z) = {} & \sum_{i=0}^{N-1} \eta_{i,N-1} W^*{}_{i,N-1}(x) \\ & + \int_0^z \lambda(t)(L\sum_{i=0}^{N-1} \eta_{i,N-1} W^*{}_{i,N-1}(z) \\ & + N_l \sum_{i=0}^{N-1} \eta_{i,N-1} W^*{}_{i,N-1}(z))dt. \end{aligned} \tag{12}$$

3. Numerical Applications

In this section, we solved four examples using the proposed method. The numerical results also demonstrate the proposed scheme's accuracy and efficiency.

Example 1 (Kasi and Sreenivasulu [6]). Considers the following tenth order boundary value problem:

$$v^{(10)} + v = -10(2z \sin z - 9 \cos z), \quad -1 \leq z \leq 1 \tag{13}$$

$$v(-1) = v(1) = 0,\ v'(1) = -v'(-1) = 2\cos 1,\ v''(-1) = v'''(1) = 2\cos 1 - 4\sin 1$$

$$v'''(-1) = -v'''(1) = 6\cos 1 + 6\sin 1,\ v^{(4)}(-1) = v^{(4)}(1) = -12\cos 1 + 8\sin 1. \tag{14}$$

The exact solution for the problem is

$$v = (z^2 - 1)\cos z. \tag{15}$$

The correction functional for the boundary value problem (13) and (14) is given as

$$v_{i+1} = v_i(z) + \int_0^z \lambda(t)\left(v^{(10)} + v + 10(2t \sin t - 9 \cos t)\right)dt, \tag{16}$$

where $\lambda(t) = \dfrac{(-1)^{10}(t-z)^9}{9!}$ is the Lagrange multiplier.

Applying the modified variational iteration technique using the Chebyshev polynomials of the fourth kind, we assume an approximate solution of the form

$$v_{n,9}(z) = \sum_{i=0}^{9} a_{i,9} W_{i,9}(z). \tag{17}$$

Hence, we get the following iterative formula:

$$v_{n+1,N-1}(z) = \sum_{i=0}^{9} a_{i,9}W_{i,9}(z) + \int_0^z \frac{(t-z)^9}{9!}\left(\frac{d^{10}}{dt^{10}}\left(\sum_{i=0}^{9} a_{i,9}W_{i,9}(t)\right) + \sum_{i=0}^{9} a_{i,9}W_{i,9}(t) + 10(2t\sin t - 9\cos t)\right)dt \quad (18)$$

$$v_{n+1,N-1}(z) = a_{0,9}W_{0,9}(z) + a_{1,9}W_{1,9}(z) + a_{2,9}W_{2,9}(z) + a_{3,9}W_{3,9}(z) + a_{4,9}W_{4,9}(z) + a_{5,9}W_{5,9}(z) + a_{6,9}W_{6,9}(z) + a_{7,9}W_{7,9}(z) + a_{8,9}W_{8,9}(z) + a_{9,9}W_{9,9}(z) + \int_0^z \frac{(t-z)^9}{9!}\left(\frac{d^{10}}{dt^{10}}\left(\sum_{i=0}^{9} a_{i,9}W_{i,9}(t)\right) + \sum_{i=0}^{9} a_{i,9}W_{i,9}(t) + 10(2t\sin t - 9\cos t)\right)dt. \quad (19)$$

Using Equation (7) and application of the boundary conditions (14), the values of the unknown constants can be determined as follows:

$$a_{0,9} = -0.4399013822, \quad a_{1,9} = -0.249494616,$$

$$a_{2,9} = 0.2494687761, \quad a_{3,9} = 0.02991538274,$$

$$a_{4,9} = -0.02993609110, \quad a_{5,9} = -0.000638919406,$$

$$a_{6,9} = 0.000633743317, \quad a_{7,9} = 0.000055222284,$$

$$a_{8,9} = 0.0000055222284, \quad a_{9,9} = 0.00000000.$$

Consequently, the series solution is given as

$$v(z) = -0.9999999999 + 4.795096536 10^{-10} z^{18} - 1.151854037 10^{-11} z^{16} - 1.10^{-21} z^{15} + 2.099146446 10^{-9} z^{14} - 2.776608679 10^{-7} z^{12} + 0.00002507716049 z^{10} - 0.00141369047 0z^{8} + 0.0430555555 3z^{6} - 0.5416666666 z^{4} + 1.50000000 z^{2} - \frac{1}{121645100408832000} z^{20} + 2.10^{-12} z^{5} - 5.00 10^{-11} z^{3} - 1.340 10^{-10} z. \quad (20)$$

Example 2 (Kasi and Sreenivasulu [6]). Considers the following tenth-order boundary value problem:

$$v^{(10)} - (z^2 - 2z)v = 10\cos z - (z-1)^3 \sin z, \quad -1 \le z \le 1 \tag{21}$$

with boundary conditions

$$v(-1) = 2\sin 1,\ v(1) = 0, v'(-1) = -2\cos 1 - \sin 1,\ v'(1) = \sin 1,$$

$$v''(-1) = 2\cos 1 - 2\sin 1,\ v''(1) = 2\cos 1,\ v'''(-1) = 2\cos 1 + 3\sin 1,$$

$$v'''(1) = -\ 3\sin 1,\ v^{(4)}(-1) = -4\cos 1 + 2\sin,\ v^{(4)}(1) = -\ 4\cos 1. \tag{22}$$

The exact solution for the problem is

$$v(z) = (z-1)\sin z. \tag{23}$$

The correction functional for the boundary value problem (21) and (22) is given as

$$v_{i+1} = v_i(x) + \int_0^z \lambda(t)(v^{(10)} - (t^2 - 2t)v - 10\cos t + (t-1)^3 \sin t)dt, \tag{24}$$

where $\lambda(t) = \dfrac{(-1)^{10}(t-z)^9}{9!}$ is the Lagrange multiplier.

Applying the modified variational iteration technique using the Chebyshev polynomials of the fourth kind, we assume an approximate solution of the form

$$v_{n,9}(z) = \sum_{i=0}^{9} a_{i,9} W_{i,9}(z). \tag{25}$$

Hence, we get the following iterative formula:

$$v_{n+1,N-1}(z) = \sum_{i=0}^{9} a_{i,9} W_{i,9}(z) + \int_0^z \frac{(t-z)^9}{9!}\left(\frac{d^{10}}{dt^{10}}\left(\sum_{i=0}^{9} a_{i,9} W_{i,9}(t)\right)\right.$$

$$\left. - (t^2 - 2t)\sum_{i=0}^{9} a_{i,9} W_{i,9}(t) - 10\cos t + (t-1)^3 \sin t\right)dt, \tag{26}$$

$$v_{n+1,N-1}(z) = a_{0,9}W_{0,9}(z) + a_{1,9}W_{1,9}(z) + a_{2,9}W_{2,9}(z) + a_{3,9}W_{3,9}(z)$$
$$+ a_{4,9}W_{4,9}(z) + a_{5,9}W_{5,9}(z) + a_{6,9}W_{6,9}(z) + a_{7,9}W_{7,9}(z)$$
$$+a_{8,9}W_{8,9}(z)+a_{9,9}W_{9,9}(z)+\int_0^z \frac{(t-z)^9}{9!}\left(\frac{d^{10}}{dt^{10}}(\sum_{i=0}^{9} a_{i,9}W_{i,9}(t))\right.$$
$$\left. - (t^2 - 2t)\sum_{i=0}^{9} a_{i,9}W_{i,9}(t) - 10\cos t + (t-1)^3 \sin t\right)dt. \quad (27)$$

Using Equation (7) and application of the boundary conditions (22), the values of the unknown constants can be determined as follows

$$a_{0,9} = 0.8801362005, \quad a_{1,9} = -0.6503257357,$$
$$a_{2,9} = 0.190701144, \quad a_{3,9} = 0.02920552360,$$
$$a_{4,9} = -0.009400920074, \quad a_{5,9} = -0.00037659810,$$
$$a_{6,9} = 0.000123222795, \quad a_{7,9} = 0.00002281755,$$
$$a_{8,9} = -7.69667210^{-7}, \quad a_{9,9} = -5.382310^{-9}.$$

Consequently, the series solution is given as

$$v(z) = -2.50521083810^{-8}z^{12} - 1.6059043810^{-10}z^{13} + 1.60590438610^{-10}z^{14}$$
$$+ 7.647163210^{-13}z^{15} - 7.6471636710^{-13}z^{16} - 2.811458510^{-10}z^{17}$$
$$- \frac{1}{851515702861824000}z^{22} + \frac{1}{362880}z^{10} + 2.50521084110^{-8}z^{11}$$
$$+ 2.8114555610^{-15}z^{18} - 4.110261310^{-18}z^{20} + 8.22089710^{-18}z^{19}$$
$$+4.30604259810^{-18}z^{21} - 0.0000027553760 0z^{9} - 0.0001984126720z^{8}$$
$$+ 0.0001984128376z^{7} + 0.008333333344z^{6} - 0.008333333405z^{5}$$
$$- 0.166666666z^{4} + 0.1666666662z^{3} + 0.9999999997z^{2}$$
$$- 0.9999999985z - 5.7350000010^{-10}. \quad (28)$$

Example 3 (Kasi and Sreenivasulu [6])**.** Considers the following tenth order boundary value problem:

$$v^{(10)} - v'' + vz = (-8 + z - z^2)e^z, \quad 0 \le z \le 1 \tag{29}$$

with boundary conditions

$$v(0) = 1, \quad v(1) = 0, \quad v'(0) = 0, \quad v'(1) = -e, \quad v''(0) = -1, \quad v''(1) = -2e,$$

$$v'''(0) = -2, \quad v'''(1) = -3e, \quad v^{(4)}(0) = -3, \quad v^{(4)}(1) = -4e. \tag{30}$$

The exact solution of the example is

$$v(z) = (1 - z)e^z. \tag{31}$$

The correction functional for the boundary value problem (29) and (30) is given as

$$v_{i+1} = v_i(x) + \int_0^z \lambda(t)(v^{(10)} - v'' + vt - (-8 + t - t^2)e^t)dt, \tag{32}$$

where $\lambda(t) = \frac{(-1)^{10}(t - z)^9}{9!}$ is the Lagrange multiplier.

Applying the modified variational iteration technique using the shifted Chebyshev polynomials of the fourth kind, we assume an approximate solution of the form

$$v^*_{n,p}(z) = \sum_{i=0}^{9} a_{i,9}W^*_{i,9}(z). \tag{33}$$

Hence, we get the following iterative formula:

$$v^*_{i+1,N-1}(z) = \sum_{i=0}^{9} a_{i,9}W^*_{i,9}(z) + \int_0^z \frac{(t - z)^9}{9!}$$

$$\times \left(\frac{d^{10}}{dt^{10}}\left(\sum_{i=0}^{9} a_{i,9}W^*_{i,9}(t)\right) - \frac{d^2}{dt^2}\left(\sum_{i=0}^{9} a_{i,9}W^*_{i,9}(t)\right) \right.$$

$$\left. + t\left(\sum_{i=0}^{9} a_{i,9}W^*_{i,9}(t)\right) - (-8 + t - t^2)e^t \right)dt, \tag{34}$$

$$v^*_{i+1, N-1}(z) = a_{0,9}W^*_{0,9}(z) + a_{1,9}W^*_{1,9}(z) + a_{2,9}W^*_{2,9}(z) + a_{3,9}W^*_{3,9}(z)$$

$$+ a_{4,9}W^*_{4,9}(z) + a_{5,9}W^*_{5,9}(z) + a_{6,9}W^*_{6,9}(z) + a_{7,9}W^*_{7,9}(z)$$

$$+ a_{8,9}W^*_{8,9}(z) + a_{9,9}W^*_{9,9}(z) + \int_0^z \frac{(t-z)^9}{9!}$$

$$\times \left(\frac{d^{10}}{dt^{10}} \left(\sum_{i=0}^{9} a_{i,9}W^*_{i,9}(t)\right) - \frac{d^2}{dt^2} \left(\sum_{i=0}^{9} a_{i,9}W^*_{i,9}(t)\right) \right.$$

$$\left. + t\left(\sum_{i=0}^{9} a_{i,9}W^*_{i,9}(t)\right) - (-8 + t - t^2)e^t \right) dt. \qquad (35)$$

Using Equation (8) and application of the boundary conditions (30), the values of the unknown constants can be determined as follows:

$a_{0,9} = 0.9029960441$, $a_{1,9} = -0.1578129315$, $a_{2,9} = -0.07004843010$,

$a_{3,9} = -0.01008059536$, $a_{4,9} = -0.00089644024$, $a_{5,9} = -0.00005815092$,

$a_{6,9} = -0.000002971276$, $a_{7,9} = -1.2375010^{-7}$, $a_{8,9} = -0.078810^{-9}$,

$a_{9,9} = -8.4210^{-9}$.

Consequently, the series solution is given as

$$v^*(z) = 5.175685837\ 10^{-16}z^{19} - 1.2483912810^{-15}z^{18} - 2.2520910110^{-14}z^{17}$$

$$-7.1670579710^{-13}z^{16} - 1.07067296610^{-11}z^{15} - 1.49118504910^{-10}z^{14}$$

$$-1.927086204110^{-8}z^{13} - 2.29644329610^{-8}z^{12} - 2.29644329610^{-8}z^{12}$$

$$-2.50521082110^{-7}z^{11} + 4.30604259810^{-18}z^{21} - 0.000002480158732z^{10}$$

$$-0.001190658048z^7 + 0.006944300241z^6 + 0.033333382 56z^5$$

$$-0.1250000052z^4 - 0.3333333216z^3 - 0.5000000049z^2 + 3.752000010^{-10}z$$

$$+ 3.29223198210^{-17}z^{20} - 0.00002207252480z^9 - 0.000173500065z^8$$

$$+ 1.000000000. \qquad (36)$$

Example 4 (Ali et al. [2] and Iqbal et al. [4]). Considers the following tenth order boundary value problem:

$$v^{(10)} = -(80 + 19z + z^2)e^z, \quad 0 \le z \le 1 \tag{37}$$

with boundary conditions

$$v(0) = 0, \quad v(1) = 0, \quad v''(0) = 0, \quad v''(1) = -4e, \quad v^{(4)}(0) = -8,$$

$$v^{(4)}(1) = -16e, \quad v^{(6)}(0) = -24, \quad v^{(6)}(1) = -36e, \quad v^{(8)}(0) = -48,$$

$$v^{(4)}(1) = -64e. \tag{38}$$

The exact solution of the example is

$$v(z) = z(1-z)e^z. \tag{39}$$

The correct functional for the boundary value problem (37) and (38) is given as

$$v_{i+1} = v_i(x) + \int_0^z \lambda(t)(v^{(10)} + (80 + 19t + t^2)e^t)dt, \tag{40}$$

where $\lambda(t) = \dfrac{(-1)^{10}(t-z)^9}{9!}$ is the Lagrange multiplier.

Applying the modified variational iteration technique using the shifted Chebyshev polynomials of the fourth kind, we assume an approximate solution of the form

$$v^*{}_{n,9} = \sum_{i=0}^{9} a_{i,9} W^*{}_{i,9}(z). \tag{41}$$

Hence, we get the following iterative formula:

$$v^*{}_{i+1,N-1}(z) = \sum_{i=0}^{9} a_{i,9} W^*{}_{i,9}(z) + \int_0^z \frac{(t-z)^9}{9!}$$

$$\times \left(\frac{d^{10}}{dt^{10}} (\sum_{i=0}^{9} a_{i,9} W^*{}_{i,9}(t)) + (80 + 19t - t^2)e^t \right) dt, \tag{42}$$

$$v^*_{i+1,\,N-1}(z) = a_{0,\,9}W^*{}_{0,\,9}(z) + a_{1,\,9}W^*{}_{1,\,9}(z) + a_{2,\,9}W^*{}_{2,\,9}(z)$$

$$+\, a_{3,9}W^*{}_{3,9}(z) + a_{4,9}W^*{}_{4,9}(z) + a_{5,9}W^*{}_{5,9}(z) + a_{6,9}W^*{}_{6,9}(z)$$

$$+\, a_{7,\,9}W^*{}_{7,\,9}(z) + a_{8,\,9}W^*{}_{8,\,9}(z) + a_{9,\,9}W^*{}_{9,\,9}(z)$$

$$+\int_0^z \frac{(t-z)^9}{9!}\left(\frac{d^{10}}{dt^{10}}\left(\sum_{i=0}^{9} a_{i,9}W^*{}_{i,9}(t)\right) + (80 + 19t + t^2)e^t\right)dt. \tag{43}$$

Using Equation (8) and application of the boundary conditions (38), the values of the unknown constants can be determined as follows:

$a_{0,\,9} = 0.862961228,$ $a_{1,\,9} = 0.1293313161,$ $a_{2,\,9} = -\,0.07699657358,$

$a_{3,9} = -\,0.02277570009,$ $a_{4,9} = -\,0.002982417993,$ $a_{5,9} = -\,0.0002537042443,$

$a_{6,\,9} = -\,0.00001597640481,$ $a_{7,\,9} = -\,7.85740449810^{-7},$

$a_{8,\,9} = -\,2.94239326210^{-7},$ $a_{9,\,9} = -\,6.6227691410^{-10}.$

Consequently, the series solution is given as

$$v^*(z) = -\frac{1}{11432810188000}z^{18} - \frac{1}{2032499589120}z^{17} - \frac{1}{93405312000}$$

$$z^{16} - \frac{1}{6406022400}z^{15} - \frac{1}{518918400}z^{14} - \frac{1}{43545600}z^{13} - \frac{1}{3991680}z^{12}$$

$$-\frac{1}{403200}z^{11} - \frac{1}{45360}z^{10} - 0.0001736119193z^9 - 0.001190476191z^8$$

$$-\,0.006944435244z^7 - 0.03333333334z^6 - 0.1250000441z^5$$

$$-\,0.3333333331z^4 - 0.4999999078z^3 - 0.9999999432z. \tag{44}$$

4. Tables

Table 1 (Error estimates)**:** The result of the proposed method compared Galerkin method with septic B-splines (Kasi and Sreenivasulu [6])

z	**Exact solution**	**Approximate solution**	**Absolute error by the proposed method**	**GMSB-s Error**
– 0.8	– 0.2508144153	– 0.2508144149	4.000000e-10	5.483627e-06
– 0.6	– 0.5282147935	– 0.5282147939	4.000000e-10	9.536743e-07
– 0.4	– 0.7736912350	– 0.7736912349	1.000000e-10	8.702278e-06
– 0.2	– 0.9408639147	– 0.9408639150	3.000000e-10	2.980232e-07
0.0	– 1.0000000000	– 0.9999999999	1.000000e-10	1.955032e-05
0.2	– 0.9408639147	– 0.9408639150	3.000000e-10	2.920628e-05
0.4	– 0.7736912350	– 0.7736912351	1.000000e-10	2.169609e-05
0.6	– 0.5282147935	– 0.5282147941	6.000000e-10	7.390976e-06
0.8	– 0.2508144153	– 0.2508144151	2.000000e-10	7.450581e-07

Table 2 (Error estimates)**:** The result of the proposed method compared Galerkin method with septic B-splines (Kasi and Sreenivasulu [6])

z	**Exact solution**	**Approximate solution**	**Absolute error by the proposed method**	**GMSB-s Error**
– 0.8	1.2912409640	1.291240961	3.000000e-09	4.649162e-06
– 0.6	0.9034279574	0.9034279559	1.500000e-09	1.329184e-05
– 0.4	0.5451856792	0.5451856781	1.100000e-09	2.050400e-05
– 0.2	0.2384031970	0.2384031960	1.000000e-09	9.477139e-06
0.0	0.0000000000	– 5.735000e-10	5.735000e-10	2.731677e-06
0.2	– 0.1589354646	– 0.1589354650	4.000000e-10	1.458824e-05
0.4	– 0.2336510054	– 0.2336510054	0.000000e-00	2.110004e-05
0.6	– 0.2258569894	– 0.2258569893	1.000000e-10	1.908839e-05
0.8	– 0.1434712182	– 0.1434712180	2.000000e-10	1.342595e-05

Table 3 (Error estimates)**:** The result of the proposed method compared Galerkin method with septic B-splines (Kasi and Sreenivasulu [6])

z	Exact solution	Approximate solution	Absolute error by the proposed method	GMSB-s Error
0.1	0.9946538262	0.9946538263	1.000000e-10	1.537800e-05
0.2	0.9771222064	0.9771222065	1.000000e-10	4.452467e-05
0.3	0.9449011656	0.9449011652	4.000000e-10	3.331900e-05
0.4	0.8950948188	0.8950948185	3.000000e-10	3.552437e-05
0.5	0.8243606355	0.8243606352	3.000000e-10	9.477139e-06
0.6	0.7288475200	0.7288475199	1.000000e-10	2.586842e-05
0.7	0.6041258121	0.6041258120	1.000000e-10	3.975630e-05
0.8	0.4451081856	0.4451081853	3.000000e-10	3.531575e-05
0.9	0.2459603111	0.2459603107	4.000000e-10	2.214313e-05

Table 4 (Error estimates)**:** The result of the proposed method compared with reproducing kernel Hilbert space method by (Ali et al. [2], & Iqbal et al. [4])

z	Exact solution	Absolute error by the proposed method	AE (RKHSM) Ali et al. [2]	AE (NPCSM) Iqbal et al. [4]	AE (PCSM) Iqbal et al. [4]
0.2	0.195424441	1.070000e-08	3.330000e-08	2.433000e-07	3.982000e-04
0.4	0.358037927	1.730000e-08	7.031000e-08	3.986000e-07	6.663000e-04
0.6	0.358037927	1.730000e-08	6.076000e-08	4.428000e-07	7.598000e-04
0.8	0.356086549	1.060000e-08	2.682000e-08	3.328000e-07	5.885000e-04

5. Conclusion

The modified variational iteration technique with Chebyshev and shifted Chebyshev polynomials of the fourth kind was successfully applied in this paper to obtain numerical solutions to tenth order boundary value problems. Chebyshev polynomials of the fourth kind and shifted Chebyshev polynomials of the fourth kind are combined with the Variational iteration method in the modification. The method produces rapidly converging series solutions, which are occur in physical problems. Tables 1, 2, 3 and 4 show that when compared to methods in the literature, the proposed method produces a better result. Finally, the numerical results demonstrated that the present method is a powerful mathematical tool for solving the class of problems under consideration.

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