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CLASSIFICATION OF ROTA-BAXTER OPERATORS ON D_8

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Abstract

Rota-Baxter operators on Lie groups were introduced recently as integrals of Rota-Baxter operators on Lie algebras with applications to integrable systems. In [5], Das and Rathee listed Rota-Baxter operators on the dihedral group D_8 which are extension of some homomorphism on the unique cyclic subgroup with order 4 of D_8. The aim of this paper is to classify Rota-Baxter operators on D_8. With the aid of Matlab procedures, it is proved that there exist 56 Rota-Baxter operators on D_8 including 18 splitting Rota-Baxter operators and 28 Rota-Baxter operators that are group endomorphisms. The Rota-Baxter endomorphisms form a semigroup with respect to composition.

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1. Introduction

In 1960, Rota-Baxter operators for commutative algebras was first considered in [3] by Baxter. Subsequently, various authors have contributed to the development of the theory of Rota-Baxter operators; see [9] for further details. In 2021, Guo et al. [10] introduced the notion of Rota-Baxter operators on groups and Lie groups and gave some basic examples and properties of these operators. A group G with a Rota-Baxter operator is called a *Rota-Baxter group*.

Soon after the intial breakthrough in [8], quite a few studies were carried out in this direction (see [1, 2, 4, 5, 6, 7, 11, 12]). More specifically, Bardakov and Gubarev [1] have investigated the relationship between skew left braces and Rota-Baxter groups, and shown that every Rota-Baxter group gives rise to a skew left brace and every left brace can be embedded into a Rota-Baxter group. In 2023, Bardakov and Gubarev [2] have given different constructions of Rota-Baxter operators on a group. In particular, they have proved that all Rota-Baxter operators on all sporadic simple groups are *splitting*. In the same year, Catino et al. [4] defined Rota-Baxter operators for Clifford semigroups and extended some results in [1] to Clifford semigroups. In [5], Das and Rathee investigated the extensions and automorphisms of Rota-Baxter groups, and in particular, in [5, Example 6.7], they listed Rota-Baxter operators on the dihedral group D_8 which are extension of some homomorphism on the unique cyclic subgroup with order 4 of D_8. On the other hand, Gao, Guo, Liu and Zhu [6] constructed free Rota-Baxter groups and Goncharov [7] investigated Rota-Baxter operators on cocommutative Hopf algebras, respectively. More recently, Li and Wang introduced Rota-Baxter systems and studied the relationship between Rota-Baxter systems and Rota-Baxter groups in [11]. Now the theory of Rota-Baxter groups has become a very active topic.

As mentioned earlier, Bardakov and Gubarev proved that all Rota-Baxter operators on 26 sporadic simple groups are splitting. A natural question is the following: what is the structure of Rota-Baxter operators on other groups? This problem seems very difficult in general case. In this paper, by using some key facts on Rota-Baxter operators on groups given in the texts [2] and [10], we have determined and classified all the Rota-Baxter operators on the dihedral group D_8 with the help of Matlab procedures. In particular, we have proved that there exist 56 Rota-Baxter operators on D_8 including 18 splitting Rota-Baxter operators and 28 Rota-Baxter endomorphisms. Moreover, the Rota-Baxter endomorphisms on D_8 form a semigroup with respect to the composition of mappings. The results of this paper show that the structure of Rota-Baxter operators on general groups is very complicated.

The paper is organized as follows. In Section 2, we first recall some necessary results on Rota-Baxter operators on groups and some basic properties of the dihedral group D_8. In particular, it is showed that the size of the image of a Rota-Baxter operator on D_8 may be 8, 4, 2 or 1 (see Lemmas 2.1 and 2.4 (1)). Then we have determined the Rota-Baxter operators on D_8 by considering the sizes of the images of these operators in Sections 3-5. More specifically, Section 3 explores the Rota-Baxter operators on D_8 whose images have 8 elements and shows that there are 12 candidates of this kind of operators. In Section 4, Rota-Baxter operators on D_8 whose images have 4 elements are considered and 28 candidates of this class of operators are described. Section 5 is devoted to Rota-Baxter operators on D_8 whose images contain at most 2 elements and 16 candidates of Rota-Baxter operators on D_8 are obtained. In the final section, among other things we prove that the above mentioned 56 candidates are all really Rota-Baxter operators, where there are 28 Rota-Baxter endomorphisms and 18 splitting Rota-Baxter operators.

2. Preliminaries

In this section, we shall recall some necessary results on Rota-Baxter operators on groups and some basic properties of the dihedral group D_8. Let $(G, \cdot)$ be a group with the identity e. From [10], a map $B : G \to G$ is called a *Rota-Baxter operator* of weight of 1 on G if

$$B(g)B(h) = B(gB(g)hB(g)^{-1}), \tag{2.1}$$

for all $g, h \in G$. In this case,

$$\ker B = \{x \in G \,|\, B(x) = e\} \text{ and } \operatorname{Im} B = \{B(x) \,|\, x \in G\},$$

are called the *kernel and image* of B, respectively. In the sequel, we shall call Rota-Baxter operators of weight of 1 *Rota-Baxter operators* for simplicity. Now let us recall some properties of Rota-Baxter operators on groups.

Lemma 2.1 (Lemmas 5 and 6 in [2], also see [10]). *Let B be a Rota-Baxter operator on a group G with identity e. Then $B(e) = e$. Moreover, $\ker B$ and $\operatorname{Im} B$ are subgroups of G.*

Lemma 2.2 (Proposition 2.6 in [1]). *Let B be a Rota-Baxter operator on a finite group G. Define a new multiplication $\circ_B$ on G as follows:*

$$g \circ_B h = gB(g)hB(g)^{-1} \text{ for all } g, h \in G.$$

Then $(G, \circ_B)$ is also a group and B is a group homomorphism from $(G, \circ_B)$ to $(G, \cdot)$. As a consequence, we have $(G, \circ_B)/\ker B \cong (\operatorname{Im} B, \cdot)$ and so $|\ker B| = \frac{|G|}{|\operatorname{Im} B|}$.

Lemma 2.3 (Lemma 7 in [2]). *Let B be a Rota-Baxter operator on a finite group G with identity e and $g \in G$. If $B(g) = e$, then $B(h) = B(gh)$ for any $h \in G$. In particular, if*

$$G = \coprod_{i \in I} (\ker B) g_i,$$

is the decomposition of G into the disjoint union of right cosets with respect to the subgroup $\ker B$, then $B(x) = B(y)$ if x and y lie in the same coset.

Now, we shall give some known results and properties of the dihedral group:

$$D_8 = \langle a, b \mid a^4 = b^2 = e, bab^{-1} = a^{-1} \rangle = \{e, a, a^2, a^3, b, ba, ba^2, ba^3\}.$$

We use e to denote the identity of D_8 throughout the remaining part of the paper.

Lemma 2.4. *With the above notation, we have the following well known facts:*

(1) *The subgroups of* D_8 *are*

$H_0 = \{e, b\}$, $H_1 = \{e, ba\}$, $H_2 = \{e, ba^2\}$, $H_3 = \{e, ba^3\}$, $N_0 = \{e\}$,

$N_1 = \{e, a^2\}$, $N_2 = \{e, a, a^2, a^3\}$, $N_3 = \{e, a^2, b, a^2b\}$,

$N_4 = \{e, a^2, ba, ba^3\}$, $N_5 = D_8$,

and the normal subgroups of D_8 *are* N_0, N_1, N_2, N_3, N_4, *and* N_5.

(2) *The center of* D_8 *is* $\{e, a^2\}$ *and* $ba = a^3b$, $a^2b = ba^2$, $ba^3 = ab$.

(3) *The conjugacy classes of* D_8 *are* $\{e\}$, $\{a^2\}$, $\{a, a^3\}$, $\{b, ba^2\}$, *and* $\{ba, ba^3\}$.

Lemma 2.5. *Let B be a Rota-Baxter operator on* D_8.

(1) *If* $x, y \in D_8$ *and* $B(x)y = yB(x)$, *then* $B(x)B(y) = B(xy)$. *In particular,* $B(x)B(a^2) = B(xa^2)$ *for all* $x \in D_8$.

(2) $B(x)^2 \in (\{B(a^2), e\} \cap \{a^2, e\})$ *for all* $x \in D_8$.

(3) *If* $\mathrm{Im}B \cap \{a, a^3\} \neq \emptyset$, *then* $B(a^2) = a^2$.

Proof. (1) Let $x, y \in D_8$. Then by (2.1), we have

$$B(x)B(y) = B(xB(x)yB(x)^{-1}) = B(xyB(x)B(x)^{-1}) = B(xy).$$

Since a^2 lies in the center of D_8 by Lemma 2.4 (2), it follows that $B(x)a^2 = a^2B(x)$ and so $B(x)B(a^2) = B(xa^2)$ for all $x \in D_8$.

(2) Let $x \in D_8$. we have $B(x)^2 = B(x)B(x) = B(xB(x)xB(x)^{-1})$. Observe that $B(e) = e$ by Lemma 2.1, it follows that $B(e)^2 = e$. Since $B(a)aB(a)^{-1} \in \{a, a^3\}$ by Lemma 2.4 (3), we have

$$B(a)^2 \in B(a\{a, a^3\}) = \{B(a^2), B(a^4)\} = \{B(a^2), B(e)\} = \{B(a^2), e\}.$$

By similar arguments, we can see that $B(x)^2 \in \{B(a^2), e\}$ for all $x \in D_8$. Moreover, it is easy to check that $g^2 \in \{a^2, e\}$ for all $g \in D_8$. Thus (2) follows.

(3) If $\mathrm{Im}B \cap \{a, a^3\} \neq \emptyset$, then $B(x_0) \in \{a, a^3\}$ for some $x_0 \in D_8$. This implies that $B(x_0)^2 = (a^3)^2 = a^2 \in \{B(a^2), e\}$ by item (2) above. This implies that $B(a^2) = a^2$. □

3. Rota-Baxter Operators B on D_8 with $|\mathrm{Im}B| = 8$

In view of Lemma 2.1, the kernel and image of a Rota-Baxter operator on a group G are always subgroups of G. So by Lemma 2.4 (1) the images of Rota-Baxter operators on D_8 may be H_i and N_j, where $i = 0, 1, 2, 3$ and $j = 0, 1, 2, 3, 4, 5$. We shall determine all the Rota-Baxter operators on D_8 by considering their images. In this section, we study the Rota-Baxter operators B on D_8 with $|\mathrm{Im}B| = 8$ (i.e., $\mathrm{Im}B = N_5 = D_8$).

Proposition 3.1. *Let B be a Rota-Baxter operator on D_8 with* $\mathrm{Im}B = D_8$. *Then B is one of the following permutations on D_8:*

$$B_1 = (ba, ba^3),\ B_2 = (b, ba^2),\ B_3 = (a, a^3),$$

$$B_4 = (a, a^3)(b, ba^2)(ba, ba^3),\ B_5 = (a, b)(a^3, ba^2)(ba, ba^3),$$

$$B_6 = (a, b, a^3, ba^2),\ B_7 = (a, ba^2, a^3, b),$$

$$B_8 = (a, ba^2)(a^3, b)(ba, ba^3),\ B_9 = (a, ba, a^3, ba^3),$$

$$B_{10} = (a, ba)(a^3, ba^3)(b, ba^2),\ B_{11} = (a, ba^3, a^3, ba),$$

$$B_{12} = (a, ba^3)(a^3, ba)(b, ba^2).$$

Moreover, $B_6^{-1} = B_7$ *and* $B_9^{-1} = B_{11}$.

Proof. Let B be a Rota-Baxter operator on D_8 with $\mathrm{Im}B = D_8$. Then B is bijective, and we have

$$B(e) = e,\ B(a^2) = a^2,$$

by Lemmas 2.1 and 2.5 (3), respectively. This implies that $B(a) \in \{a, a^3, b, ba, ba^2, ba^3\}$. We consider the following cases:

Case 1. $B(a) \in \{a, a^3\}$. In this case, we have $B(a)B(b) = B(aB(a)bB(a)^{-1}) = B(ba)$ by (2.1), Lemma 2.4 (2) and simple calculations. If $B(b) \in \{a, a^3\}$, then

$$B(ba) = B(a)B(b) \in \{a, a^3\}\{a, a^3\} = \{a^2, e\}.$$

This is impossible as B is bijective and $B(e) = e,\ B(a^2) = a^2$. So $B(b) \in \{b, ba, ba^2, ba^3\}$. If $B(b) \in \{ba, ba^3\}$, then

$$e = B(b)^2 = B(b)B(b) = B(bB(b)bB(b)^{-1}) = B(a)^2,$$

a contradiction. This implies that $B(b) \in \{b, ba^2\}$, and so $B(b)B(a) = B(ba^3)$ by (2.1), Lemma 2.4 (2) and simple calculations. From the above statements, we have

$$B(a) \in \{a, a^3\},\ B(b) \in \{b, ba^2\},\ B(a)B(b) = B(ba),\ B(b)B(a) = B(ba^3). \tag{3.1}$$

If $B(a) = a,\ B(b) = b,$ then we have

$$B(ba^2) = B(b)B(a^3) = ba^2 \text{ and } B(a^3) = B(a)B(a^2) = a^3$$

by Lemma 2.5 (1), and $B(ba) = ab = ba^3$ and $B(ba^3) = ba$ by (3.1), respectively. In this situation, we have

$$B(e) = e,\ B(a) = a,\ B(a^2) = a^2,\ B(a^3) = a^3,$$

$$B(b) = b,\ B(ba) = ba^3,\ B(ba^2) = ba^2,\ B(ba^3) = ba.$$

That is to say, $B = (ba, ba^3)$. Similarly, we can obtain the following facts: If $B(a) = a,\ B(b) = ba^2,$ then $B = (b, ba^2)$, if $B(a) = a^3,\ B(b) = b,$ then $B = (a, a^3)$, if $B(a) = a^3, B(b) = ba^2,$ then $B = (a, a^3)(b, ba^2)(ba, ba^3)$.

Case 2. $B(a) \in \{b, ba^2\}$. In this case, by (2.1) and Lemma 2.4(2) we have $B(a)B(b) = B(ab) = B(ba^3)$. In view of Lemma 2.5 (1) and the fact $B(a^2) = a^2$, we obtain that

$$B(a^3) = B(a)B(a^2) = B(a)a^2 \in \{ba^2, b\}.$$

This implies that $B(b) \notin \{ba^2, b\}$ as B is bijective. If $B(b) \in \{ba, ba^3\}$, then we can see that $B(b)B(a) = B(ba^3)$ by (2.1) again. This shows that $B(a)B(b) = B(b)B(a)$. However, it is easy to check that $xy \neq yx$ for all $x \in \{b, ba^2\}$ and $y \in \{ba, ba^3\}$. A contradiction. Observe that

$B(e) = e$, $B(a^2) = a^2$ and the fact that B is bijective, it follows that $B(b) \in \{a, a^3\}$. This yields that $B(b)B(a) = B(ba)$ by (2.1). From the above discussions, we have

$$B(a) \in \{b, ba^2\}, B(b) \in \{a, a^3\}, B(a)B(b) = B(ba^3), B(b)B(a) = B(ba). \tag{3.2}$$

If $B(a) = b$ and $B(b) = a$, then by (3.2), we have $B(ba) = ab = ba^3$ and $B(ba^3) = ba$. In view of Lemma 2.5 (1), we obtain that $B(ba^2) = B(b)$ $B(a^2) = aa^2 = a^3$ and $B(a^3) = B(a)B(a^2) = ba^2$. In this situation, $B = (a, b)(a^3, ba^2)(ba, ba^3)$. Similarly, we can obtain the following facts: If $B(a) = b$, $B(b) = a^3$, then $B = (a, b, a^3, ba^2)$, if $B(a) = ba^2$, $B(b) = a$, then $B = (a, ba^2, a^3, b)$, if $B(a) = ba^2$, $B(b) = a^3$, then $B = (a, ba^2)(a^3, b)$ (ba, ba^3).

Case 3. $B(a) \in \{ba, ba^3\}$. By similar arguments used in Case 2, we can show that

$$B(a) \in \{ba, ba^3\}, B(b) \in \{ba^2, b\}, B(a)B(b) = B(ba), B(b)B(a) = B(ba^3).$$

In this situation, we have

$$B = (a, ba)(a^3, ba^3)(b, ba^2), B = (a, ba, a^3, ba^3),$$

$$B = (a, ba^3)(a^3, ba)(b, ba^2),$$

or $B = (a, ba^3, a^3, ba)$. □

4. Rota-Baxter Operators B on D_8 with $|\mathrm{Im}B| = 4$

In this section, we study the Rota-Baxter operators B on D_8 with $|\mathrm{Im}\, B| = 4$. In this case, $\mathrm{Im}\, B$ may be N_2, N_3 or N_4 by Lemma 2.4 (1).

4.1. Rota-Baxter operators B on D_8 with Im$B = \{e, a, a^2, a^3\}$

Let B be a Rota-Baxter operator on D_8 with $\mathrm{Im}B = N_2 = \langle a \rangle = \{e, a, a^2, a^3\}$. Then $B(a), B(b) \in \{e, a, a^2, a^3\}$, and so

$$B(e) = e,\ B(a^i) = B(a)^i,\ B(ba^i) = B(b)B(a)^i,\ B(a^2) = a^2, \qquad (4.1)$$

for all positive integer i by Lemmas 2.1 and 2.5 (1), (3). In view of Lemma 2.2, we have $|\ker B| = \frac{8}{4} = 2$. Since $B(a^2) = a^2 \neq e$, we have $a^2 \notin \ker B$. By Lemma 2.4 (1) and the fact that the kernel of B is a subgroup of D_8 (see Lemma 2.1), we have the following cases:

Case 1: $\ker B = \{e, b\}$**.** The right cosets of G with respect to the subgroup $\{e, b\}$ are

$$\{e, b\},\ \{a, ba\},\ \{a^2, ba^2\},\ \{a^3, ba^3\}.$$

In view of Lemma 2.3 and (4.1), we have

$$B(e) = B(b) = e,\ B(a) = B(ba),\ B(ba^2) = B(a^2) = a^2,$$

$$B(ba^3) = B(a^3) = B(a)^3,$$

which implies that $B(a) \notin \{e, a^2\}$ as $\mathrm{Im}B = \langle a \rangle = \{e, a, a^2, a^3\}$. If $B(a) = a$, then B is

$$B_{13} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a & a^2 & a^3 & e & a & a^2 & a^3 \end{pmatrix}.$$

If $B(a) = a^3$, then B is

$$B_{14} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a^3 & a^2 & a & e & a^3 & a^2 & a \end{pmatrix}.$$

Case 2: $\ker B = \{e, ba\}$**.** By similar arguments in Case 2, B is one of the followings:

$$B_{15} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a & a^2 & a^3 & a^3 & e & a & a^2 \end{pmatrix},$$

$$B_{16} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a^3 & a^2 & a & a & e & a^3 & a^2 \end{pmatrix}.$$

Case 3: $\ker B = \{e, ba^2\}$**.** In this case, B is one of the followings:

$$B_{17} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a & a^2 & a^3 & a^2 & a^3 & e & a \end{pmatrix},$$

$$B_{18} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a^3 & a^2 & a & a^2 & a & e & a^3 \end{pmatrix}.$$

Case 4: $\ker B = \{e, ba^3\}$**.** In this case, B is one of the followings:

$$B_{19} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a & a^2 & a^3 & a & a^2 & a^3 & e \end{pmatrix},$$

$$B_{20} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a^3 & a^2 & a & a^3 & a^2 & a & e \end{pmatrix}.$$

4.2. Rota-Baxter operators B on D_8 with $\mathrm{Im}B = \{e, a^2, b, a^2b\}$

Let B be a Rota-Baxter operator on D_8 with $\mathrm{Im}B = N_3 = \langle a^2, b\rangle = \{e, a^2, b, a^2b\}$. Then $B(x) \in \{e, a^2, b, a^2b\}$, and so

$$B(e) = e,\ B(xa^2) = B(x)B(a^2),\ B(xb) = B(x)B(b) \tag{4.2}$$

by Lemmas 2.1 and 2.5 (1) for all $x \in D_8$. This implies that

$$B(a^3) = B(a)B(a^2),\ B(ba) = B(a^3b) = B(a^3)B(b) = B(a)B(a^2)B(b), \tag{4.3}$$

$$B(ba^2) = B(a^2b) = B(a^2)B(b),\ B(ba^3) = B(ab) = B(a)B(b). \qquad (4.4)$$

In view of Lemma 2.2, $|\ker B| = \frac{8}{4} = 2$. By Lemma 2.4 (1), we have the following cases:

Case 1: $\ker B = \{e, a^2\}$**.** The right cosets of G with respect to $\{e, a^2\}$ are:

$$\{e, a^2\}, \{a, a^3\}, \{b, ba^2\}, \{ba^3, ba\}.$$

In view of Lemma 2.3 and (4.3)-(4.4), we have

$$B(e) = B(a^2) = e,\ B(a) = B(a^3),\ B(b) = B(ba^2),$$

$$B(ba^3) = B(ba) = B(a)B(b).$$

Since $\mathrm{Im}B = \{e, a^2, b, a^2b\}$ and $\ker B = \{e, a^2\}$, we have $B(a), B(b) \in \{a^2, b, a^2b\}$ and $B(a) \neq B(b)$. Thus B is one of the followings:

$$B_{21} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a^2 & e & a^2 & b & ba^2 & b & ba^2 \end{pmatrix},$$

$$B_{22} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a^2 & e & a^2 & ba^2 & b & ba^2 & b \end{pmatrix},$$

$$B_{23} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & b & e & b & a^2 & ba^2 & a^2 & ba^2 \end{pmatrix},$$

$$B_{24} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & b & e & b & ba^2 & a^2 & ba^2 & a^2 \end{pmatrix},$$

$$B_{25} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & ba^2 & e & ba^2 & a^2 & b & a^2 & b \end{pmatrix},$$

$$B_{26} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & ba^2 & e & ba^2 & b & a^2 & b & a^2 \end{pmatrix}.$$

Case 2: $\ker B = \{e, b\}$. The right cosets of G with respect to $\{e, b\}$ are:

$$\{e, b\}, \{a, ba\}, \{a^2, ba^2\}, \{a^3, ba^3\}.$$

In view of Lemma 2.3 and (4.3)-(4.4), we have $B(b) = B(e) = e$ and

$$B(a)B(a^2) = B(a^3) = B(ba^3) = B(a)B(b) = B(a).$$

This gives $B(a^2) = e$ and so $a^2 \in \ker B$. A contradiction.

Case 3: $\ker B = \{e, ba\}$. The right cosets of G with respect to $\{e, ba\}$ are:

$$\{e, ba\}, \{a, ba^2\}, \{a^2, ba^3\}, \{a^3, b\}. \tag{4.5}$$

By Lemma 2.3, we have $B(a^2) = B(ba^3)$. As $\ker B = \{e, ba\}$, we have $B(a^2) \neq e$. Assume that $B(a^2) = B(ba^3) \in \{b, ba^2\}$. Since $ba^3(bba^3b^{-1}) = ba^3a^3b = ba^2b = a^2$ and

$$ba^3(ba^2ba^3(ba^2)^{-1}) = ba^3ba^2ba^3ba^2 = baab = a^2,$$

we have

$$B(a^2) = B(ba^3B(ba^3)ba^3B(ba^3)^{-1}) = B(ba^3)B(ba^3) = e,$$

and so $a^2 \in \ker B$. A contradiction. Thus $B(a^2) = B(ba^3) = a^2$ as $\operatorname{Im} B = \{e, a^2, b, ba^2\}$. Since $|\operatorname{Im} B| = 4$, we have $B(a), B(b) \notin \{e, a^2\}$ by Lemma 2.3 and (4.5), and so $B(a), B(b) \in \{b, ba^2\}$ and $B(a) \neq B(b)$. In view of (4.3)-(4.4), B is one of the followings:

$$B_{27} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & b & a^2 & ba^2 & ba^2 & e & b & a^2 \end{pmatrix},$$

$$B_{28} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & ba^2 & a^2 & b & b & e & ba^2 & a^2 \end{pmatrix}.$$

Case 4: $\ker B = \{e, ba^2\}$**.** The right cosets of G with respect to $\{e, ba^2\}$ are:

$$\{e, ba^2\}, \{a, ba^3\}, \{a^2, b\}, \{a^3, ba\}.$$

In view of (4.3)-(4.4) and Lemma 2.3, we have $B(a)B(b) = B(ba^3) = B(a)$, and so $B(b) = e$. This gives that $b \in \ker B$. A contradiction.

Case 5: $\ker B = \{e, ba^3\}$**.** Exchange the roles of a^3 and a in Case 4, we can obtain that B is one of the followings:

$$B_{29} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & b & a^2 & ba^2 & b & a^2 & ba^2 & e \end{pmatrix},$$

$$B_{30} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & ba^2 & a^2 & b & ba^2 & a^2 & b & e \end{pmatrix}.$$

4.3. Rota-Baxter operators B on D_8 with $\mathrm{Im}B = \{e, a^2, ba, ba^3\}$

Denote $c = a^3$ and $d = ba$. Observe that

$$D_8 = \langle c, d \rangle = \{e, c, c^2, c^3, d, dc, dc^2, dc^3\},$$

it follows that $\{e, c^2, d, dc^2\} = \{e, a^2, ba, ba^3\}$. Replacing a and b by c and d in Subsection 4.2, respectively, we have the following candidate Rota-Baxter operators on D_8 whose images are $\{e, a^2, ba, ba^3\}$:

$$B_{31} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a^2 & e & a^2 & ba & ba^3 & ba & ba^3 \end{pmatrix},$$

$$B_{32} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a^2 & e & a^2 & ba^3 & ba & ba^3 & ba \end{pmatrix},$$

$$B_{33} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & ba & e & ba & a^2 & ba^3 & a^2 & ba^3 \end{pmatrix},$$

$$B_{34} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & ba & e & ba & ba^3 & a^2 & ba^3 & a^2 \end{pmatrix},$$

$$B_{35} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & ba^3 & e & ba^3 & a^2 & ba & a^2 & ba \end{pmatrix},$$

$$B_{36} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & ba^3 & e & ba^3 & ba & a^2 & ba & a^2 \end{pmatrix},$$

$$B_{37} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & ba & a^2 & ba^3 & e & ba & a^2 & ba^3 \end{pmatrix},$$

$$B_{38} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & ba^3 & a^2 & ba & e & ba^3 & a^2 & ba \end{pmatrix},$$

$$B_{39} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & ba & a^2 & ba^3 & a^2 & ba^3 & e & ba \end{pmatrix},$$

$$B_{40} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & ba^3 & a^2 & ba & a^2 & ba & e & ba^3 \end{pmatrix}.$$

5. Rota-Baxter Operators B on D_8 with $|\mathrm{Im}B| \leq 2$

In this section, we study the Rota-Baxter operators B on D_8 with $|\mathrm{Im}B| \leq 2$. In this case, $\mathrm{Im}B$ may be N_0, N_1 or H_0, H_1, H_2, H_3 by Lemma 2.4 (1).

Let B be a Rota-Baxter operator on D_8 with $\mathrm{Im}B = N_1 = \langle a^2 \rangle = \{e, a^2\}$. In view of Lemma 2.2, we have $|\ker B| = \frac{8}{2} = 4$. By Lemma 2.4(1), we have the following cases:

Case 1: $\ker B = \{e, a, a^2, a^3\}$**.** In this case, since $\mathrm{Im}B = \{e, a^2\}$, it follows that B is

$$B_{41} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & e & e & e & a^2 & a^2 & a^2 & a^2 \end{pmatrix}.$$

Case 2: $\ker B = \{e, a^2, b, ba^2\}$**.** In this case, B is

$$B_{42} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a^2 & e & a^2 & e & a^2 & e & a^2 \end{pmatrix}.$$

Case 3: $\ker B = \{e, a^2, ba, ba^3\}$**.** In this case, B is

$$B_{43} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a^2 & e & a^2 & a^2 & e & a^2 & e \end{pmatrix}.$$

Let B be a Rota-Baxter operator on D_8 with $\mathrm{Im} B = \langle ba^i \rangle = \{e, ba^i\} = H_i$, $i = 0, 1, 2, 3$. Then by similar arguments as above, we obtain that B is one of the followings:

$$B_{44} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & e & e & e & b & b & b & b \end{pmatrix},$$

$$B_{45} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & b & e & b & e & b & e & b \end{pmatrix},$$

$$B_{46} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & b & e & b & b & e & b & e \end{pmatrix},$$

$$B_{47} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & e & e & e & ba & ba & ba & ba \end{pmatrix},$$

$$B_{48} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & ba & e & ba & e & ba & e & ba \end{pmatrix},$$

$$B_{49} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & ba & e & ba & ba & e & ba & e \end{pmatrix},$$

$$B_{50} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & e & e & e & ba^2 & ba^2 & ba^2 & ba^2 \end{pmatrix},$$

$$B_{51} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & ba^2 & e & ba^2 & e & ba^2 & e & ba^2 \end{pmatrix},$$

$$B_{52} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & ba^2 & e & ba^2 & ba^2 & e & ba^2 & e \end{pmatrix},$$

$$B_{53} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & e & e & e & ba^3 & ba^3 & ba^3 & ba^3 \end{pmatrix},$$

$$B_{54} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & ba^3 & e & ba^3 & e & ba^3 & e & ba^3 \end{pmatrix},$$

$$B_{55} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & ba^3 & e & ba^3 & ba^3 & e & ba^3 & e \end{pmatrix}.$$

Finally, let B be a Rota-Baxter operator on D_8 with $\mathrm{Im}B = \{e\} = N_0$. Then B is

$$B_{56} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & e & e & e & e & e & e & e \end{pmatrix}.$$

6. The Classification of Rota-Baxter Operators on D_8

By the statements in the previous sections, D_8 has no more than 56 Rota-Baxter operators, namely B_i, $i = 1, 2, \ldots, 56$. In this section, among other things we shall show that these 56 candidates are all really Rota-Baxter operators.

Let G be a group and denote the set of all maps from G to G by $\mathcal{T}(G)$. For each $B \in \mathcal{T}(G)$, define $\widetilde{B} \in \mathcal{T}(G)$ as follows: $\widetilde{B}(g) = g^{-1}B(g^{-1})$ for all $g \in G$. Then $\widetilde{\widetilde{B}} = B$. In fact, for $g \in G$, we have

$$\widetilde{\widetilde{B}}(g) = g^{-1}\widetilde{B}(g^{-1}) = g^{-1}gB(g) = B(g).$$

Now, define a binary relation ρ on $\mathcal{T}(G)$ as follows: for all $B, C \in \mathcal{T}(G)$,

$$B \rho C \text{ if and only if } C = \widetilde{B} \text{ or } C = B.$$

Obviously, ρ is an equivalence on $\mathcal{T}(G)$. Furthermore, we have the following lemma by direct calculations. Observe that on D_8, $B_j = \widetilde{B_i}$ if and only if

$$B_j(a) = a^3 B_i(a^3),\ B_j(a^3) = aB_i(a) \text{ and } B_j(x) = xB_i(x),$$

for all $x \in D_8 \setminus \{a, a^3\}$.

Lemma 6.1. *The 56 candidate Rota-Baxter operators B_1-B_{56} can be divided into the following 28 ρ-classes*:

$\{B_1, B_{42}\}, \{B_2, B_{43}\}, \{B_3, B_{56}\}, \{B_4, B_{41}\}, \{B_5, B_{36}\}, \{B_6, B_{55}\},$

$\{B_7, B_{49}\}, \{B_8, B_{34}\}, \{B_9, B_{45}\}, \{B_{10}, B_{23}\}, \{B_{11}, B_{51}\}, \{B_{12}, B_{25}\},$

$\{B_{13}, B_{21}\}, \{B_{14}, B_{44}\}, \{B_{15}, B_{32}\}, \{B_{16}, B_{47}\}, \{B_{17}, B_{22}\}, \{B_{18}, B_{50}\},$

$\{B_{19}, B_{31}\}, \{B_{20}, B_{53}\}, \{B_{24}, B_{39}\}, \{B_{26}, B_{38}\}, \{B_{27}, B_{35}\}, \{B_{28}, B_{48}\},$

$\{B_{29}, B_{54}\}, \{B_{30}, B_{33}\}, \{B_{37}, B_{46}\}, \{B_{40}, B_{52}\}.$

Define another binary relation σ on $\mathcal{T}(G)$ as follows: for all $B, C \in \mathcal{T}(G)$,

$B \sigma C$ if and only if there exists an automorphism φ on G such that $\varphi C = B\varphi$. (6.1)

Then σ is also an equivalence on $\mathcal{T}(G)$. The following lemma lists the automorphisms of D_8, which is well-known and can be proved easily.

Lemma 6.2. *The automorphisms of* D_8 *are listed as follows:*

$$\phi_1 = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a & a^2 & a^2 & b & ba & ba^2 & ba^3 \end{pmatrix},$$

$$\phi_2 = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a & a^2 & a^3 & ba & ba^2 & ba^3 & b \end{pmatrix},$$

$$\phi_3 = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a & a^2 & a^3 & ba^2 & ba^3 & b & ba \end{pmatrix},$$

$$\phi_4 = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a & a^2 & a^3 & ba^3 & b & ba & ba^2 \end{pmatrix},$$

$$\phi_5 = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a^3 & a^2 & a & b & ba^3 & ba^2 & ba \end{pmatrix},$$

$$\phi_6 = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a^3 & a^2 & a & ba & b & ba^3 & ba^2 \end{pmatrix},$$

$$\phi_7 = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a^3 & a^2 & a & ba^2 & ba & b & ba^3 \end{pmatrix},$$

$$\phi_8 = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a^3 & a^2 & a & ba^3 & ba^2 & ba & b \end{pmatrix}.$$

By Lemma 6.2 and routine calculations, we have the following result. Observe that

$$\phi^{-1}B\phi = \begin{pmatrix} e\phi & a\phi & a^2\phi & a^3\phi & b\phi & (ba)\phi & (ba^2)\phi & (ba^3)\phi \\ e(B\phi) & a(B\phi) & a^2(B\phi) & a^3(B\phi) & b(B\phi) & (ba)(B\phi) & (ba^2)(B\phi) & (ba^3)(B\phi) \end{pmatrix},$$

for any bijective transformation ϕ on D_8 and $B \in \mathcal{T}(D_8)$.

Lemma 6.3. *The 56 candidate Rota-Baxter operators* B_1*-*B_{56} *on* D_8 *can be divided into the following 18* σ*-classes*:

$$\{B_1, B_2\}, \{B_3\}, \{B_4\}, \{B_5, B_8, B_{10}, B_{12}\}, \{B_6, B_7, B_9, B_{11}\},$$

$$\{B_{13}, B_{15}, B_{17}, B_{19}\}, \{B_{14}, B_{16}, B_{18}, B_{20}\}, \{B_{21}, B_{22}, B_{31}, B_{32}\},$$

$$\{B_{23}, B_{25}, B_{34}, B_{36}\}, \{B_{24}, B_{26}, B_{33}, B_{35}\}, \{B_{27}, B_{30}, B_{38}, B_{39}\},$$

$$\{B_{28}, B_{29}, B_{37}, B_{40}\}, \{B_{41}\}, \{B_{42}, B_{43}\}, \{B_{44}, B_{47}, B_{50}, B_{53}\},$$

$$\{B_{45}, B_{49}, B_{51}, B_{55}\}, \{B_{46}, B_{48}, B_{52}, B_{54}\}, \{B_{56}\}.$$

Remark 6.4. We can obtain Lemma 6.3 by using Matlab. To this aim, we need the matrix representation of D_8. In fact, if we let

$$a = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

then the subgroup of $\mathrm{GL}(4, \mathbb{R})$ generated by a and b is isomorphic to D_8. By using this representation and corresponding procedures (see A.1 in Appendix), We can check whether any two candidate Rota-Baxter operators are σ-equivalent. For example, if we check

$$B = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a^2 & e & a^2 & b & ba^2 & b & ba^2 \end{pmatrix},$$

$$R = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a^2 & e & a^2 & ba^2 & b & ba^2 & b \end{pmatrix},$$

we only need to enter

$$[e, a, a\hat{}2, a\hat{}3, b, b*a, b*a\hat{}2, b*a\hat{}3; e, a\hat{}2, e, a\hat{}2, b, b*a\hat{}2, b, b*a\hat{}2]$$

$$[e, a, a\hat{}2, a\hat{}3, b, b*a, b*a\hat{}2, b*a\hat{}3; e, a\hat{}2, e, a\hat{}2, b*a\hat{}2, b, b*a\hat{}2, b]$$

after the cursor in Matlab. When you have entered this line, the sentence "*B* is equivalent to *R*" is displayed.

Denote the least equivalence of $\mathcal{T}(G)$ containing ρ and σ by δ. That is, $\delta = \rho \vee \sigma$ in the lattice of the equivalences in $\mathcal{T}(G)$. In view of Lemmas 6.1 and 6.3, we have the following result:

Lemma 6.5. *The 56 candidate Rota-Baxter operators on D_8 can be divided into the following 9 δ-classes:*

$$\{B_1, B_2, B_{42}, B_{43}\}, \{B_3, B_{56}\}, \{B_4, B_{41}\},$$

$$\{B_5, B_8, B_{10}, B_{12}, B_{23}, B_{25}, B_{34}, B_{36}\},$$

$$\{B_6, B_7, B_9, B_{11}, B_{45}, B_{49}, B_{51}, B_{55}\},$$

$$\{B_{13}, B_{15}, B_{17}, B_{19}, B_{21}, B_{22}, B_{31}, B_{32}\},$$

$$\{B_{14}, B_{16}, B_{18}, B_{20}, B_{44}, B_{47}, B_{50}, B_{53}\},$$

$$\{B_{24}, B_{26}, B_{27}, B_{30}, B_{33}, B_{35}, B_{38}, B_{39}\},$$

$$\{B_{28}, B_{29}, B_{37}, B_{40}, B_{46}, B_{48}, B_{52}, B_{54}\},$$

In order to achieve our purpose, we also need to state some known results.

Lemma 6.6 (Lemma 8 in [2], also see [10])**.** *Let B be a Rota-Baxter operator on a group G. Then the operator on G defined by the rule that*

$$\widetilde{B}(g) = g^{-1}B(g^{-1}) \text{ for all } g \in G,$$

is also a Rota-Baxter operator on G.

Lemma 6.7 (Lemma 9 in [2])**.** *Let B be a Rota-Baxter operator on a group G and φ be an automorphism of G. Then $\varphi^{-1}B\varphi$ is also a Rota-Baxter operator on G.*

Let G be a group. According to [2], if H and L are two subgroups of G such that $G = HL$ and $H \cap L = \{e\}$, then we call (H, L) an *exact pair* of G. In this case, we can define $B_{H,L}$ in $\mathcal{T}(G)$ as follows: $B_{H,L} : G \to G, hl \mapsto l^{-1}$. Moreover, we say an element B in $\mathcal{T}(G)$ *splitting* if $B = B_{H,L}$ for some exact pair (H, L) of G.

Lemma 6.8 (Example 15 in [2], also see [10]). *Let* (H, L) *be an exact pair of a group* G. *Then* $B_{H,L}$ *is a Rota-Baxter operator on* G. *In this case,* $\ker B = H$ *and* $\mathrm{Im} B = L$.

In view of Lemmas 2.4 (1) and 6.8, we have the following lemma by direct calculations.

Lemma 6.9. *The exact pairs of* D_8 *can be listed as follows:*

$$(N_0, D_8), (H_0, N_2), (H_1, N_2), (H_2, N_2), (H_3, N_2), (H_1, N_3), (H_3, N_3), (H_0, N_4), (H_2, N_4);$$

$$(N_2, H_0), (N_4, H_0), (N_2, H_1), (N_3, H_1), (N_2, H_2), (N_4, H_2), (N_2, H_3), (N_3, H_3), (D_8, N_0).$$

The corresponding splitting Rota-Baxter operators are:

$$B_3, B_{14}, B_{16}, B_{18}, B_{20}, B_{28}, B_{29}, B_{37}, B_{40};$$

$$B_{44}, B_{46}, B_{47}, B_{48}, B_{50}, B_{52}, B_{53}, B_{54}, B_{56}.$$

Proposition 6.10. *The transformations* B_1*-*B_{56} *mentioned above are all Rota-Baxter operators on* D_8.

Proof. By Lemmas 6.5-6.9, to check the 56 candidate Rota-Baxter operators, we only need to check the following 6 candidates:

$$B_1, B_4, B_5, B_6, B_{13}, B_{24}. \tag{6.2}$$

We can check these 6 candidates one by one certainly. But to avoid lengthy calculations, we can realize this verification process by using Matlab. By using the representation given in Remark 6.4 and corresponding procedures (see A.2 in Appendix), we can obtain that each item in (6.2) is a Rota-Baxter operator on D_8, and so all 56 candidates are really Rota-Baxter operators on D_8. For example, if we check that

$$B_1 = (ba, ba^3) = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a & a^2 & a^3 & b & ba^3 & ba^2 & ba \end{pmatrix},$$

is a Rota-Baxter operator, we only need to enter the second line

$$[e;\ a;\ a\hat{\ }2;\ a\hat{\ }3;\ b;\ b * a\hat{\ }3;\ b * a\hat{\ }2;\ b * a],$$

of B_1 after the cursor in Matlab. When you have entered this line, the sentence "The mapping is a Rota-Baxter operator" is displayed.

Remark 6.11. In [2, Theorem 53], it has been shown that all Rota-Baxter operators on 26 sporadic simple groups are necessarily splitting. However, by Lemma 6.9 and Proposition 6.10, there exist non-splitting Rota-Baxter operators in D_8.

Now, we consider another equivalence on the set of Rota-Baxter operators in D_8. To this aim, we need the notion of skew left braces from [8].

Definition 6.12. A skew left brace is a triple $(G, \cdot, \circ)$ such that $(G, \cdot)$ and $(G, \circ)$ are groups and

$$a \circ (b \cdot c) = (a \circ b) \cdot a^{-1} \cdot (a \circ c),$$

holds for all $a, b, c \in G$, where a^{-1} denotes the inverse of a in the group $(G, \cdot)$.

Bardakov and Gubarev [1] have obtained the relationship between skew left braces and Rota-Baxter groups. In particular, they have proved that Rota-Baxter groups give rise to skew left braces.

Lemma 6.13 (Proposition 3.1 in [1])**.** *Let $(G, \cdot)$ be a group and B be a Rota-Baxter operator on G. For all $x, y \in G$, define $x \circ_B y = xB(x)$ $yB(x)^{-1}$. Then $(G, \cdot, \circ_B)$ forms a skew left brace. In this case, $(G, \cdot, \circ_B)$ is called the skew left brace induced by the group $(G, \cdot)$ and the Rota-Baxter operator B.*

Let R and B be two Rota-Baxter operators on a group $(G, \cdot)$. If we define $R \sim B$ if the skew left brace $(G, \cdot, \circ_R)$ is isomorphic to the skew left brace $(G, \cdot, \circ_B)$, then $\sim$ forms an equivalence relation on the set of Rota-Baxter operators in $(G, \cdot)$. For this equivalence, we have the following result.

Lemma 6.14 (Corollary 3.12 in [12]). *Let R and B be two Rota-Baxter operators on a group $(G, \cdot)$. Then $R \sim B$ if and only if there exists an automorphism on $(G, \cdot)$ such that, for all $g \in G$, $\varphi(R(g))^{-1} B(\varphi(g))$ lies in the center of $(G, \cdot)$.*

By (6.1) and Lemma 6.14, one can obtain the following corollary:

Corollary 6.15. *Let R and B be two Rota-Baxter operators on a group $(G, \cdot)$. If $R \sigma B$, then $R \sim B$. In particular, if the center of G is trivial, then $R \sigma B$ if and only if $R \sim B$.*

Since the center of D_8 is $\{e, a^2\}$, by using Lemmas 6.2, 6.3, 6.14 and Corollary 6.15, we can obtain the following result by routine calculations:

Lemma 6.16. *The 56 Rota-Baxter operators B_1-B_{56} on D_8 can be divided into the following 7 $\sim$-classes:*

$$\{B_1, B_2, B_3, B_4\},$$

$$\{B_5, B_8, B_{10}, B_{12}, B_6, B_7, B_9, B_{11}\},$$

$$\{B_{13}, B_{15}, B_{17}, B_{19}, B_{14}, B_{16}, B_{18}, B_{20}\},$$

$$\{B_{21}, B_{22}, B_{31}, B_{32}, B_{44}, B_{47}, B_{50}, B_{53}\},$$

$$\{B_{23}, B_{25}, B_{34}, B_{36}, B_{45}, B_{49}, B_{51}, B_{55}\},$$

$$\{B_{24}, B_{26}, B_{33}, B_{35}, B_{27}, B_{30}, B_{38}, B_{39}, B_{28}, B_{29}, B_{37}, B_{40}, B_{46}, B_{48}, B_{52}, B_{54}\},$$

$$\{B_{41}, B_{42}, B_{43}, B_{56}\}.$$

Remark 6.17. We can also obtain Lemma 6.16 by using Matlab. In fact, by using the representation given in Remark 6.4 and Matlab procedures (see A.3 in Appendix), we can check whether any two Rota-Baxter operators are ~-equivalent. For example, if we check

$$B = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a & a^2 & a^3 & b & ba^3 & ba^2 & ba \end{pmatrix},$$

$$R = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a^3 & a^2 & a & b & ba & ba^2 & ba^3 \end{pmatrix},$$

we only need to enter

$$[e,\ a,\ a\hat{}2,\ a\hat{}3,\ b,\ b*a,\ b*a\hat{}2,\ b*a\hat{}3;$$

$$e,\ a,\ a\hat{}2,\ a\hat{}3,\ b,\ b*a\hat{}3,\ b*a\hat{}2,\ b*a],$$

$$[e,\ a,\ a\hat{}2,\ a\hat{}3,\ b,\ b*a,\ b*a\hat{}2,\ b*a\hat{}3;$$

$$e,\ a\hat{}3,\ a\hat{}2,\ a,\ b,\ b*a,\ b*a\hat{}2,\ b*a\hat{}3]$$

after the cursor in Matlab. When you have entered this line, the sentence "B and R satisfy the relation ~." is displayed.

In the following statements, we shall list the skew left braces induced by D_8 and its Rota-Baxter operators

$$B_1,\ B_5,\ B_{13},\ B_{44},\ B_{45},\ B_{46},\ B_{56},$$

respectively. By Lemmas 6.13 and 6.16, up to isomorphism, these skew left braces are the only skew left braces which can be induced by D_8 and its Rota-Baxter operators.

Case 1. The skew left brace $(D_8, \cdot, \circ_{B_1})$ induced by D_8 and B_1, where

$$x \circ_{B_1} y = xB_1(x)yB_1(x)^{-1},$$

for all $x, y \in D_8$:

$\circ_{B_1}$	e	a	a^2	a^3	b	ba	ba^2	ba^3
e	e	a	a^2	a^3	b	ba	ba^2	ba^3
a	a	a^2	a^3	e	ba	ba^2	ba^3	b
a^2	a^2	a^3	e	a	ba^2	ba^3	b	ba
a^3	a^3	e	a	a^2	ba^3	b	ba	ba^2
b	b	ba^3	ba^2	ba	e	a^3	a^2	a
ba	ba	b	ba^3	ba^2	a	e	a^3	a^2
ba^2	ba^2	ba	b	ba^3	a^2	a	e	a^3
ba^3	ba^3	ba^2	ba	b	a^3	a^2	a	e

In this case, $(D_8, \circ_{B_1}) \cong (D_8, \cdot)$.

Case 2. The skew left brace $(D_8, \cdot, \circ_{B_5})$ induced by D_8 and B_5, where

$$x \circ_{B_5} y = xB_5(x)yB_5(x)^{-1},$$

for all $x, y \in D_8$:

$\circ_{B_5}$	e	a	a^2	a^3	b	ba	ba^2	ba^3
e	e	a	a^2	a^3	b	ba	ba^2	ba^3
a	a	e	a^3	a^2	ba^3	ba^2	ba	b
a^2	a^2	a^3	e	a	ba^2	ba^3	b	ba
a^3	a^3	a^2	a	e	ba	b	ba^3	ba^2
b	b	ba	ba^2	ba^3	a^2	a^3	e	a
ba	ba	b	ba^3	ba^2	a	e	a^3	a^2
ba^2	ba^2	ba^3	b	ba	e	a	a^2	a^3
ba^3	ba^3	ba^2	ba	b	a^3	a^2	a	e

In this case, $(D_8, \circ_{B_5}) \cong (D_8, \cdot)$.

Case 3. The skew left brace $(D_8, \cdot, \circ_{B_{13}})$ induced by D_8 and B_{13}, where

$$x \circ_{B_{13}} y = xB_{13}(x)yB_{13}(x)^{-1},$$

for all $x, y \in D_8$:

$\circ_{B_{13}}$	e	a	a^2	a^3	b	ba	ba^2	ba^3
e	e	a	a^2	a^3	b	ba	ba^2	ba^3
a	a	a^2	a^3	e	ba	ba^2	ba^3	b
a^2	a^2	a^3	e	a	ba^2	ba^3	b	ba
a^3	a^3	e	a	a^2	ba^3	b	ba	ba^2
b	b	ba	ba^2	ba^3	e	a	a^2	a^3
ba	ba	ba^2	ba^3	b	a	a^2	a^3	e
ba^2	ba^2	ba^3	b	ba	a^2	a^3	e	a
ba^3	ba^3	b	ba	ba^2	a^3	e	a	a^2

In this case, $(D_8, \circ_{B_{13}}) \cong \mathbb{Z}_2 \times \mathbb{Z}_4$.

Case 4. The skew left brace $(D_8, \cdot, \circ_{B_{44}})$ induced by D_8 and B_{44}, where

$$x \circ_{B_{44}} y = xB_{44}(x)yB_{44}(x)^{-1},$$

for all $x, y \in D_8$:

$\circ_{B_{44}}$	e	a	a^2	a^3	b	ba	ba^2	ba^3
e	e	a	a^2	a^3	b	ba	ba^2	ba^3
a	a	a^2	a^3	e	ba^3	b	ba	ba^2
a^2	a^2	a^3	e	a	ba^2	ba^3	b	ba
a^3	a^3	e	a	a^2	ba	ba^2	ba^3	b
b	b	ba^3	ba^2	ba	e	a^3	a^2	a
ba	ba	b	ba^3	ba^2	a^3	a^2	a	e
ba^2	ba^2	ba	b	ba^3	a^2	a	e	a^3
ba^3	ba^3	ba^2	ba	b	a	e	a^3	a^2

In this case, $(D_8, \circ_{B_{44}}) \cong \mathbb{Z}_2 \times \mathbb{Z}_4$.

Case 5. The skew left brace $(D_8, \cdot, \circ_{B_{45}})$ induced by D_8 and B_{45}, where

$$x \circ_{B_{45}} y = xB_{45}(x)yB_{45}(x)^{-1},$$

for all $x, y \in D_8$:

$\circ_{B_{45}}$	e	a	a^2	a^3	b	ba	ba^2	ba^3
e	e	a	a^2	a^3	b	ba	ba^2	ba^3
a	a	e	a^3	a^2	ba^3	ba^2	ba	b
a^2	a^2	a^3	e	a	ba^2	ba^3	b	ba
a^3	a^3	a^2	a	e	ba	b	ba^3	ba^2
b	b	ba	ba^2	ba^3	e	a	a^2	a^3
ba	ba	b	ba^3	ba^2	a^3	a^2	a	e
ba^2	ba^2	ba^3	b	ba	a^2	a^3	e	a
ba^3	ba^3	ba^2	ba	b	a	e	a^3	a^2

In this case, $(D_8, \circ_{B_{45}}) \cong (D_8, \cdot)$.

Case 6. The skew left brace $(D_8, \cdot, \circ_{B_{46}})$ induced by D_8 and B_{46}, where

$$x \circ_{B_{46}} y = xB_{46}(x)yB_{46}(x)^{-1},$$

for all $x, y \in D_8$:

$\circ_{B_{46}}$	e	a	a^2	a^3	b	ba	ba^2	ba^3
e	e	a	a^2	a^3	b	ba	ba^2	ba^3
a	a	e	a^3	a^2	ba^3	ba^2	ba	b
a^2	a^2	a^3	e	a	ba^2	ba^3	b	ba
a^3	a^3	a^2	a	e	ba	b	ba^3	ba^2
b	b	ba^3	ba^2	ba	e	a^3	a^2	a
ba	ba	ba^2	ba^3	b	a^3	e	a	a^2
ba^2	ba^2	ba	b	ba^3	a^2	a	e	a^3
ba^3	ba^3	b	ba	ba^2	a	a^2	a^3	e

In this case, $(D_8, \circ_{B_{46}}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Case 7. The skew left brace $(D_8, \cdot, \circ_{B_{56}})$ induced by D_8 and B_{56}, where

$$x \circ_{B_{56}} y = xB_{13}(x)yB_{56}(x)^{-1},$$

for all $x, y \in D_8$:

$\circ_{B_{56}}$	e	a	a^2	a^3	b	ba	ba^2	ba^3
e	e	a	a^2	a^3	b	ba	ba^2	ba^3
a	a	a^2	a^3	e	ba^3	b	ba	ba^2
a^2	a^2	a^3	e	a	ba^2	ba^3	b	ba
a^3	a^3	e	a	a^2	ba	ba^2	ba^3	b
b	b	ba	ba^2	ba^3	e	a	a^2	a^3
ba	ba	ba^2	ba^3	b	a^3	e	a	a^2
ba^2	ba^2	ba^3	b	ba	a^2	a^3	e	a
ba^3	ba^3	b	ba	ba^2	a	a^2	a^3	e

In this case, $(D_8, \circ_{B_{56}}) \cong (D_8, \cdot)$.

In the end of this paper, we determine so-called Rota-Baxter endomorphisms on D_8. Recall that a *Rota-Baxter endomorphism* (respectively, *automorphism*) on a group is a Rota-Baxter operator which is also an endomorphism (respectively, automorphism). In the sequel, we consider Rota-Baxter endomorphisms on D_8. The following result is useful.

Lemma 6.18 (Proposition 21 in [2]). *Let G be a group. If G has a Rota-Baxter automorphism, then G is abelian. On the other hand, if B is an endomorphism on G such that* ImB *is an abelian subgroup of G, then B is a Rota-Baxter endomorphism.*

In view of the first part of Lemma 6.18 and the fact that D_8 is non-ableian, there is no Rota-Baxter automorphism on D_8. On the other hand, the kernel of a Rota-Baxter endomorphism on D_8 is certainly a normal subgroup of D_8. By Lemma 2.4 (1) and the discussions in Sections 3-5, the candidates for Rota-Baxter endomorphisms on D_8 are B_{21}-B_{26}, B_{31}-B_{36}, and B_{41}-B_{56}. Again by using the representation given in Remark 6.4 and Matlab procedures (see A.4 in Appendix), we can show these candidates are really Rota-Baxter endomorphisms on D_8. For example, if we check

$$B_{21} = \begin{pmatrix} e & a & a^2 & a^3 & b & ba & ba^2 & ba^3 \\ e & a^2 & e & a^2 & b & ba^2 & b & ba^2 \end{pmatrix},$$

is an endomorphism, we only need to enter the second line

$$[e;\ a\hat{}2;\ e;\ a\hat{}2;\ b;\ b * a\hat{}2;\ b;\ b * a\hat{}2]$$

of B_{21} after the cursor in Matlab. When you have entered this line, the sentence "The mapping is a Rota-Baxter endomorphism" is displayed.

Let B be an endomorphism on D_8 which is not an automorphism. Then $\mathrm{Im}B$ is a subgroup of D_8 and $\mathrm{Im}B \neq D_8$, and so $\mathrm{Im}B$ is abelian by Lemma 2.4 (1). By the second part of Lemma 6.18, B is a Rota-Baxter endomorphism on D_8. Thus non-automorphism endomorphisms are exactly Rota-Baxter endomorphisms on D_8. It is obvious that the composition of any two non-automorphism endomorphisms is again a non-automorphism endomorphism on D_8. In view of the statements in the previous paragraph and Lemma 6.2, we have the following corollary:

Corollary 6.19. *The semigroup of endomorphisms on* D_8 *is*

$$\mathrm{End}D_8 = \{\phi_1, \ldots, \phi_6;\ B_{21}, \ldots, B_{26};\ B_{31}, \ldots, B_{36};\ B_{41}, \ldots, B_{56}\},$$

which contains 34 elements, and the set of Rota-Baxter endomorphisms on D_8 *forms a 28-element subsemigroup of* $\mathrm{End}D_8$.

Remark 6.20. We observe that the set of bijective Rota-Baxter operators

$$U = \{B_1, B_2, \dots, B_{12}\},$$

does not form a subgroup of the symmetric group $\mathrm{Sym}(D_8)$ on D_8. In fact, U does not contain the identity permutation on D_8.

Now we summarize our results as a theorem.

Theorem 6.21. *The set of Rota-Baxter operators on* D_8 *is* $\{B_1, B_2, \dots, B_{56}\}$ *in which*

$$B_{21}, \dots, B_{26};\ B_{31}, \dots, B_{36};\ B_{41}, \dots, B_{56},$$

are 28 Rota-Baxter endomorphisms and

$$B_3, B_{14}, B_{16}, B_{18}, B_{20}, B_{28}, B_{29}, B_{37}, B_{40};$$

$$B_{44}, B_{46}, B_{47}, B_{48}, B_{50}, B_{52}, B_{53}, B_{54}, B_{56},$$

are 18 *splitting Rota-Baxter operators.*

We end the paper with some perspective and outstanding questions. In this paper, we have determined and classified all the Rota-Baxter operators on D_8 by some known facts on Rota-Baxter operators of groups in [2] and [10] together with necessary Matlab procedures. From the results obtained in this paper, we can recognize that the construction of Rota-Baxter operators on a group is complicated even if the construction of the corresponding group is relative simple. In the present paper, we have only considered the group D_8. This suggests the following problem naturally.

Problem 6.22. How to generalize the results in this paper to all dihedral groups D_{2n}?

By the comments before Corollary 6.19, any non-automorphism endomorphism on D_8 is a Rota-Baxter operator. In fact, all finite inner abelian groups (Recall that a group is called *inner abelian* if any proper subgroup of this group is abelian) have this property by Lemmas 2.1 and 6.18. This suggests the following problem.

Problem 6.23. How to characterize those groups in which every non-automorphism endomorphism is a Rota-Baxter operator?

In the statements following Remark 6.17, up to isomorphism we have determined all 7 skew left braces which can be induced by D_8 and its Rota-Baxter operators. By using GAP ([13]), one can know that there are 12 skew left brace structures over D_8 up to isomorphism. Hence, there are now 5 skew left brace structures on D_8 that cannot be obtained from Rota-Baxter operators. So the following problem seems meaningful.

Problem 6.24. Determine all 5 skew left brace structures on D_8 that cannot be obtained from Rota-Baxter operators.

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http://www.gapsystem.org

■

A. Appendix

A.1. The codes to check whether two candidate Rota-Baxter operators are σ-equivalent

---

```
e=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1];
a=[0 0 0 1;1 0 0 0;0 1 0 0;0 0 1 0];
b=[0 0 0 1;0 0 1 0;0 1 0 0;1 0 0 0];
A=[e,a,a^2,a^3,b,b*a,b*a^2,b*a^3];
T=[e,a,a^2,a^3,b,b*a,b*a^2,b*a^3;
e,a,a^2,a^3,b*a,b*a^2,b*a^3,b;
e,a,a^2,a^3,b*a^2,b*a^3,b,b*a;
e,a,a^2,a^3,b*a^3,b,b*a,b*a^2;
e,a^3,a^2,a,b,b*a^3,b*a^2,b*a;
e,a^3,a^2,a,b*a,b,b*a^3,b*a^2;
e,a^3,a^2,a,b*a^2,b*a,b,b*a^3;
e,a^3,a^2,a,b*a^3,b*a^2,b*a,b];
B=input ('Please enter operator B.');
R=input ('Please enter operator R.');
c=[1;5;9;13;17;21;25;29];
u=0;
Z=zeros (4,32);
G= [A;Z];
H= [A;Z];
for i=1:8
F= [A;T(c(i,:):c(i,:)+3,1:32)];
for j=1:8
```

```
for k=1:8
if F(5:8,c(j,:):c(j,:)+3)==A(:,c(k,:):c(k,:)+3)
G(5:8,c(j,:):c(j,:)+3)=B(5:8,c(k,:):c(k,:)+3);
else
end
end
end
for b=1:8
for d=1:8
if R(5:8,c(b,:):c(b,:)+3)==A(:,c(d,:):c(d,:)+3)
H(5:8,c(b,:):c(b,:)+3)=F(5:8,c(d,:):c(d,:)+3);
else
end
end
end
if G= =H
u=u+1;
else
end
end
if u>0
disp('B and R are equivalent.')
else
disp ('B and R are not equivalent.')
end
```

A.2. The codes to check that a mapping is a Rota-Baxter operator

---

```
e=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1];
a=[0 0 0 1;1 0 0 0;0 1 0 0;0 0 1 0];
b=[0 0 0 1;0 0 1 0;0 1 0 0;1 0 0 0];
A=[e;a;a^2;a^3;b;b*a;b*a^2;b*a^3];
B=input ('Please enter the matrix composed by the images of each element corresponding to matrix A.');
c=[1;5;9;13;17;21;25;29];
O=zeros(8,8);
R=ones(8,8);
for i=1:8
for j=1:8
A(c(i,:):c(i,:)+3,:)*B(c(i,:):c(i,:)+3,:)*A(c(j,:):c(j,:)+3,:)
*inv(B(c(i,:):c(i,:)+3,:));
for k=1:8
if A(c(i,:):c(i,:)+3,:)*B(c(i,:):c(i,:)+3,:)*A(c(j,:):c(j,:)+3,:)
*inv(B(c(i,:):c(i,:)+3,:))==A(c(k,:):c(k,:)+3,:)
s=k;
else
end
end
if B(c(i,:):c(i,:)+3,:)*B(c(j,:):c(j,:)+3,:)==B(c(s,:):c(s,:)+3,:)
O(i,j)=1;
else
end
end
end
```

```
if O= =R
disp('The mapping is a Rota-Baxter operator.')
else
disp('The mapping is not a Rota-Baxter operator.')
end
```

A.3. The codes to check whether two Rota-Baxter operators are ~-equivalent

---

```
e=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1];
a=[0 0 0 1;1 0 0 0;0 1 0 0;0 0 1 0];
b=[0 0 0 1;0 0 1 0;0 1 0 0;1 0 0 0];
A=[e,a,a^2,a^3,b,b*a,b*a^2,b*a^3];
T=[e,a,a^2,a^3,b,b*a,b*a^2,b*a^3;
e,a,a^2,a^3,b*a,b*a^2,b*a^3,b;
e,a,a^2,a^3,b*a^2,b*a^3,b,b*a;
e,a,a^2,a^3,b*a^3,b,b*a,b*a^2;
e,a^3,a^2,a,b,b*a^3,b*a^2,b*a;
e,a^3,a^2,a,b*a,b,b*a^3,b*a^2;
e,a^3,a^2,a,b*a^2,b*a,b,b*a^3;
e,a^3,a^2,a,b*a^3,b*a^2,b*a,b];
B=input ('Please enter operator B.');
R=input ('Please enter operator R.');
c=[1;5;9;13;17;21;25;29];
u= zeros (8);
p=0;
q= zeros (1,8);
```

```
Z=zeros (4,32);
E= zeros (1,8);
I=ones (1,8);
G=[A;Z];
H=[A;Z];
for r=1:8
F=[A;T(c(r,:):c(r,:)+3,1:32)];
for i=1:8
for j=1:8
if R(5:8,c(i,:):c(i,:)+3)==A(:,c(j,:):c(j,:)+3)
H(5:8,c(i,:):c(i,:)+3)=F(5:8,c(j,:):c(j,:)+3);
else
end
end
end
for k=1:8
for s=1:8
if F(5:8,c(k,:):c(k,:)+3)==A(:,c(s,:):c(s,:)+3)
G(5:8,c(k,:):c(k,:)+3)=B(5:8,c(s,:):c(s,:)+3);
else
end
end
end
for t=1:8
P= inv(G(5:8,c(t,:):c(t,:)+3))* H(5:8,c(t,:):c(t,:)+3);
if P==e
```

```
u(r,t)=1;
else
end
if P==a^2
u(r,t)=1;
else
end
end
if u(r,1:8)==I
q(:,r)=r;
p=p+1;
else
end
end
if p>=1
disp ('B and R satisfy the relation ~.')
else
disp ('B and R do not satisfy the relation ~.')
end
```

A.4. The codes to check that a Rota-Baxter operator is an endomorphism

---

```
e=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1];
a=[0 0 0 1;1 0 0 0;0 1 0 0;0 0 1 0];
b=[0 0 0 1;0 0 1 0;0 1 0 0;1 0 0 0];
A=[e;a;a^2;a^3;b;b*a;b*a^2;b*a^3];
B=input ('Please enter the matrix composed by the images of each element corresponding to matrix A.');
c=[1;5;9;13;17;21;25;29];
O=zeros(8,8);
R=ones(8,8);
for i=1:8
for j=1:8
C=A(c(i,:):c(i,:)+3,:)*A(c(j,:):c(j,:)+3,:);
D=B(c(i,:):c(i,:)+3,:)*B(c(j,:):c(j,:)+3,:);
for k=1:8
if C= =A(c(k,:):c(k,:)+3,:)
t=k;
else
end
end
if D= =B(c(t,:):c(t,:)+3,:)
O(i,j)=1;
else
end
```

```
end
end
if O= =R
disp ('The mapping is a Rota--Baxter endomorphism.')
else
disp ('The mapping is not a Rota--Baxter endomorphism.')
end
```