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SPINOR FRAMEWORK IN LINEARIZED GRAVITY: A PERSPECTIVE FORM THE ENERGY MOMENTUM TENSOR

ELISA VARANI

Università Cattolica del Sacro Cuore
Milan
Italy
e-mail: elisa.varani@unicatt.it

Abstract

We want to analyze the energy-momentum tensor obtained for spinors in general relativity in the linear gravity approximation.

For this study, we take as a reference the theory of spinors in general relativity, the formalism of the spin connection, covariant derivatives refer to that theory.

We consider linear gravity with $g_{uv} = \eta_{\mu v} + h_{\mu v}$, here we assume $h_{\mu v}$ as a first step of perturbation from flat spacetime, recursively we apply the same approximation till to $nh_{\mu v}$ contributions.

In this process, we examine the energy momentum for a Dirac particle and calculate the tensor connected to each step of perturbation from flat to $nh_{\mu v}$.

The energy-momentum tensor has recursive behaviour: in particular, at each step towards a little more curved space-time, a new contribution must be added to the energy-momentum tensor calculated in the former step.

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In this approximation the tensor $T^{\mu\nu}$ linearly depends on the metric, this may then appear to be a clue that leads to a form of quantization of the energy if we assume the metric is quantized.

The analysis continues in more detail, we see in particular that there are intakes related to the spin connection so that the full energy momentum approximation is

$$T^{n\mu\nu} \cong \hat{T}^{\mu\nu} - \frac{n}{2}(h^{\mu}_{a}\hat{T}^{a\nu} + h^{\nu}_{a}\hat{T}^{a\mu}) - n\frac{i}{4}(h^{\mu}_{a}h^{\nu}_{c|b}S^{abc} + h^{\nu}_{a}h^{\mu}_{c|b}S^{abc}).$$

Starting from additional contributions to the energy momentum tensor we try to attribute a new physical meaning to the cosmological constant. In the second part of the paper, we consider the tensor as a source of the wave equation and study different solutions according to distinctive terms of the tensor.

1. Introduction

1.1. Spinors and quantum gravity

Quantum gravity is an attempt to include general relativity and quantum mechanics principles.

Here we consider spinors as the encounter point of these two descriptions. Spinors enter the scene of general relativity thanks to the equivalence principle. We need a flat inertial reference frame where we can describe spinors as a representation of the Lorentz group: conforming to the equivalence principle we can set up a system of inertial coordinates so that the effects of the gravitational field are canceled out, we can find shelter from gravitation and set up spinors in locally flat spacetime.

Spinors are the quintessence of quantum mechanics.

Let's consider that for each point of curved spacetime we can introduce a flat reference system, we reset the memory of the process for a moment and then we return to realize that if we have arrived in the new reference frame, we owe it to a small jump $h_{\mu\nu}$ from the previous flat space time. During this path what's happened to the energy momentum tensor related to fermions? We see that this one is getting the same contributions at each step.

This result considers the spinor energy momentum tensor, as we can derive from the Dirac theory, in the general relativity background. Torsion is admitted in this theory even if the affine connection is symmetric [1], [2].

We find out that there is a relation between the changing in the metric tensor and the energy momentum tensor for the spinor field, the next question is how this combination is realized.

This mechanism involves gravitational waves.

Gravitational waves in linearized gravity are developed from the Einstein field equation, the perturbation $h_{\mu\upsilon}$ propagates as a wave according to the equation:

$$\Box h_{\mu\upsilon} = -\frac{16\pi G}{c^2} T^{\mu\upsilon}. \tag{1}$$

In the following analysis, we consider the terms of the energy-momentum tensor as sources of gravitational waves, we get non-homogeneous differential equations. At first order we get a differential equation typical for a standing wave if we choose appropriate boundary conditions.

A standing string has normal modes of vibration so that the energy is quantized, here the vibration does not concern a string but concerns the metric and the normal modes are referred to different perturbations of the flat spacetime.

If we consider the spin connection contributions to the energy momentum tensor, we obtain two other types of non-homogeneous equations that have a derivative term of the metric and these equations are related to forced and damped oscillations, in the next paragraphs we show in detail how these types of oscillations can generate opposite currents and the creation of whirlpools.

In the end, the curved space-time turns out to be obtained from the sum of many approximations of the size $h_{\mu\upsilon}$, the source for the metric perturbation is the energy momentum tensor associated with the metric tensor.

The curved space-time is generated by the normal modes of vibration of the metric having their source in the energy momentum tensor $T^{\mu\upsilon}$, this is a possible way to realize the coupling between the metric field and the spinor field; the peculiar behaviour of the spinor field is achieved through the interaction with the gravitational field creating opposite contributions to the metric tensor.

Curvature measurements are obtained from detection of energy perturbations; we get an example of curvature energy spectra for hyperbolic paraboloid (see Figure 1) [3].

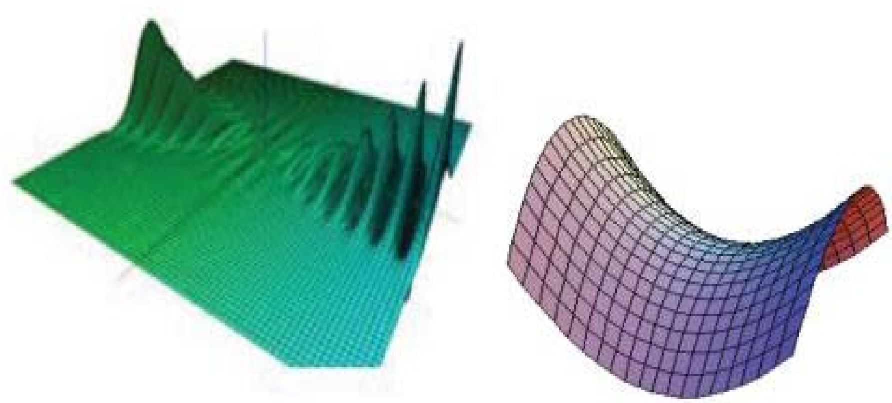

Figure 1. Curvature energy spectra.

2. Materials and Methods

2.1. Energy momentum tensor for the Dirac field in curved space time and in weak gravity

We start with the Lagrangian for the spinor field in the flat Minkowski space time.

$$L_f = \frac{i}{2}\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - \frac{i}{2}\partial_{\mu}{}^{\mu}\psi - m\bar{\psi}\psi. \tag{2}$$

For the calculation of the energy momentum tensor, we follow the Noether's theorem:

$$\hat{T}^{\mu\upsilon} = \frac{\partial L_f}{\partial\left(\partial_{\mu}\psi\right)}\partial_{\upsilon}\psi + \frac{\partial L_f}{\partial\left(\partial_{\mu}\bar{\psi}\right)}\partial_{\upsilon}\bar{\psi} - \delta_{\upsilon}^{\mu}L_f. \tag{3}$$

We get the energy momentum tensor in the explicit symmetric form:

$$\hat{T}^{\mu\upsilon} = \frac{1}{4}[\,\bar{\psi}i\gamma^{\mu}\partial^{\upsilon}\psi - \overline{\partial^{\upsilon}\psi}i\gamma^{\mu}\psi + \bar{\psi}i\gamma^{\upsilon}\partial^{\mu}\psi - \overline{\partial^{\mu}\psi}i\gamma^{\upsilon}\psi\,]. \tag{4}$$

Let’s consider the energy-momentum tensor in curved spacetime with the covariant derivative.

$$D_{\upsilon}\psi \equiv \psi\,\|\upsilon = \partial_{\upsilon}\psi + \Gamma_{\upsilon}\psi; \quad \overline{D^{\upsilon}\psi} = \partial_{\upsilon}\bar{\psi} - \Gamma_{\upsilon}\bar{\psi}, \tag{5}$$

$$\Gamma_{\upsilon} = \frac{i}{4}e_{b\mu}\nabla_{\upsilon}e_{a}^{\mu}\hat{\alpha}^{ab}; \quad \nabla_{\upsilon}e_{a}^{\mu} = \partial_{\upsilon}e_{a}^{\mu} + \Gamma_{\upsilon_{\lambda}}^{\mu}e_{a}^{\lambda}. \tag{6}$$

Christoffel symbols are so defined

$$\Gamma_{\beta\gamma}^{\alpha} = \begin{Bmatrix}\alpha\\ \beta\gamma\end{Bmatrix} = \frac{1}{2}g^{\alpha\lambda}(\,\partial_{\beta}g_{\lambda\gamma} + \partial_{\gamma}g_{\lambda\beta} - \partial_{\lambda}g_{\beta\gamma}\,). \tag{7}$$

According to the Noether theorem now we obtain:

$$T^{\mu\upsilon} = \frac{1}{4}[\,\bar{\psi}i\gamma^{\mu}D^{\upsilon}\psi - \overline{D^{\upsilon}\psi}i\gamma^{\mu}\psi + \bar{\psi}i\gamma^{\mu}D^{\mu}\psi - \overline{D^{\mu}\psi}i\gamma^{\upsilon}\psi\,]. \tag{8}$$

The tensor has the same form but covariant derivatives instead of the ordinary ones

$$\hat{T}^{\mu\upsilon} = \frac{1}{4}[\,\bar{\psi}i\gamma^{\mu}\partial^{\upsilon}\psi - \overline{\partial^{\upsilon}\psi}i\gamma^{\mu}\psi + \bar{\psi}i\gamma^{\upsilon}\partial^{\mu}\psi - \overline{\partial^{\mu}\psi}i\gamma^{\upsilon}\psi\,]. \tag{9}$$

Now we consider the tensor in weak gravity approximation,

$$g_{u\upsilon} = \eta_{\mu\upsilon} + h_{\mu\upsilon}, \tag{10}$$

$$\gamma^{\mu} = \hat{\gamma}^{\mu} - \frac{1}{2} h_a^{\mu} \hat{\gamma}^a. \tag{11}$$

We make explicit calculation with (11) and we get

$$T^{\mu\nu} = \hat{T}^{\mu\nu} + \frac{1}{4}\Big[\overline{\psi} i\Big(-\frac{1}{2} h_a^{\mu}\hat{\gamma}^a\Big) D^{\nu}\psi - \overline{D^{\nu}\psi} i\Big(-\frac{1}{2} h_a^{\mu}\hat{\gamma}^a\Big)\psi + \overline{\psi} i\Big(-\frac{1}{2} h_a^{\nu}\hat{\gamma}^a\Big) D^{\mu}\psi$$

$$- \overline{D^{\mu}\psi} i\Big(-\frac{1}{2} h_a^{\nu}\hat{\gamma}^a\Big)\psi\Big], \tag{12}$$

$$T^{\mu\nu} = \hat{T}^{\mu\nu} + \frac{1}{4}\Big\{ -\frac{1}{2} h_a^{\mu}[\overline{\psi} i(\hat{\gamma}^a)\partial^{\nu}\psi - \overline{\partial^{\nu}\psi} i(\hat{\gamma}^a)\psi] - \frac{1}{2} h_a^{\nu}[\overline{\psi} i(\hat{\gamma}^a)\partial^{\mu}\psi$$

$$- \overline{\partial^{\mu}\psi} i(\hat{\gamma}^a)\psi]\Big\}, \tag{13}$$

$$T^{\mu\nu} = \hat{T}^{\mu\nu} - \frac{1}{2} h_a^{\mu}\hat{T}^{a\nu} - \frac{1}{2} h_a^{\nu}\hat{T}^{a\mu} - \frac{1}{2} h_a^{\mu}\frac{i}{4}[\overline{\psi}(\hat{\gamma}^a)\Gamma^{\nu}\psi + \overline{\psi}\Gamma^{\nu}(\hat{\gamma}^a)\psi]$$

$$- \frac{1}{2} h_a^{\nu}\frac{i}{4}[\overline{\psi}(\hat{\gamma}^a)\Gamma^{\nu}\psi + \overline{\psi}\Gamma^{\mu}(\hat{\gamma}^a)\psi], \tag{14}$$

$$T^{\mu\nu} = \hat{T}^{\mu\nu} - \frac{1}{2} h_a^{\mu}\hat{T}^{a\nu} - \frac{1}{2} h_a^{\nu}\hat{T}^{a\mu} - \frac{1}{2} h_a^{\mu}\frac{i}{4}[\overline{\psi}\{\hat{\gamma}^a, \Gamma^{\nu}\}\psi]$$

$$- \frac{1}{2} h_a^{\nu}\frac{i}{2}[\overline{\psi}\{\hat{\gamma}^a, \Gamma^{\mu}\}\psi], \tag{15}$$

$$\Gamma^{\nu} \cong \frac{i}{4} h^{\nu}_{c|b}\hat{\sigma}^{bc};\ \Gamma^{\mu} \cong \frac{i}{4} h^{\mu}_{c|b}\hat{\sigma}^{bc}, \tag{16}$$

$$T^{\mu\nu} = \hat{T}^{\mu\nu} - \frac{1}{2} h_a^{\mu}\hat{T}^{a\nu} - \frac{1}{2} h_a^{\nu}\hat{T}^{a\mu} + \frac{i}{4} h_a^{\mu} h^{\nu}_{c|b}\frac{i}{4}[\overline{\psi}\{\hat{\gamma}^a, \hat{\sigma}^{bc}\}\psi]$$

$$+ \frac{i}{4} h_a^{\nu} h^{\mu}_{c|b}\frac{i}{4}[\overline{\psi}\{\hat{\gamma}^a, \hat{\sigma}^{bc}\}\psi], \tag{17}$$

$$S^{abc} = -\frac{i}{4}[\overline{\psi}\{\hat{\gamma}^a, \hat{\sigma}^{bc}\}\psi] \text{ spin angular momentum tensor}, \tag{18}$$

$$T^{\mu\nu} \cong \hat{T}^{\mu\nu} - \frac{1}{2} h_a^{\mu}\hat{T}^{a\nu} - \frac{1}{2} h_a^{\nu}\hat{T}^{a\mu} + \frac{i}{4} h_a^{\mu} h^{\nu}_{c|b} S^{abc} - \frac{i}{4} h_a^{\nu} h^{\mu}_{c|b} S^{abc}. \tag{19}$$

If we ignore the spin contribution, we can write the energy momentum tensor approximated as

$$T^{\mu v} \cong \hat{T}^{\mu v} - \frac{1}{2} h_a^{\mu} \hat{T}^{av} - \frac{1}{2} h_a^{v} \hat{T}^{a\mu}. \tag{20}$$

So if we start with a flat space-time and then we introduce a little deformation in the metric the energy momentum tensor is changing of a factor $-N^{\mu v}$ each time.

$$N^{\mu v} = \frac{1}{2} (h_a^{\mu} \hat{T}^{av} + h_a^{v} \hat{T}^{a\mu}), \tag{21}$$

$N^{\mu v}$ is symmetric for μv

$$T^{\mu v} \cong \hat{T}^{\mu v} - N^{\mu v}. \tag{22}$$

We approximate the metric, each time putting ourselves in a new reference frame, starting from the flat we proceed with discrete path of the size $h_{\mu v}$ to a little more curved space time (Figure 2).

The result at the first step is the following equation:

$$T'^{\mu v} \cong \hat{T}^{\mu v} - \frac{1}{2} h_a^{\mu} \hat{T}^{av} - \frac{1}{2} h_a^{v} \hat{T}^{a\mu}. \tag{23}$$

Second step

$$T''^{\mu v} \cong T'^{\mu v} - \frac{1}{2} h_a^{\mu} T'^{av} - \frac{1}{2} h_a^{v} T'^{a\mu} \cong \hat{T}^{\mu v} - h_a^{\mu} \hat{T}^{av} - h_a^{v} \hat{T}^{a\mu} \tag{24}$$

Third step

$$T'''^{\mu v} \cong \hat{T}^{\mu v} - \frac{3}{2} h_a^{\mu} \hat{T}^{av} - \frac{3}{2} h_a^{v} \hat{T}^{a\mu}. \tag{25}$$

And so on for n-times,

$$T^{n\mu v} \cong \hat{T}^{\mu v} - \frac{n}{2} h_a^{\mu} \hat{T}^{av} - \frac{n}{2} h_a^{v} \hat{T}^{a\mu}, \tag{26}$$

here n is not a world index, it only indicates the n-steps of the path.

At each step we sum up the effect of the metric approximation with the factor $-N^{\mu\upsilon} = -\frac{1}{2} h_a^{\mu} \hat{T}^{a\upsilon} - \frac{1}{2} h_a^{\upsilon} \hat{T}^{a\mu}$ inherited from the previous energy-momentum tensor.

We have completely ignored second order contributions.

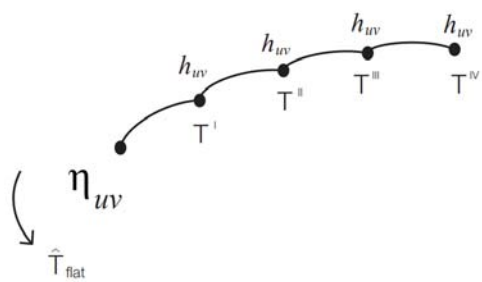

Figure 2. Iteration sequence to approximate the energy-momentum tensor through discrete paths of size $h_{\mu\upsilon}$ up to slightly more curved space-time.

Now if we include the spinor angular momentum, we have to consider the tensor

$$T^{n\mu\upsilon} \cong \hat{T}^{\mu\upsilon} - \frac{n}{2} (h_a^{\mu} \hat{T}^{a\upsilon} + h_a^{\upsilon} \hat{T}^{a\mu}) + n \frac{i}{4} h_a^{\mu} h_{c|b}^{\upsilon} \frac{i}{4} [\overline{\psi} \{ \hat{\gamma}^a , \hat{\sigma}^{bc} \} \psi]$$

$$+ n \frac{i}{4} h_a^{\upsilon} h_{c|b}^{\mu} \frac{i}{4} [\overline{\psi} \{ \hat{\gamma}^a , \hat{\sigma}^{bc} \} \psi], \tag{27}$$

$$T^{n\mu\upsilon} \cong \hat{T}^{\mu\upsilon} - \frac{n}{2} (h_a^{\mu} \hat{T}^{a\upsilon} + h_a^{\upsilon} \hat{T}^{a\mu}) - n \frac{i}{4} (h_a^{\mu} h_{c|b}^{\upsilon} S^{abc} + h_a^{\upsilon} h_{c|b}^{\mu} S^{abc}). \tag{28}$$

First step with spinor:

$$T'^{\mu\upsilon} = \hat{T}^{\mu\upsilon} - \frac{1}{2} h_a^{\mu} \hat{T}^{a\upsilon} - \frac{1}{2} h_a^{\upsilon} \hat{T}^{a\mu} + \frac{i}{4} h_a^{\mu} h_{c|b}^{\upsilon} \frac{i}{4} [\overline{\psi} \{ \hat{\gamma}^a , \hat{\sigma}^{bc} \} \psi]$$

$$+\frac{i}{4}h_a^{v}h_{c|b}^{\mu}\frac{i}{4}[\overline{\psi}\{\hat{\gamma}^a, \hat{\sigma}^{bc}\}\psi]. \tag{29}$$

$$T''^{\mu v} = T'^{\mu v} - \frac{1}{2}h_a^{\mu}T'^{av} - \frac{1}{2}h_a^{v}T'^{a\mu} + \frac{i}{4}h_a^{\mu}h_{c|b}^{v}\frac{i}{4}[\overline{\psi}\{\hat{\gamma}^a, \hat{\sigma}^{bc}\}\psi]$$
$$+\frac{i}{4}h_a^{v}h_{c|b}^{\mu}\frac{i}{4}[\overline{\psi}\{\hat{\gamma}^a, \hat{\sigma}^{bc}\}\psi], \tag{30}$$

$$T''^{\mu v} = T'^{\mu v} - \frac{1}{2}h_a^{\mu}T'^{av} - \frac{1}{2}h_a^{v}T^{a\mu} + \frac{i}{4}h_a^{\mu}h_{c|b}^{v}\frac{i}{4}[\overline{\psi}\{\hat{\gamma}^a, \hat{\sigma}^{bc}\}\psi]$$
$$+\frac{i}{4}h_a^{v}h_{c|b}^{\mu}\frac{i}{4}[\overline{\psi}\{\hat{\gamma}^a, \hat{\sigma}^{bc}\}\psi]$$
$$-\frac{1}{2}h_a^{\mu}(\hat{T}^{av} - \frac{1}{2}h_a^{a}\hat{T}^{bv} - \frac{1}{2}h_b^{v}\hat{T}^{ba} + \frac{i}{4}h_k^{a}h_{c|b}^{v}\frac{i}{4}[\overline{\psi}\{\hat{\gamma}^k, \hat{\sigma}^{bc}\}\psi]$$
$$+\frac{i}{4}h_k^{v}h_{c|b}^{a}\frac{i}{4}[\overline{\psi}\{\hat{\gamma}^k, \hat{\sigma}^{bc}\}\psi])$$
$$-\frac{1}{2}h_a^{v}(\hat{T}^{a\mu} - \frac{1}{2}h_b^{\mu}\hat{T}^{b\mu} - \frac{1}{2}h_b^{\mu}\hat{T}^{ba} + \frac{i}{4}h_k^{a}h_{c|b}^{\mu}\frac{i}{4}[\overline{\psi}\{\hat{\gamma}^k, \hat{\sigma}^{bc}\}\psi]$$
$$+\frac{i}{4}h_k^{\mu}h_{c|b}^{a}\frac{i}{4}[\overline{\psi}\{\hat{\gamma}^k, \hat{\sigma}^{bc}\}\psi])$$
$$+\frac{i}{4}h_a^{\mu}h_{c|b}^{v}\frac{i}{4}[\overline{\psi}\{\hat{\gamma}^a, \hat{\sigma}^{bc}\}\psi] + \frac{1}{4}h_a^{v}h_{c|b}^{\mu}\frac{i}{4}[\overline{\psi}\{\hat{\gamma}^a, \hat{\sigma}^{bc}\}\psi], \tag{31}$$

$$T'^{av} = \hat{T}^{av} - \frac{1}{2}h_b^{a}\hat{T}^{bv} - \frac{1}{2}h_b^{v}\hat{T}^{ba} + \frac{i}{4}h_k^{a}h_{c|b}^{v}\frac{i}{4}[\overline{\psi}\{\hat{\gamma}^k, \hat{\sigma}^{bc}\}\psi]$$
$$+\frac{i}{4}h_k^{v}h_{c|b}^{a}\frac{i}{4}[\overline{\psi}\{\hat{\gamma}^k, \hat{\sigma}^{bc}\}\psi], \tag{32}$$

$$T'^{a\mu} = \hat{T}^{a\mu} - \frac{1}{2}h_b^{\mu}\hat{T}^{b\mu} - \frac{1}{2}h_b^{\mu}\hat{T}^{ba} + \frac{i}{4}h_k^{a}h_{c|b}^{\mu}\frac{i}{4}[\overline{\psi}\{\hat{\gamma}^k, \hat{\sigma}^{bc}\}\psi]$$
$$+\frac{i}{4}h_k^{\mu}h_{c|b}^{a}\frac{i}{4}[\overline{\psi}\{\hat{\gamma}^k, \hat{\sigma}^{bc}\}\psi]. \tag{33}$$

We don't consider second order contribution like $h_a^v h_b^\mu$ but include $h_a^\mu h_{c|b}^v$

$$T''^{\mu v} \cong \hat{T}^{\mu v} - \frac{1}{2} h_a^\mu \hat{T}^{av} - \frac{1}{2} h_b^v \hat{T}^{a\mu} + \frac{i}{4} h_a^\mu h_{c|b}^v \frac{i}{4} [\overline{\psi}\{\hat{\gamma}^a, \hat{\sigma}^{bc}\}\psi]$$

$$+ \frac{i}{4} h_a^v h_{c|b}^\mu \frac{i}{4} [\overline{\psi}\{\hat{\gamma}^a, \hat{\sigma}^{bc}\}\psi]$$

$$- \frac{1}{2} h_a^\mu (\hat{T}^{av} + \dots) - \frac{1}{2} h_a^v (\hat{T}^{a\mu} + \dots) + \frac{i}{4} h_a^\mu h_{c|b}^v \frac{i}{4} [\overline{\psi}\{\hat{\gamma}^a, \hat{\sigma}^{bc}\}\psi]$$

$$+ \frac{i}{4} h_a^v h_{c|b}^\mu \frac{i}{4} [\overline{\psi}\{\hat{\gamma}^a, \hat{\sigma}^{bc}\}\psi], \tag{34a}$$

34b. $$T''^{\mu v} \cong \hat{T}^{\mu v} - h_b^\mu \hat{T}^{av} - h_b^v \hat{T}^{a\mu} + 2\frac{i}{4} h_a^\mu h_{c|b}^v \frac{i}{4} [\overline{\psi}\{\hat{\gamma}^a, \hat{\sigma}^{bc}\}\psi]$$

$$+ 2\frac{i}{4} h_a^v h_{c|b}^\mu \frac{i}{4} [\overline{\psi}\{\hat{\gamma}^a, \hat{\sigma}^{bc}\}\psi]$$

At the moment we don't consider these kinds of contribution

$$-\frac{1}{2} h_a^\mu \left(-\frac{1}{2} h_b^\mu \hat{T}^{bv} + \frac{i}{4} h_k^a h_{c|b}^v \frac{i}{4} [\overline{\psi}\{\hat{\gamma}^k, \hat{\sigma}^{bc}\}\psi] \right), \tag{35}$$

because they are second order or more, so we write:

$$T''^{\mu v} \cong \hat{T}^{\mu v} - (h_a^\mu \hat{T}^{av} + h_a^v \hat{T}^{a\mu}) - 2\frac{i}{4} (h_a^\mu h_{c|\,b}^v S^{abc} + h_a^v h_{c|\,b}^\mu S^{abc}). \tag{34c}$$

We can see from Equation (28) that the spinor contribution $\frac{i}{4} h_a^\mu h_{c|\,b}^v \frac{i}{4} [\psi\{\hat{\lambda}^a, \hat{\sigma}^{bc}\}\psi]$ is different from zero only if the derivative $h_{c|b}^v \neq 0$, that to say the coupling between the gravitational field h_a^μ and torsion is effective only if the deformation of the metric is not constant.

$$h_{c|b}^v = h_{c|\,b}^v \partial_b h_c^v \neq 0. \tag{35}$$

3. Results

3.1. Energy-momentum tensor depending on discrete steps in the metric

The main results we get is this equation describing the energy momentum tensor related to the changing in the metric and to torsion

$$T^{n\mu v} \cong \hat{T}^{\mu v} - \frac{n}{2}\left(h_a^\mu \hat{T}^{av} + h_a^v \hat{T}^{a\mu}\right) + n\,\frac{i}{4}\left(h_a^\mu h_{c|b}^v + h_a^v h_{c|b}^\mu\right) S^{abc}. \quad (36)$$

We write the equation in a more compact way

$$T^{n\mu v} \cong \hat{T}^{\mu v} n N^{\mu v} + n\,\frac{i}{4}\,B^{uv}. \quad (37)$$

The first term $\hat{T}^{\mu v}$ is about Dirac particle in flat spacetime

$$N^{\mu v} = \frac{1}{2}\left(h_a^\mu \hat{T}^{av} + h_a^v \hat{T}^{a\mu}\right). \quad (38)$$

This term represents the coupling between the deformed gravitational space and the Dirac particle, it's symmetric.

The contribution to the energy momentum tensor is negative increasing at each step, as if spinors could lose energy in the coupling with the gravitational field.

$$B^{uv} = \left(h_a^\mu h_{c|b}^v S^{abc} + h_a^v h_{c|b}^\mu S^{abc}\right), \quad (39)$$

B^{uv} is symmetric for the world indices, but it contains the spinor angular momentum antisymmetric for the flat indices of the local reference frame.

Here we have the coupling between the gravitational field and the spinor angular momentum through the derivative $h_{c|b}^v$.

It's apparently negative. Let's check $\frac{i}{4}\,S^{abc}$.

Here we have to deal with

$$M^{abc} = \{\hat{\gamma}^a, \hat{\sigma}^{bc}\} \tag{40}$$

and specify non vanishing terms.

We consider Dirac matrices so defined:

$$\hat{\gamma}^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \ \hat{\gamma}^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{41}$$

$$\hat{\sigma}^{ij} = \frac{i}{2}\left[\hat{\gamma}^i, \hat{\gamma}^j\right] = \varepsilon_{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}. \tag{42}$$

The only non-vanishing terms of $M^{abc} = \{\hat{\gamma}^a, \hat{\sigma}^{bc}\}$ are

$$M^{0bc} = 2\varepsilon_{bck}\begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix} \tag{43}$$

index 0 time like; *bck* flat spacelike

$$M^{b0c} = -M^{0bc} = -2\varepsilon_{bck}\begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix}, \tag{44}$$

$$M^{kbc} = 2\varepsilon_{bck}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{45}$$

So if we make explicit $-n\frac{i}{4}B^{uv} = -n\frac{i}{4}(h^\mu_a h^v_{c|b} + h^v_a h^\mu_{c|b})S^{abc}$, (46)

we get three kinds of contributions:

$$-n\frac{i}{4}(h^\mu_a h^v_{c|b} + h^v_a h^\mu_{c|b})S^{abc} = -n\frac{i}{4}(h^\mu_0 h^v_{c|b} + h^v_0 h^\mu_{c|b})S^{0bc}$$

$$= +n\frac{i}{4}(h^\mu_0 h^v_{c|b} + h^v_0 h^\mu_{c|b})\frac{i}{4}[\overline{\psi}M^{0bc}\psi]$$

$$= -n\frac{n}{8}(h^\mu_0 h^v_{c|b} + h^v_0 h^\mu_{c|b})\varepsilon_{bck}\overline{\psi}\begin{pmatrix} \sigma^k & 0 \\ 0 & -0^k \end{pmatrix}\psi \tag{46.I}$$

$$-n\frac{i}{4}(h^{\mu}_{a}h^{v}_{c|b}+h^{v}_{a}h^{\mu}_{c|b})S^{abc}=-n\frac{i}{4}(h^{\mu}_{a}h^{v}_{c|0}+h^{v}_{a}h^{\mu}_{c|0})S^{a0c}$$

$$=+n\frac{i}{4}(h^{\mu}_{a}h^{v}_{c|0}+h^{v}_{a}h^{\mu}_{c|0})\frac{i}{4}[\overline{\psi}M^{a0c}\psi]$$

$$=+\frac{n}{8}(h^{\mu}_{a}h^{v}_{c|0}+h^{v}_{a}h^{\mu}_{c|0})\varepsilon_{ack}\overline{\psi}\begin{pmatrix}\sigma^{k} & 0\\ 0 & -\sigma^{k}\end{pmatrix}\psi, \tag{46.II}$$

here index a is flat spacelike.

$$-n\frac{i}{4}(h^{\mu}_{a}h^{v}_{c|b}+h^{v}_{a}h^{\mu}_{c|b})S^{abc}=-n\frac{i}{4}(h^{\mu}_{k}h^{v}_{c|b}+h^{v}_{k}h^{\mu}_{c|b})S^{kbc}$$

$$=+n\frac{i}{4}(h^{\mu}_{k}h^{v}_{c|b}+h^{v}_{k}h^{\mu}_{c|b})\frac{i}{4}[\overline{\psi}M^{kbc}\psi]$$

$$=-\frac{n}{8}(h^{\mu}_{k}h^{v}_{c|b}+h^{v}_{k}h^{\mu}_{c|b})\varepsilon_{bck}\overline{\psi}\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\psi. \tag{46.III}$$

The energy momentum tensor has a recursive form and for each successive approximation we can detect a negative factor associated with the energy momentum tensor for the Dirac field in flat spacetime, this contribution is multiplied by n at each step.

In addition, we get positive and negative contributions from spinor angular momentum, we mean this could be associated to expansion and compression or growing and damping deformation of the metric.

Spinor as source of gravitational waves loses energy in creating curvature and torsion of the metric.

3.2. Cosmological constant reinterpreted

We suggest that the terms calculated explicitly in the energy momentum tensor could be a declaration about the cosmological constant and dark energy: spinor and torsion enter in the mechanism of expansion.

We consider the field equation with the cosmological constant

$$G^{\mu v}+\Lambda g^{\mu v}=-\frac{8\pi G}{c^{2}}\hat{T}^{\mu v}. \tag{47}$$

Now if we start with the field equation without cosmological constant and develop the energy momentum tensor we get

$$G^{\mu v} = -\frac{8\pi G}{c^2}\hat{T}^{\mu v}, \tag{48}$$

$$G^{\mu v} = -\frac{8\pi G}{c^2}(\hat{T}^{\mu v} - N^{\mu v} + \frac{i}{4}B^{uv}), \tag{49}$$

$$G^{\mu v} + \frac{8\pi G}{c^2}(-N^{\mu v} + \frac{i}{4}B^{uv}) = -\frac{8\pi G}{c^2}\hat{T}^{\mu v}, \tag{50}$$

$$\Lambda g^{\mu v} = \frac{8\pi G}{c^2}(-N^{\mu v} + \frac{i}{4}B^{uv}), \tag{51}$$

$$\Lambda(\eta^{\mu v} + h^{\mu v}) = \frac{8\pi G}{c^2}[-\frac{1}{2}(h_a^{\mu}\hat{T}^{av} + h_a^{v}\hat{T}^{a\mu}) + \frac{i}{4}(h_a^{\mu}h_{c|b}^{v} + h_a^{v}h_{c|b}^{\mu})S^{abc}]. \tag{52}$$

The cosmological constant is positive and create acceleration in the expansion of the universe, the value is associated to vacuum energy or to dark energy. There is a problem with the interpretation of the cosmological constant as vacuum energy: the value of the constant calculated from quantum field theory is too big and would produce acceleration too high compared with measurements [4].

In this frame, the cosmological constant is associated to spinor coupling to the gravitational field and to torsion interaction.

Torsion is implicated in the expansion of the universe and could replace the role of dark energy to explain repulsive force contrasting gravity [5].

The cosmological constant could assume different value for each step of expansion

$$\Lambda g^{\mu v} = \frac{8\pi G}{c^2}(-nN^{\mu v} + n\frac{i}{4}B^{uv}). \tag{53}$$

Here we have attractive decelerating term (minus sign) and a mixed term of torsion with positive (repulsive) and negative contribution, creating different acceleration of the expansion.

3.3. Gravitational waves sources

We consider different contributions as wave source and look forward to the solutions as a superpositions of the two effects.

$$\Box h^{\mu\upsilon} = -\frac{16\pi G}{c^2}\left[\hat{T}^{\mu\upsilon} - \frac{1}{2}(h_a^{\mu}\hat{T}^{a\upsilon} + h_a^{\upsilon}\hat{T}^{a\mu}) + \frac{i}{4}(h_a^{\mu}h_{c|b}^{\upsilon} + h_a^{\upsilon}h_{c|b}^{\mu})S^{abc}\right],$$

(54)

$$h^{\mu\upsilon} = \frac{\partial^2}{\partial t^2} - \nabla^2, \tag{55}$$

$$\Box h^{\mu\upsilon} = -\frac{16\pi G}{c^2}\left[-\frac{1}{2}(h_a^{\mu}\hat{T}^{a\upsilon} + h_a^{\upsilon}\hat{T}^{a\mu})\right]. \tag{56}$$

This is a non-homogeneous wave equation like the elastic string wave equation- describing tension in a fixed string due to an external force: $h^{\mu\upsilon}$ could be interpreted as the tension in the string and the non-homogeneous term $-\frac{1}{2}(h_a^{\mu}\hat{T}^{a\upsilon} + h_a^{\upsilon}\hat{T}^{a\mu})$ is assumed to be the force acting on the string forcing vibrations.

$$\Box h^{\mu\upsilon} = -\frac{16\pi G}{c^2}\left[\frac{i}{4}(h_a^{\mu}h_{c|b}^{\upsilon} + h_a^{\upsilon}h_{c|b}^{\upsilon})S^{abc}\right]. \tag{57}$$

Here we have three terms.

$$\Box h^{\mu\upsilon} = -C\frac{1}{8}(h_0^{\mu}h_{c|b}^{\upsilon} + h_0^{\upsilon}h_{c|b}^{\upsilon})\varepsilon_{bck}\overline{\psi}\begin{pmatrix}\sigma^k & 0\\ 0 & -\sigma^k\end{pmatrix}\psi \tag{58}$$

$$-\frac{16\pi G}{c^2} = C,$$

cbk are flat spacelike indices

$$\Box h^{\mu\upsilon} = +C\frac{1}{8}(h_a^{\mu}h_{c|0}^{\upsilon} + h_a^{\upsilon}h_{c|0}^{\mu})\varepsilon_{ack}\overline{\psi}\begin{pmatrix}\sigma^k & 0\\ 0 & -\sigma^k\end{pmatrix}\psi; \tag{59}$$

0 is flat time like index

$$\Box h^{\mu\upsilon} = -C\frac{n}{8}(h_k^{\mu}h_{c|b}^{\upsilon} + h_k^{\upsilon}h_{c|b}^{\mu})\varepsilon_{bck}\overline{\psi}\begin{pmatrix}0 & 1\\ -I & 0\end{pmatrix}\psi. \tag{60}$$

We notice that the nonhomogeneous terms contain first order partial derivatives, like for heat equation covered by vacuum wave equation. These first order terms are associated to damping or growth phenomena, depending on the plus or minus sign; there are opposite currents creating whirlpool, torsion from spinor becomes torsion of the metric.

So, the picture we can figure out from (56) is about background stationary waves due to spinor fields related to discrete expansion. In addition to this, Equation (57) describes the torsion effects arising when spinors are present in curved spacetime. Stationary wave plus whirlpool are the effects of the deformation of the metric due to the presence of spinors.

4. Summary and conclusion

In this investigation, we have assumed the approximation of linear gravity with progressive steps till to $nh_{\mu\upsilon}$ so that $g_{uv} = \eta_{\mu\upsilon} + nh_{\mu\upsilon}$. For each perturbation of the size $h_{\mu\upsilon}$ the energy momentum tensor is so written

$$T^{n\mu\upsilon} \cong \hat{T}^{\mu\upsilon} + n[-\frac{1}{2}(h_a^{\mu}\hat{T}^{a\upsilon} + h_a^{\upsilon}\hat{T}^{a\mu}) + \frac{i}{4}(h_a^{\mu}h_{c|b}^{\upsilon} + h_a^{\upsilon}h_{c|b}^{\mu})S^{abc}]. \tag{61}$$

This is a model of discrete expansion driven through gravitational waves originating from spinors.

But how this mechanism starts? We could suppose vacuum fluctuation for the first deformation $h_{\mu\upsilon}$, spinors coupled to gravitational field assume new energy contributions and the presence of spinors deforms the metric.

How are the discrete levels of the metric connected? Gravitational wave originates from spinors having their source in the energy momentum tensor, the tensor changes modifying the gravitational field with the perturbation of the size $h_{\mu\upsilon}$.

In this model, we are not fixing the size of $h_{\mu\upsilon}$, could it be at Planck scale? Let's calculate what happens with $l_p = \sqrt{\frac{hG}{c^3}}$, if we consider the size of the space deformation of $h_{\mu\upsilon}$ of the order of l_p, we are describing the universe at the Planck time $t_p \cong 10^{-43}s$, according with the cosmological picture of the early beginnings of the Universe.

What is the fuel to expand to different levels? The fuel is the energy tensor of the previous level, this is the source for the wave deforming the metric curvature.

If we consider the discrete steps used for this model of expansion, we can distinguish the contribution of two kinds of waves (56) and (57).

We get two different phenomenology according to the solutions of the differential equations: the former source contribution $h_a^{\mu}\hat{T}^{a\upsilon} + h_a^{\upsilon}\hat{T}^{a\mu}$ allows gravitational waves creating a new size of the metric as expansion.

This torsional source $\frac{i}{4}(h_a^{\mu}h_{c|b}^{\upsilon} + h_a^{\upsilon}h_{c|b}^{\mu})S^{abc}$, contributes to other metric changesets related to vortices creation through growing and damping metric perturbation.

The spiral shape of galaxies could be a good example of the effects of this mechanism on a huge scale.

We propose a different interpretation of the cosmological constant, expansion is due to the additional term of the energy-momentum tensor.

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Appendix A

Contribution from I) in details:

A1. $$-\frac{n}{8}(h_0^\mu h_{c|b}^v + h_0^v h_{c|b}^\mu)\varepsilon_{bck}\overline{\psi}\begin{pmatrix}\sigma_k & 0\\ 0 & -\sigma^k\end{pmatrix}\psi$$

$$= -\frac{n}{8}h_0^\mu(h_{2|1}^v - h_{1|2}^v)\overline{\psi}\begin{pmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{pmatrix}\psi - \frac{n}{8}h_0^\mu(h_{1|3}^v - h_{3|1}^v)\overline{\psi}\begin{pmatrix}\sigma^2 & 0\\ 0 & -\sigma^2\end{pmatrix}$$

$$-\frac{n}{8}h_0^\mu(h_{3|2}^v - h_{2|3}^v)\overline{\psi}\begin{pmatrix}\sigma^1 & 0\\ 0 & -\sigma^1\end{pmatrix}\psi$$

+symmetric part.

If $h_{c|b}^v = h_{b|c}^v$ this contribution is zero.

Contribution from II) in details:

A2. $$+\frac{n}{8}(h_a^\mu h_{c|0}^v + h_a^v h_{c|0}^\mu)\varepsilon_{ack}\overline{\psi}\begin{pmatrix}\sigma^k & 0\\ 0 & -\sigma^k\end{pmatrix}\psi$$

$$= +\frac{n}{8}(h_1^\mu h_{2|0}^v - h_2^\mu h_{1|0}^v)\overline{\psi}\begin{pmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{pmatrix}\psi + \frac{n}{8}h_0^\mu(h_3^\mu h_{1|0}^v - h_1^\mu h_{3|0}^v)\overline{\psi}\begin{pmatrix}\sigma^2 & 0\\ 0 & -\sigma^2\end{pmatrix}$$

$$+\frac{n}{8}(h_2^\mu h_{3|0}^v - h_3^\mu h_{2|0}^v)\overline{\psi}\begin{pmatrix}\sigma^1 & 0\\ 0 & -\sigma^1\end{pmatrix}\psi$$

+symmetric part.

If $h_{c|0}^v$ terms are zero, the related contribution A2 vanishes.

If the metric h_c^v does not change with respect to time in the tetrads frame, this term is zero.

Contribution from III) in details:

A3. $-\frac{n}{8}(h_k^{\mu}h_{c|b}^{v}+h_k^{\mu}h_{c|b}^{v})\varepsilon_{bck}\overline{\psi}\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\psi$

$$=-\frac{n}{8}h_3^{\mu}(h_{2|1}^{v}-h_{1|2}^{v})\overline{\psi}\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\psi-\frac{n}{8}h_2^{\mu}(h_{1|3}^{v}-h_{3|1}^{v})\overline{\psi}\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}\psi$$

$$-\frac{n}{8}h_1^{\mu}(h_{3|2}^{v}-h_{2|3}^{v})\overline{\psi}\begin{pmatrix}0 & l\\ -l & 0\end{pmatrix}\psi$$

+symmetric part.

If $h_{c|b}^{v}=h_{b|c}^{v}$ this contribution is zero.

APPENDIX B

Gravitational wave with source $h^{\mu v}=-\frac{16\pi G}{c^2}\hat{T}^{\mu v}$, the solution is so written [6]:

$$h^{\mu v}(\vec{x},t)=-\frac{4\pi G}{c^2}\int\frac{\hat{T}^{\mu v}}{r}(\vec{x}',t_{ret})d^3x', \qquad \text{(B0)}$$

$r=|\vec{x}-\vec{x}'|, \quad t_{ret}=t-\frac{r}{c}.$

We resume the kinds of differential equations considering as variables only x and t, we get

B1. $h_{tt}-h_{xx}=hT$, non-homogeneous wave equation for an elastic string, the forcing term is the coupling between the stress energy tensor and the metric.

The solutions depend on boundary conditions we can get stationary waves or linear advection solutions.

We consider $h(0,t)=h(L,t)=0$, with $L=l_p=\sqrt{\frac{hG}{c^3}}$, to get standing waves.

B2a. $h_{tt} - h_{xx} = \pm h_x T$.

B2b. $h_{tt} - h_{xx} = \pm h_t T$.

In detail, if we consider the full equation for example 2b) $h_{tt} - h_{xx} = h_t T$ the first order derivative gives rise to damping vibration while if we consider the other kind of 2b) $h_{tt} - h_{xx} = -h_t T$ with minus sign we get a growing vibration. These two terms are opposite currents.

The same if we consider 2a) $h_{tt} - h_{xx} = \pm h_x T$, here the first order derivative is respect to space index, again the same opposite currents.