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A GENERALIZATION OF THE DU INTEGRAL INEQUALITY

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Abstract

This paper is devoted to a generalization of an integral-type inequality established by Wei-Shih Du in 2023. The generalization is motivated by the inclusion of two adaptable mathematical quantities: a parameter and a probability density function. An application to the Laplace transform is discussed.

1. Introduction

Inequality theory is the basis for many theoretical and practical advances in various scientific disciplines. It aims to provide simple tools for evaluating the behaviour of complex mathematical quantities through the use of bounds. The most classical inequalities can be found in [1, 3, 9, 10]. By generalizing these classical inequalities, we can facilitate the

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development of more sophisticated analytical techniques. This is still a hot topic and important developments have been made in this direction. We can refer to the papers in the recent special issue entitled "Current Research on Mathematical Inequalities" in [2].

In this paper, we focus on a simple inequality that emerged from an international competition called "IMC 2022", which will take place in Blagoevgrad, Bulgaria, from 1 to 7 August 2022. More information can be found in [7]. This inequality is formulated in the theorem below.

Theorem 1.1. *Let* $f : [0, 1] \to [0, +\infty)$ *be an integrable function such that, for any* $x \in [0, 1]$, $f(x)f(1-x) = 1$. *Then we have*

$$\int_0^1 f(x)dx \geq 1.$$

Some generalizations of this theorem can be found in [6, 4]. In particular, the theorem below is proved in [4, Theorem 2.5].

Theorem 1.2. *Let* $(a, b) \in \mathbb{R}^2$ *such that* $a < b$ *and* $f : [a, b] \to \mathbb{R}$ *be an integrable function. Suppose that there exists an integrable function* $g : [a, b] \to \mathbb{R}$ *such that*

$$\int_a^b |g(x)|dx \leq \int_a^b |f(x)|dx,$$

and

$$\inf_{x \in [a, b]} |f(x)g(x)| \geq c,$$

where c denotes a strictly positive constant. Then we have

$$\int_a^b |f(x)|dx \geq (b-a)\sqrt{c}.$$

The proof is derived from the arithmetic mean-geometric mean inequality and a judicious use of the assumptions made.

In this paper, we propose a generalization of Theorem 1.2. It is obtained by using a generalized version of the Hölder inequality, giving alternative techniques to those used in the proof in [4, Theorem 2.5]. The generalization is characterized by the addition of an adjustable parameter and a special function. This special function belongs to the class of a probability density functions, i.e., functions $h(x)$ which are almost surely continuous on $\mathbb{R}$, positive, and such that $\int_{-\infty}^{+\infty} h(x)dx = 1$, see [8]. An advantage of this generalization is the relaxation of the integration domain (a, b), depending on the support of the chosen probability density function. A new inequality involving a famous transform is then established using this general result.

Section 2 presents our generalized inequality, together with applications and discussion. A conclusion is given in Section 3.

2. Generalization

The theorem below presents the proposed generalization.

Theorem 2.1. *Let* $p > 1$, $f : \mathbb{R} \to \mathbb{R}$ *be a function, and* $h : \mathbb{R} \to [0, +\infty)$ *be a probability density function such that* $f(x)h(x)$ *is integrable. Suppose that there exists a function* $g : \mathbb{R} \to \mathbb{R}$ *such that* $|g(x)|^{1/(p-1)}h(x)$ *is integrable,*

$$\int_{-\infty}^{+\infty} |g(x)|^{1/(p-1)}h(x)dx \leq \int_{-\infty}^{+\infty} |f(x)|h(x)dx,$$

and

$$\inf_{x\in\mathbb{R}}|f(x)g(x)| \geq c,$$

where c denotes a strictly positive constant. Then we have

$$\int_{-\infty}^{+\infty} |f(x)|h(x)dx \geq c^{1/p}.$$

Proof. Since $\int_{-\infty}^{+\infty} h(x)dx = 1$, $h(x) \geq 0$ for any $x \in \mathbb{R}$, and, by assumption, $\inf_{x \in \mathbb{R}} |f(x)g(x)| \geq c$, we have

$$c^{1/p} = c^{1/p} \int_{-\infty}^{+\infty} h(x)dx \leq \int_{-\infty}^{+\infty} |f(x)g(x)|^{1/p} h(x)dx. \qquad (1)$$

Now, let us consider the real number $q > 1$ such that $1/p + 1/q = 1$, so $q = p/(p-1)$. It follows from the generalized version of the Hölder inequality applied with the parameters p (and q) and the probability measure $v(x) = h(x)dx$ that

$$\int_{-\infty}^{+\infty} |f(x)g(x)|^{1/p} h(x)dx \leq \left[\int_{-\infty}^{+\infty} |f(x)| h(x)dx\right]^{1/p} \left[\int_{-\infty}^{+\infty} |g(x)|^{q/p} h(x)dx\right]^{1/q}. \qquad (2)$$

Using $q/p = 1/(p-1)$ and the assumption $\int_{-\infty}^{+\infty} |g(x)|^{1/(p-1)} h(x)dx \leq \int_{-\infty}^{+\infty} |f(x)| h(x)dx$, we get

$$\left[\int_{-\infty}^{+\infty} |f(x)| h(x)dx\right]^{1/p} \left[\int_{-\infty}^{+\infty} |g(x)|^{q/p} h(x)dx\right]^{1/q}$$

$$\leq \left[\int_{-\infty}^{+\infty} |f(x)| h(x)dx\right]^{1/p} \left[\int_{-\infty}^{+\infty} |f(x)| h(x)dx\right]^{1/q}$$

$$= \left[\int_{-\infty}^{+\infty} |f(x)| h(x)dx\right]^{1/p+1/q} = \int_{-\infty}^{+\infty} |f(x)| h(x)dx. \qquad (3)$$

By combining Equations (1), (2) and (3), we obtain

$$\int_{-\infty}^{+\infty} |f(x)| h(x)dx \geq c^{1/p}.$$

The desired inequality is proved. □

The proof of Theorem 1.2 is thus based on an integral-type inequality, i.e., the generalized version of the Hölder inequality, as opposed to that of [4, Theorem 2.5], which is based on the arithmetic mean-geometric mean inequality.

Taking $p = 2$ and, for $(a, b) \in \mathbb{R}^2$ such that $a < b$, the probability density function of the uniform distribution over $[a, b]$ for $h(x)$, i.e., $h(x) = 1/(b - a)$ for $x \in [a, b]$ and $h(x) = 0$ for any $x \notin [a, b]$, Theorem 2.1 becomes Theorem 1.2. In particular, all the examples developed in [6] are covered.

General but crude examples of functions $f(x)$ satisfying the assumptions of Theorem 2.1 include functions of the form $f(x) = c/g(x) + g(x)^{1/(p-1)}$ or $f(x) = \max[c/g(x), g(x)^{1/(p-1)}]$, where $g(x)$ denotes a positive function satisfying the required convergence assumptions.

Theorem 2.1 is thus a valuable generalization of Theorem 1.2, thanks to the presence of p and the probability density function $h(x)$, both of which are adaptable to different applications. In particular, thanks to this probability density function, different domains of integration can be considered in the integral terms, depending on the support of this function.

Among the possible applications of this result, let us present one involving the Laplace transform. We recall that the Laplace transform of a function $k : [0, +\infty) \to \mathbb{R}$ at $\lambda \in \mathbb{R}$ is defined by

$$\mathcal{L}(k)(\lambda) = \int_0^{+\infty} k(x) e^{-\lambda x} dx,$$

provided that it converges in the integral sense; $k(x)e^{-\lambda x}$ must be integrable over $(0, +\infty)$. See [5]. The Laplace transform is a powerful tool in differential equations, signal processing, and control theory, among other areas. In this Laplace transform setting, based on Theorem 2.1, the idea is to show how the use of an appropriate probability density function $h(x)$ can lead to new research perspectives.

Proposition 2.2. *Let* $p > 1$, $\lambda > 0$ *and* $f : [0, +\infty) \to \mathbb{R}$ *be a function such that* $f(x)e^{-\lambda x}$ *is integrable. Suppose that there exists a function* $g : [0, +\infty) \to \mathbb{R}$ *such that* $|g(x)|^{1/(p-1)}e^{-\lambda x}$ *is integrable,*

$$\mathcal{L}[|g|^{1/(p-1)}](\lambda) \leq \mathcal{L}[|f|](\lambda),$$

and

$$\inf_{x \in (0, +\infty)} |f(x)g(x)| \geq c,$$

where c denotes a strictly positive constant. Then we have

$$\mathcal{L}[|f|](\lambda) \geq \frac{c^{1/p}}{\lambda}.$$

Proof. This is a direct application of Theorem 2.1 by considering the probability density function of the exponential distribution with parameter λ for $h(x)$, i.e., $h(x) = \lambda e^{-\lambda x}$ for any $x \geq 0$, and $h(x) = 0$ for any $x < 0$ (see [8]). Based on this setting, it is sufficient to note that

$$\int_{-\infty}^{+\infty} |f(x)|h(x)dx = \int_{0}^{+\infty} |f(x)|\lambda e^{-\lambda x}dx = \lambda\mathcal{L}[|f|](\lambda),$$

and the same for the other functions involved. This yields the desired result. □

This inequality involving the Laplace transform with intermediate parameters and functions is new in literature to the best of our knowledge.

3. Conclusion

In this paper, we have generalized an integral-type inequality established by Wei-Shih Du in 2023, which itself extends an older integral-type inequality that has attracted some attention. The generalization is motivated by the inclusion of a parameter and a probability density function, which can be adapted as needed. By introducing these variables, we extend the scope and exibility of the Du original inequality, allowing it to be applied in a wider range of mathematical contexts. In particular, an application to the Laplace transform is discussed. Given the importance of this transform in many scientific disciplines, we hope that our new results will have future applications in nonlinear analysis, mathematical physics and related fields.

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