Research and Communications in Mathematics and Mathematical Sciences
Vol. 15, Issue 1, 2023, Pages 49-87
ISSN 2319-6939
Published Online on October 28, 2023

http://jyotiacademicpress.org

PARAMETRIC EXTENSIONS OF SOME REFERENCED TWO-DIMENSIONAL STRICT ARCHIMEDEAN COPULAS

CHRISTOPHE CHESNEAU

Department of Mathematics
LMNO
University of Caen-Normandie
14032 Caen
France
e-mail: chesneau.christophe@gmail.com

Abstract

Copulas are multi-dimensional functions used to describe the dependence or association between variables separately from their marginal distributions. Among the numerous types of copulas, the strict Archimedean copulas are the most popular. However, the same short list of Archimedean copulas always attracts attention; a lot of strict Archimedean copulas are often discarded because they have limited tail dependence or insufficient flexibility in their shapes. In this article, we attempt to rehabilitate some of the understudied strict Archimedean copulas by making their properties more attractive via new functional modifications depending on several parameters. For each of them, the main contribution is theoretical; it consists of determining the range of admissible values for the involved parameters. Then, we concentrate on the two that show the greatest promise, which have the advantages of simple expressions and adaptable dependence qualities as a result of various tuning configurations. The first one extends the famous Clayton copula and depends on

2020 Mathematics Subject Classification: 60E15, 62H99.

Keywords and phrases: Archimedean copulas, parametric transformation, symmetry, orrelation, statistical modelling.

Communicated by Suayip Yuzbasi.

Received July 18, 2023; Revised September 26, 2023

three complementary parameters. It has the particularity of allowing negative dependence, which was not the case with the original Clayton copula. The second one can be presented as a new two-parameter trigonometric copula. We investigate several of their main properties, such as symmetry, tail dependence, and correlation; the median correlation and tau of Kendall are considered. The theoretical results are supported by graphics and numerical tables.

1. Introduction

The copula approach was developed in probability theory by Abe Sklar, an American mathematician, in the 1950s and 1960s (see [28]). In a few words, this approach allows the understanding of multi-dimensional distributions. What is commonly called "copulas" can be presented as multi-dimensional functions that separate the modelling of the margins from the dependence structure of a multi-dimensional distribution. They are helpful in situations where the marginal distributions are known but the dependence structure is not. From a more statistical viewpoint, copulas can also be used to model non-linear dependence structures and simulate data from complex multi-dimensional distributions. They have applications in finance, actuarial science, medicine, biology, informatics, and environmental science, among other fields. Hence, the copula approach is crucial for modelling and analyzing multi-dimensional data. The main information on this topic can be found in [20], [22], [16] and [23]. For significant recent developments, the following list of references can be consulted: [3], [5], [1], [18], [27], [11], [6], [10], [14], [15] and [8]. Applications with modern data analysis problems can be found in [2], [31] and [26].

There are many types of copulas that are used in probability and statistics. Among them, we may mention the Gaussian and Student type copulas, which are popular choices for modelling financial data; they allow for easy modelling of tail dependence; the Farlie-Gumbel-Morgenstern (FGM) type copulas, which are appreciated for their mathematical simplicity and their controlled perturbed action on the independence copula; and the Archimedean copulas, which can be viewed as an easy, manageable family of copulas designed for modelling non-linear dependence structures. As a matter of fact, each copula has its own strengths and weaknesses, and the choice of which copula to use depends on the specific problem at hand.

Archimedean copulas are frequently used in multi-dimensional statistical analysis due to their adaptability, simplicity, and the fact that most mathematical software supports them. Indeed, the most popular Archimedean copulas are implemented in free packages in R (with the eponymous package copula), Python, Matlab, etc. They are based on the idea of a generator function, which makes it possible to build numerous different kinds of copulas with various tail dependences and correlation structures. To select the best copula from a list fixed in advance, a thorough analysis of the data is required. It is crucial to keep in mind that the choice of copula ultimately depends on the particular problem at hand. Anyway, in the family of the Archimedean copulas, it is surprising to see that the same short list of members is often considered. This list is composed by the Gumbel-Hougaard, Ali-Mikhail-Haq, Frank, Joe, and Clayton copulas. See, for instance, the studies in [12], [13], [4] and [32]. However, many of the other Archimedean copulas might be suitable for different uses and data sets. In particular, the thirteen copulas presented in Chapter 4 of [22] are the most well-known among the group of "strict" Archimedean copulas, but a lot of them have been unfairly overlooked. Indeed, they can still be more flexible with certain mathematical modifications to gain competitiveness against others. To this end, a theoretical work must come before everything. The first steps in this sense have been taken in [21], where it is indicated how to modify a generator function with scale and shape parameters to extend any of the existing Archimedean copulas. Beyond such a standard transformation, the study in [9] proposed an extension of the Clayton copula by the use of an additional power function in the generator function. With two parameters, under some interdependence conditions on these parameters, the obtained copula reveals itself to be a suitable alternative to the Clayton copula. It allows negative dependence, among other notable properties. On the other hand, the study in [7] revisited the

Nelsen strict Archimedean copula numbers 10 and 17, known as the NSA10 and NSA17 copulas. More precisely, extensions of them have been elaborated with the addition of one parameter in each of them. The presence of this parameter enhanced the range of values of the other involved parameters and allowed the reach of new levels in terms of correlation ranges.

This article provides the following theoretical contributions to this subject: In the first part, we generalize four generator functions by adding new parameters using various schemes, including the addition or composition of power functions. The baseline generator functions taken as references correspond to (a) the strict version of the Clayton generator function (see [22, Equation (4.2.1)]), (b) the Nelsen strict generator function number 10 (see [22, Equation (4.2.10)]), (c) the Nelsen strict generator function number 19 (see [22, Equation (4.2.19)]), and (d) the cosecant strict generator function in [24] (with $\theta = 1$) (see [24, page 41]). We determine the possible values of the involved parameters, making our generalized generator functions valid from a mathematical viewpoint. The established strict generator functions are able to produce numerous innovative Archimedean copulas with different targets in terms of modelling. In the second part, in order to support this claim, we focus on the two most promising copulas, which have the features of having a simple expression and attractive dependence properties. The first one generalizes the Clayton copula and depends on three parameters. The second one is a new two-parameter trigonometric copula, also depending on a tuning angle parameter. For each of them, we examine the effects of the involved parameters, the symmetry (diagonal and radial), lower and upper tail dependences, and correlation properties. We mainly concentrate on the medial correlation and a benchmark measure established by Kendall: the tau of Kendall. In particular, in contrast to the original Clayton copula, which can only model positive dependency, the proposed generalized Clayton copula has the notable property of permitting negative dependence. When appropriate, several figures and tables are offered to support the theory.

The rest of the article is organized as follows: Section 2 presents the main results on the generalized strict generator functions, along with the detailed proofs. Section 3 completes the previous section by describing the two strict Archimedean copulas mentioned. The article ends with a conclusion in Section 4.

2. Generalized Strict Generator Functions

This section presents four new generalized strict generator functions, which are the main theoretical results of the article. Before that, the precise definition of a strict generator function needs to be recalled.

2.1. Definitions

The mathematical requirements of a valid strict generator function are detailed below.

Definition 1. A function $\eta(u), u \in [0, 1]$, is said to be a strict generator function if and only if the following conditions hold:

C1: Zero value: $\eta(1) = 0$,

C2: Divergence: $\lim_{u \to 0} \eta(u) = +\infty$,

C3: Strict decrease: for any $u \in [0, 1)$, $\eta'(u) < 0$ and $\eta'(1) \leq 0$,

C4: Convexity: for any $u \in [0, 1]$, $\eta''(u) \geq 0$.

The notion of a strict generator function is at the basis of the definition of strict Archimedean copulas; any strict Archimedean copula is defined by the addition and composition of a strict generator function. Further details will be given in Section 3, mainly in Definition 2.

As a first example, the Clayton strict generator function is defined by $\eta(u) = u^{-b} - 1, u \in [0, 1]$, with $b > 0$, which is a slight modification of the one in [22, Equation (4.2.1)]. The corresponding strict Archimedean copula is known to be diagonally symmetric, invariant under truncation, and to have a modulable lower tail dependence. It is commonly used in risk management and financial modelling to assess the joint behavior of variables, such as credit risk and extreme events. Except in [9], attempts to generalize or extend it are rare.

The next result presents a generalized version of this generator function involving three parameters. Conditions on these parameters are required, but we will see later that they are easily manageable.

Proposition 2.1. *The following function defines a new strict generator function*:

$$\eta(u) = (au^{-b} - 1)^c - (a-1)^c, \quad u \in [0, 1], \tag{1}$$

for $b > 0$, $c > 0$, *and a such that*

$$a \geq \max\left(1, \frac{b+1}{bc+1}\right).$$

Proof. The proof is centered around the four conditions presented in Definition 1.

For C1: We clearly have

$$\eta(1) = (a \times 1^{-b} - 1)^c - (a-1)^c = 0.$$

For C2: Since $a \geq 1$, $b > 0$, and $c > 0$, and $\lim_{u \to 0} u^{-b} = +\infty$, we have

$$\lim_{u \to 0} \eta(u) = \lim_{u \to 0} [(au^{-b} - 1)^c - (a-1)^c] = +\infty.$$

For C3: For any $u \in [0, 1)$, we have

$$\eta'(u) = -\, abcu^{-b-1}(au^{-b} - 1)^{c-1}.$$

Since $a \geq 1$, $b > 0$, and $c > 0$, it is clear that $\eta'(u) < 0$. Moreover, it is obvious that $\eta'(1) = -\, abc(a-1)^{c-1} \leq 0$.

For C4: For any $u \in [0, 1]$, we have

$$\eta''(u) = abc(au^{-b} - 1)^c \frac{abc + a - (b+1)u^b}{u^2(a - u^b)^2}.$$

Since $b > 0$, $c > 0$, and $a \geq (b+1)/(bc+1)$, we have

$$abc + a - (b+1)u^b \geq abc + a - (b+1) = a(bc+1) - (b+1) \geq 0.$$

Since the other main sub-terms are non-negative, we have $\eta''(u) \geq 0$.

This completes the proof; the considered function $\eta(u)$ is validated as a strict generator function. □

Based on Proposition 2.1 and Equation (1), the Clayton strict generator function is obtained with $a = 1$ and $c = 1$. Thus, our generalization is motivated by the presence of a and c, which can have significant effects on the shapes and dependence properties of the copula. This aspect will be developed in detail in Section 3. In particular, we will show that, in contrast to the original Clayton copula, which can only model positive dependency, the proposed generalized Clayton copula has the unique property of permitting negative dependence.

An important remark is that, if $c \geq 1$, then $bc + 1 \geq b + 1$, and the condition on a simply becomes $a \geq 1$. In this case, the assumptions on the parameters become $a \geq 1$, $b > 0$, and $c \geq 1$, making them completely independent. This fact is clearly attractive from a statistical perspective. On the other hand, for $c \in (0, 1)$, we have $a \geq (b+1)/(bc+1)$. We will see later that this case is important to make the corresponding Archimedean copula able to capture the negative dependence.

Let us now focus on the Nelsen strict generator function number 10 (see [22, Equation (4.2.10)]). It is mainly defined with logarithmic and power functions, and, more precisely, $\eta(u) = \log(2u^{-a} - 1)$, $u \in [0, 1]$, with $a > 0$. This special strict generator has found theoretical developments in [7] and practical use in [17], mainly.

The result below uses a power function to explore a new generalization.

Proposition 2.2. *The following function defines a new strict generator function*:

$$\eta(u) = \log\left[\frac{1}{u^a}(2 + bu) - (1 + b)\right], \quad u \in [0, 1], \tag{2}$$

for $b \in [-1, 0]$, *and*

$$\frac{b^2}{(b+2)(b+1)} \leq a \leq 1.$$

Proof. The four requirements listed in Definition 1 serve as the foundation for the proof.

For C1: It is clear that

$$\eta(1) = \log\left[\frac{1}{1^a} \times (2 + b \times 1) - (1 + b)\right] = \log(1) = 0.$$

For C2: Since $a > 0$, we have $\lim_{u \to 0}(2 + bu)/u^a = +\infty$, implying that

$$\lim_{u\to 0} \eta(u) = \lim_{u\to 0} \log\left[\frac{1}{u^a}(2 + bu) - (1 + b)\right] = \lim_{x\to 0} - a\log(x) = +\infty.$$

For C3: For any $u \in [0, 1)$, we have

$$\eta'(u) = \frac{bu(1-a) - 2a}{u[2 - (b+1)u^a + bu]}.$$

Since $a \in (0, 1]$ and $b \in [-1, 0]$, we have $2 - (b+1)u^a + bu \geq 2 - (b+1) + b = 1 > 0$, and $bu(1-a) \leq 0$, implying that $\eta'(u) < 0$. On the other hand, we have $\eta'(1) = [b(1-a) - 2a]/[2 - (b+1) + b] \leq 0$.

For C4: For any $u \in [0, 1]$, with the use of multiple differentiation rules, we have

$$\eta''(u) = \frac{ab^2u^2 - a(b+1)u^a[(a-1)bu + 2(a+1)] + 4abu + 4a - b^2u^2}{u^2[2-(b+1)u^a + bu]^2}.$$

We have

$$\begin{aligned} & ab^2u^2 - a(b+1)u^a[(a-1)bu + 2(a+1)] + 4abu + 4a - b^2u^2 \\ & = ab^2u^2 + b(b+1)a(1-a)u^{a+1} - 2(b+1)a(a+1)u^a + 4abu + 4a - b^2u^2 \\ & = 4a + b(b+1)a(1-a)u^{a+1} - 2(b+1)a(a+1)u^a + 4abu - (1-a)b^2u^2 \\ & \geq 4a + b(b+1)a(1-a) - 2(b+1)a(a+1) + 4ab - (1-a)b^2 \\ & = a(2b^2 + 3b + 2) - b^2 - a^2(b+2)(b+1) \\ & = (1-a)[a(b+2)(b+1) - b^2]. \end{aligned}$$

For $a \in [b^2/[(b+2)(b+1)], 1]$, we have $a(b+2)(b+1) - b^2 \geq 0$, implying that $\eta''(u) \geq 0$. This completes the proof; it is verified that the function under consideration, $\eta(u)$, is a strict generator function. □

Based on Proposition 2.2 and Equation (2), the Nelsen strict generator function number 10 is obtained with $b = 0$. Thus, the presence of b allows for more flexibility by modulating the term u into the logarithmic term. It is worth noting that the proposed generalization is different to the one in [7], with a new power function activation.

Let us now put the light on another unexplored function: the Nelsen strict generator function number 19 (see [22, Equation (4.2.19)]). It is given by $\eta(u) = \exp(a/u) - \exp(a)$, $u \in [0, 1]$, with $a > 0$. It has found a practical interest in [17].

Proposition 2.3. *The following function defines a new strict generator function*:

$$\eta(u) = \exp\left[\frac{1}{u}(a + bu^c)\right] - \exp(a + b), \quad u \in [0, 1], \tag{3}$$

for $a > 0$, *and either*:

H1: $b \leq 0$ *and* $c \in [1, 2]$,

H2: $b > 0$ *and* $c \geq 1$.

Proof. The four conditions listed in Definition 1 are the focal point of the proof.

For C1: Under **H1** or **H2**, it is clear that

$$\eta(1) = \exp\left[\frac{1}{1}(a + b \times 1^c)\right] - \exp(a + b) = 0.$$

For C2: Under **H1** or **H2**, since $\lim_{u\to 0}(a + bu^c)/u = +\infty$, we have

$$\lim_{u\to 0} \eta(u) = \lim_{u\to 0}\left\{\exp\left[\frac{1}{u}(a + bu^c)\right] - \exp(a + b)\right\} = +\infty.$$

For C3: For any $u \in [0, 1)$, we have

$$\eta'(u) = -\frac{1}{u^2}[b(1 - c)u^c + a]\exp\left[\frac{1}{u}(a + bu^c)\right].$$

For $b(1 - c) \geq 0$, which is the case under **H1** or **H2**, we have $b(1 - c)u^c + a \geq 0$, implying that $\eta'(u) < 0$. Furthermore, it is clear that $\eta'(1) = -[b(1 - c) + a]\exp(a + b) \leq 0$.

For C4: For any $u \in [0, 1]$, we get

$$\eta''(u) = \frac{1}{u^4}\{b(1 - c)u^c[(2 - c)u + 2a] + a(a + 2u)$$

$$+ b^2(c - 1)^2 u^{2c}\}\exp\left[\frac{1}{u}(a + bu^c)\right].$$

For $b(1-c) \geq 0$ and $2-c \geq 0$, which is the case under **H1** or **H2**, we have $b(1-c)u^c[(2-c)u + 2a] \geq 0$, implying that $\eta''(u) \geq 0$.

This end the proof; all the conditions are satisfied, $\eta(u)$ is a valid strict generator function. □

Based on Proposition 2.3 and Equation (3), the Nelsen strict generator function number 19 is obtained with $b = 0$. Thus, the presence of b allows for more flexibility by modulating the power function u^c, and the exponent c tunes this power function. To the best of our knowledge, it is the first attempt at a non-trivial parametric generalization of the Nelsen strict generator function number 19.

On the other hand, beyond the standard constructions, the trigonometric strict generator functions and the corresponding Archimedean copulas have received some interest. The first work dedicated to this was the PhD thesis [24], followed by [29]. By leveraging the flexibility of trigonometric functions, these copulas enable a comprehensive analysis of complex relationships and are particularly relevant in fields such as meteorology, finance, and signal processing.

The case for the cosecant strict generator function in [24] (with $\theta = 1$) is of particular interest (see [24, page 41]). In the following proposition, we develop a parametric generalization of it.

Proposition 2.4. *The following function defines a new strict generator function*:

$$\eta(u) = \frac{1}{\cos[a(1-u)^b] - \cos(a)} - \frac{1}{1-\cos(a)}, \quad u \in [0, 1], \tag{4}$$

for $a \in (0, \pi/2]$ *and* $b \geq 1$.

Proof. The proof is centered around the four conditions presented in Definition 1.

For C1: Since $\cos(0) = 1,$ it is clear that

$$\eta(1) = \frac{1}{\cos[a(1-1)^b] - \cos(a)} - \frac{1}{1-\cos(a)} = 0.$$

For C2: Since $\lim_{u \to 0} \cos[a(1-u)^b] = \cos(a),$ we have

$$\lim_{u \to 0} \eta(u) = \lim_{u \to 0} \left\{ \frac{1}{\cos[a(1-u)^b] - \cos(a)} - \frac{1}{1-\cos(a)} \right\} = +\infty.$$

For C3: For any $u \in [0, 1),$ we have

$$\eta'(u) = \frac{-ab(1-u)^{b-1} \sin[a(1-u)^b]}{\{\cos[a(1-u)^b] - \cos(a)\}^2}.$$

We have $1 - u \in [0, 1)$ and $\{\cos[a(1-u)^b] - \cos(a)\}^2 \geq 0.$ Since $a \in (0, \pi/2]$ and $b \geq 1,$ we have $[a(1-u)^b] > 0.$ This implies that $\eta'(u) < 0.$ Moreover, it is clear that $\eta'(1) = 0.$

For C4: For any $u \in [0, 1],$ we have

$$\eta''(u) = \frac{a^2b^2(1-u)^{2b-2} \cos[a(1-u)^b]}{\{\cos[a(1-u)^b] - \cos(a)\}^2} + \frac{2a^2b^2(1-u)^{2b-2}\{\sin[a(1-u)^b]\}^2}{\{\cos[a(1-u)^b] - \cos(a)\}^3}$$

$$+ \frac{a(b-1)b(1-u)^{b-2} \sin[a(1-u)^b]}{\{\cos[a(1-u)^b] - \cos(a)\}^2}.$$

We have $1 - u \in [0, 1]$ and $\{\cos[a(1-u)^b] - \cos(a)\}^2 \geq 0.$ Since $a \in (0, \pi/2]$ and $b \geq 1,$ we have $b - 1 \geq 0, \cos[a(1-u)^b] \geq \cos(a),$ $\sin[a(1-u)^b] > 0$ and $\sin[a(1-u)^b] > 0.$ All the main being non-negative, we obtain $\eta''(u) \geq 0.$

This finishes the proof; $\eta(u)$ is effectively a strict generator function. $\square$

Based on Proposition 2.4 and Equation (4), the cosecant strict generator function in [24] (with $\theta = 1$) is obtained with $a = \pi/2$ and $b = 1$. We thus innovate by making it more flexible by introducing a and b, which have the advantage of being independent between them.

The rest of the study examines two simple, strict Archimedean copulas that can be derived from our findings.

3. Two New Copulas

The aim of this section is to show how the findings elaborated in Section 2 can be applied in a copula setting. We select only two copulas based on their simplicity, originality, and potential applicability. The study remains, however, mainly theoretically oriented.

3.1. Strict Archimedean copula

First, we must recall the exact connection between a strict generator function and the associated strict Archimedean copula. This is done in the definition below.

Definition 2. The strict Archimedean copula related to a strict generator function $\eta(u)$ is defined by

$$C(x, y) = \eta^{-1}[\eta(x) + \eta(y)], \quad (x, y) \in [0, 1]^2, \tag{5}$$

where $\eta^{-1}(u)$ denotes the inverse function of $\eta(u)$.

Hence, if $\eta(u)$ is a strict generator function of moderate complexity, and with an explicit $\eta^{-1}(u)$, we define a quite manageable strict Archimedean copula. With this in mind, and the findings of Section 2, we focus on the strict generator functions defined in Equations (1) and (4). The two others are not considered because $\eta^{-1}(u)$ has an analytical expression only for very specific values of the parameters, which makes them less manageable, despite some interest.

3.2. A generalized Clayton copula

The Clayton copula is commonly used to model positive dependence and is characterized by its lower tail dependence. In this portion, based on Equation (1), we develop a three-parameter generalization of it, beginning with the result below.

Proposition 3.1. *The following function defines a valid strict Archimedean copula*:

$$C(x, y) = a^{1/b}\left\{1 + [(ax^{-b} - 1)^c + (ay^{-b} - 1)^c - (a-1)^c]^{1/c}\right\}^{-1/b},$$

$$(x, y) \in [0, 1]^2, \qquad (6)$$

for $b > 0$, $c > 0$, *and a such that*

$$a \geq \max\left(1, \frac{b+1}{bc+1}\right).$$

Proof. The inverse function of the strict generator function $\eta(u)$ in Equation (1) is obtained by developing the following equivalence: $\eta(u) = v \Leftrightarrow u = \eta^{-1}(v)$ for any $u \in [0, 1]$. The step-by-step development is as follows:

$$\eta(u) = v \Leftrightarrow (au^{-b} - 1)^c - (a-1)^c = v$$

$$\Leftrightarrow (au^{-b} - 1)^c = v + (a-1)^c$$

$$\Leftrightarrow au^{-b} = 1 + [v + (a-1)^c]^{1/c}$$

$$\Leftrightarrow u = a^{1/b}\left\{1 + [v + (a-1)^c]^{1/c}\right\}^{-1/b}.$$

Therefore, we have

$$\eta^{-1}(v) = a^{1/b}\left\{1 + [v + (a-1)^c]^{1/c}\right\}^{-1/b}, \quad v \geq 0.$$

Owing to Definition 2, the corresponding strict Archimedean copula is obtained as

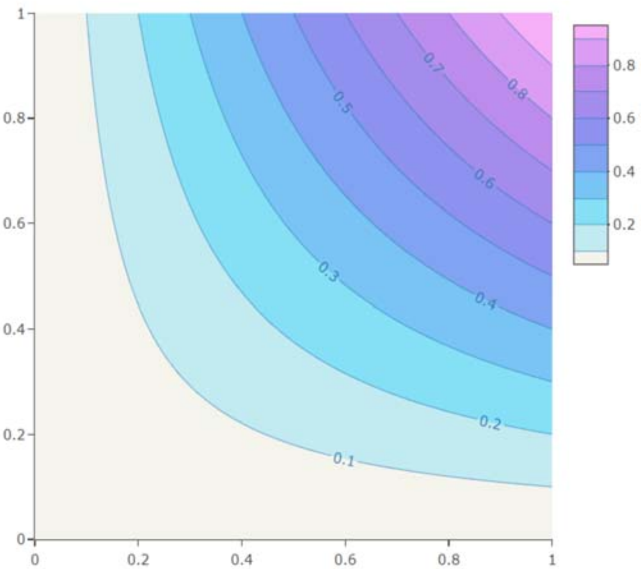
$$C(x, y) = \eta^{-1}[\eta(x) + \eta(y)]$$

$$= a^{1/b}\left\{1 + [(ax^{-b} - 1)^c + (ay^{-b} - 1)^c - (a - 1)^c]^{1/c}\right\}^{-1/b}.$$

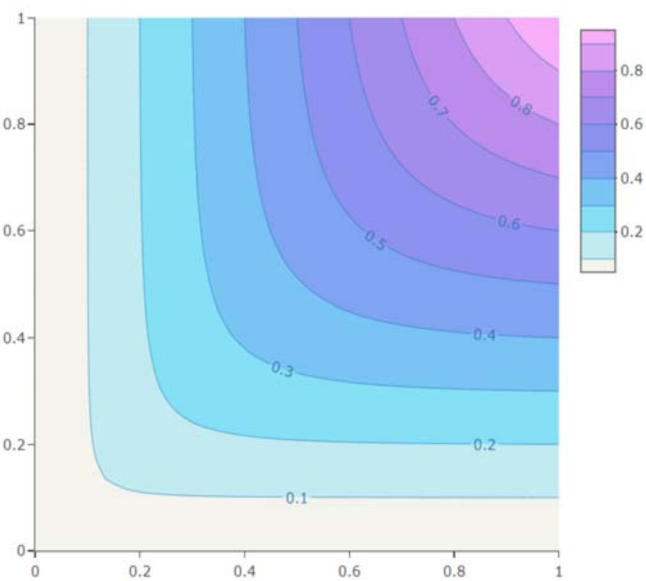
The desired copula is obtained, ending the proof. □

The Clayton copula is obtained by taking $a = 1$ and $c = 1$ in Equation (6). Let us call the copula in Equation (6) the generalized Clayton (GClay) copula. To our knowledge, it is the first three-parameter generalization of the Clayton copula with a quite implementable representation. As for any (absolutely continuous) copula, one can prove that it satisfies $C(x, 0) = C(0, y) = 0$, $C(x, 1) = x$, $C(1, y) = y$, and $\partial^2 C(x, y)/(\partial x \partial y) \geq 0$.

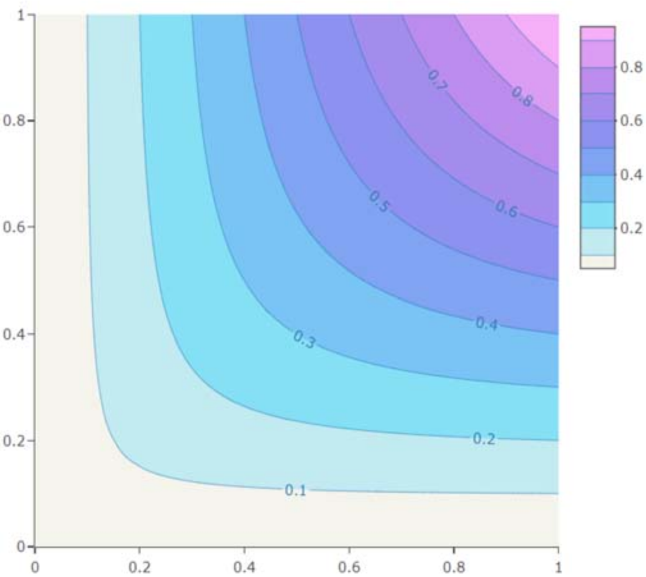
The shapes possessed by a copula determine the type of dependence exhibited by the random variables it models. With this in mind, Figure 1 displays the contour shapes of the GClay copula for various values of a, b, and c. The software R is used (see [25]).


0
0.2
0.4
0.6
0.8
1
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8

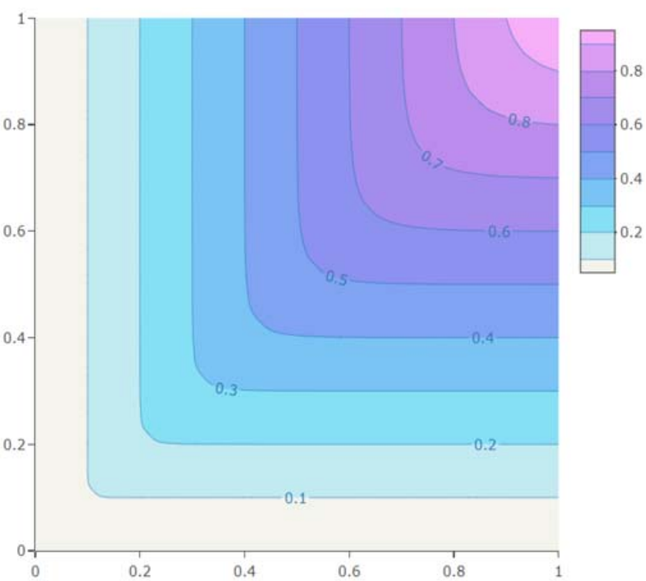
(a)


0
0.2
0.4
0.6
0.8
1
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8

(b)



(c)



(d)

Figure 1. Plots of the contour shapes of the GClay copula for (a) $a = 1$, $b = 0.1$, $c = 1$, (b) $a = 1.5$, $b = 1.5$, $c = 1.5$, (c) $a = 2$, $b = 0.5$, $c = 2$ and (d) $a = 5$, $b = 2$, $c = 5$.

Diverse contour shapes are observed, including round and almost squared forms depending on the values of the parameters. This illustrates the versatility of the GClay copula.

The corresponding GClay copula density is given as

$$c(x, y) = \frac{\partial^2}{\partial x \partial y} C(x, y) = \frac{a^{1/b+2}}{xy(a - x^b)(a - y^b)} (ax^{-b} - 1)^c (ay^{-b} - 1)^c$$

$$\times [(ax^{-b} - 1)^c + (ay^{-b} - 1)^c - (a-1)^c]^{1/c-2}$$

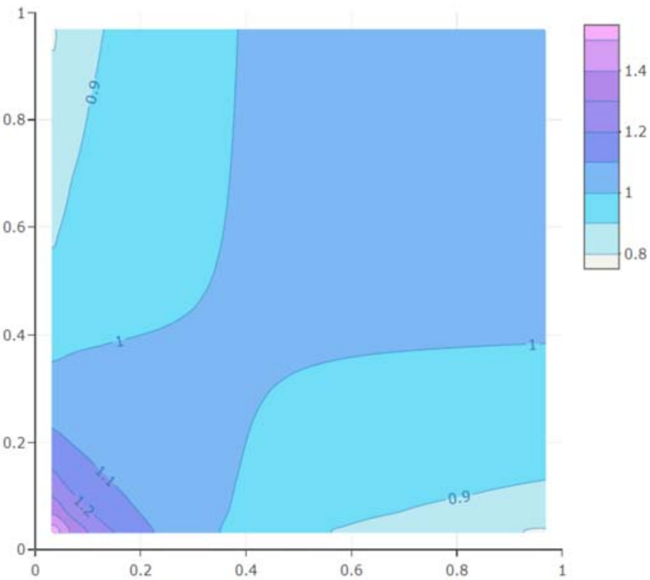
$$\times \left\{[(ax^{-b} - 1)^c + (ay^{-b} - 1)^c - (a-1)^c]^{1/c} + 1\right\}^{-1/b-2}$$

$$\times \left\{[(ax^{-b} - 1)^c + (ay^{-b} - 1)^c - (a-1)^c]^{1/c}\right.$$

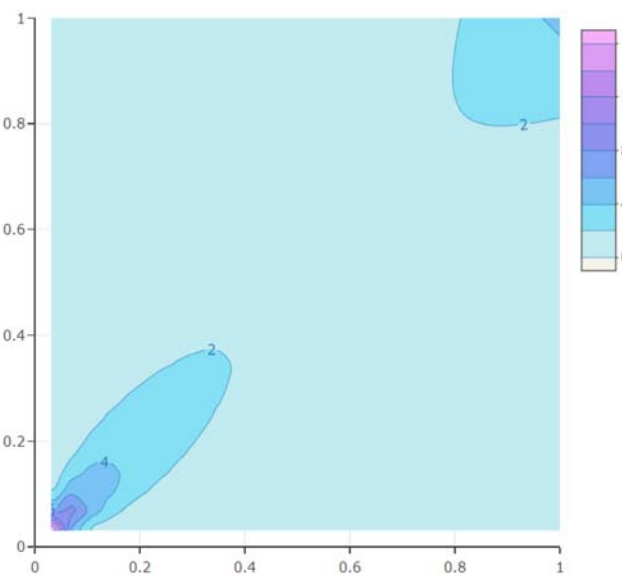
$$\left. + b\left(c[(ax^{-b} - 1)^c + (ay^{-b} - 1)^c - (a-1)^c]^{1/c} + c - 1\right)\right\},$$

$$(x, y) \in [0, 1]^2.$$

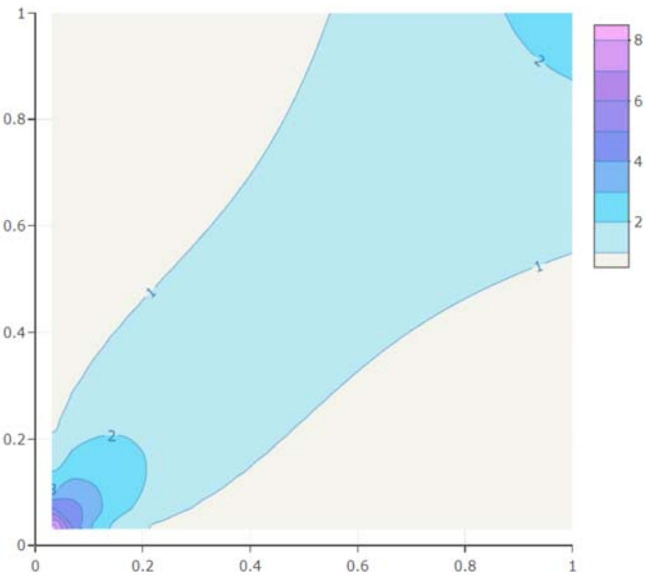
More versatile are the shapes of this function, and more adaptable is the dependence structure of the copula. To get an idea of this aspect, Figure 2 displays the shapes of the GClay copula density for various values of a, b, and c.


1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
0.9
1
1
1.1
1.2
0.9
1.4
1.2
1
0.8

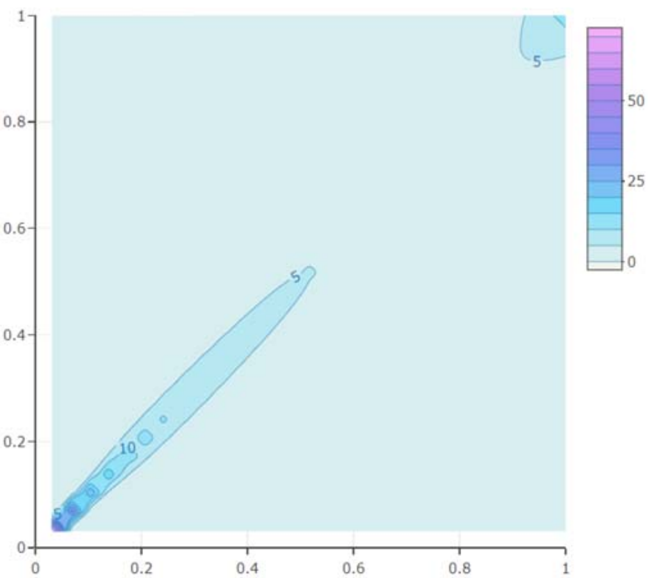
(a)


1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
2
2
4
16
12
8
4
0

(b)



(c)



(d)

Figure 2. Plots of the shapes of the GClay copula density for (a) $a = 1$, $b = 0.1$, $c = 1$, (b) $a = 1.5$, $b = 1.5$, $c = 1.5$, (c) $a = 2$, $b = 0.5$, $c = 2$, and (d) $a = 5$, $b = 2$, $c = 5$.

The GClay copula density demonstrates a grant contour shape diversity, illustrating the interest of the proposed dependence model.

On the other hand, the survival GClay copula is obtained as

$$\hat{C}(x, y) = x + y - 1 + C(1 - x, 1 - y)$$

$$= x + y - 1 + a^{1/b}\Big[1 + \{[a(1 - x)^{-b} - 1]^c$$

$$+ [a(1 - y)^{-b} - 1]^c - (a - 1)^c\}^{1/c}\Big]^{-1/b}, \quad (x, y) \in [0, 1]^2.$$

As the GClay copula, the survival GClay copula adds a new three-parameter copula to the body of knowledge.

The GClay copula is obviously diagonally symmetric. However, since there exists (x, y) such that $\hat{C}(x, y) \neq C(x, y)$, the GClay copula is not radially symmetric.

The Fréchet-Hoeffding bounds can be applied. Hence, for any $(x, y) \in [0, 1]^2$, the GClay copula satisfies the min-max-inequalities: $\max(x + y - 1, 0) \leq C(x, y) \leq \min(x, y)$, i.e.,

$$\max(x + y - 1, 0) \leq a^{1/b}\left\{1 + [(ax^{-b} - 1)^c + (ay^{-b} - 1)^c - (a - 1)^c]^{1/c}\right\}^{-1/b}$$

$$\leq \min(x, y).$$

These two-dimensional inequalities can be thought of as standalone multivariate analysis tools that are not limited to the copula topic.

Using standard limit techniques, the lower tail dependence parameter of the GClay copula is calculated as

$$\lambda_L = \lim_{x \to 0} \frac{C(x, x)}{x} = \lim_{x \to 0} \frac{a^{1/b}\left\{1 + [2(ax^{-b} - 1)^c - (a - 1)^c]^{1/c}\right\}^{-1/b}}{x}$$

$$= 2^{-1/(bc)}.$$

Hence, the GClay copula has a lower tail dependence depending on b and c. By putting $a = 1$ and $c = 1$, we refined the well-known lower tail dependence result for the Clayton copula.

The upper tail dependence parameter follows as

$$\lambda_U = \lim_{x \to 1} \frac{1 - 2x + C(x, x)}{1 - x}$$

$$= \lim_{x \to 1} \frac{1 - 2x + a^{1/b}\left\{1 + [2(ax^{-b} - 1)^c - (a - 1)^c]^{1/c}\right\}^{-1/b}}{1 - x} = 0.$$

As a result, the GClay copula has no upper tail dependence.

The medial correlation of the GClay copula is

$$M = 4C\left(\frac{1}{2}, \frac{1}{2}\right) - 1$$

$$= 4a^{1/b}\left\{1 + [2(a2^b - 1)^c - (a - 1)^c]^{1/c}\right\}^{-1/b} - 1.$$

The roles of a, b, and c are relatively complex, but from a computational standpoint, the medial correlation is quite doable. A numerical study of M reveals that it can be negative, meaning that the GClay copula is also suitable to model negative dependence. This fact will be developed in more depth below with the tau of Kendall.

For any strict Archimedean copula with a strict generator function $\eta(u)$, the general definition of the tau of Kendall is

$$\tau = 1 + 4\int_0^1 \frac{\eta(u)}{\eta'(u)} du. \tag{7}$$

In the setting of the GClay copula, we obtain

$$\tau = 1 - 4\int_0^1 \frac{(au^{-b} - 1)^c - (a - 1)^c}{abcu^{-b-1}(au^{-b} - 1)^{c-1}} du.$$

Due to its great level of complexity, it cannot be further improved analytically.

Table 1 presents some of its numerical values for various values of a, b and c satisfying $a \geq 1$, $b > 0$ and $c \geq 1$.

Table 1. Values of the tau of Kendall of the GClay copula for selected values of a, b and c: the case $c \geq 1$

$a = \rightarrow$	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
$b = 0.5$, $c = 2$	0.6	0.5166	0.4776	0.4531	0.4361	0.4234	0.4137	0.4058	0.3994	0.394	0.3895
$b = 1.5$, $c = 1.5$	0.619	0.5954	0.5835	0.5756	0.5699	0.5655	0.562	0.5591	0.5567	0.5547	0.553
$b = \rightarrow$	0.1	0.3	0.5	0.7	0.9	1.1	1.3	1.5	1.7	1.9	2.1
$a = 1$, $c = 2$	0.5238	0.5652	0.6	0.6296	0.6552	0.6774	0.697	0.7143	0.7297	0.7436	0.7561
$a = 2$, $c = 1$	0.0476	0.1304	0.2	0.2593	0.3103	0.3548	0.3939	0.4286	0.4595	0.4872	0.5122
$c = \rightarrow$	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
$a = 1$, $b = 2$	0.5	0.5455	0.5833	0.6154	0.6429	0.6667	0.6875	0.7059	0.7222	0.7368	0.75
$a = 1.5$, $b = 1.5$	0.4286	0.4619	0.4919	0.5188	0.5433	0.5655	0.5858	0.6043	0.6214	0.6371	0.6516

From this table, in the case $c \geq 1$ and the considered values of the parameters, we see that $\tau \in [0, 0.75]$. This range of values is quite acceptable, making the GClay copula suitable to model various kinds of positive dependence.

Table 2 presents some numerical values of the tau of Kendall of the GClay copula for various values of a, b and c, with $c \in (0, 1)$. We chose a as $a = (b+1)/(bc+1)$ and we vary b and c.

Table 2. Values of the tau of Kendall of the GClay copula for selected values of a, b and c, the case $c \in (0, 1)$

$b = \rightarrow$	0.1	0.3	0.5	0.7	0.9	1.1	1.3	1.5	1.7	1.9	2.1
$c = 0.1$	− 0.3109	− 0.2577	− 0.2131	− 0.175	− 0.1421	− 0.1132	− 0.0876	− 0.0646	− 0.0437	− 0.0247	− 0.0072
$c = 0.5$	− 0.1997	− 0.124	− 0.0581	-2×10^{-4}	0.0512	0.0972	0.1387	0.1763	0.2105	0.2419	0.2708
$c = 0.9$	− 0.0253	0.0609	0.1339	0.1963	0.2504	0.2977	0.3394	0.3764	0.4095	0.4393	0.4662

From this table, in the case $c \in (0, 1)$ and the considered values of the parameters, it is interesting to see that $\tau \in [-0.31, 0.47]$. This proved numerically that the GClay copula can also model negative dependence, contrary to the original Clayton copula. This is a real plus for dependence modelling. Overall, we have $\tau \in [-0.31, 0.75]$, making the GClay copula competitive on the correlation range aspect.

On the other hand, for any strict Archimedean copula with a strict generator function $\eta(u)$, the general definition of the distribution function of Kendall is

$$K(u) = u - \frac{\eta(u)}{\eta'(u)}, \quad u \in [0, 1]. \tag{8}$$

In the setting of the GClay copula, we obtain

$$K(u) = u + \frac{(au^{-b} - 1)^c - (a-1)^c}{abcu^{-b-1}(au^{-b} - 1)^{c-1}}, \quad u \in [0, 1].$$

Figure 3 presents some of its plots for various values of a, b, and c satisfying $a \geq 1$, $b > 0$, and $c \geq 1$.

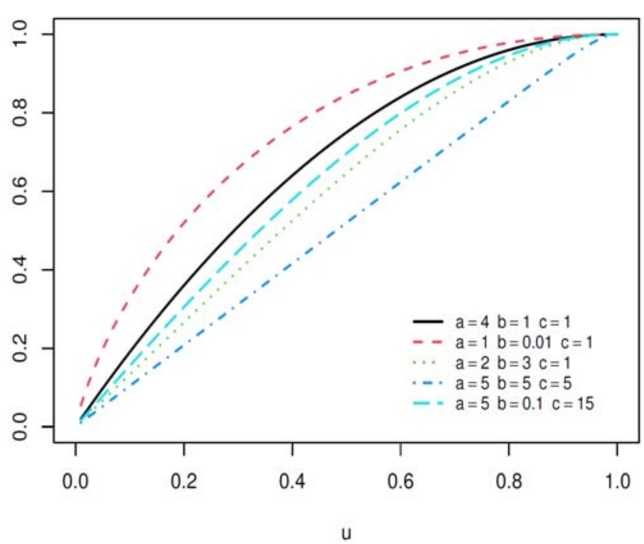


Figure 3. Graphics for the distribution function of Kendall of the GClay copula for various values of a, b and c; the case $c \geq 1$.

Various concave shapes are observed, demonstrating a relative functional flexibility. Figure 4 completes Figure 3 by presenting some plots of the GClay distribution function of Kendall for various values of a, b, and c for the case $c \in (0, 1)$. We chose a as $a = (b+1)/(bc+1)$ and we vary b and c.

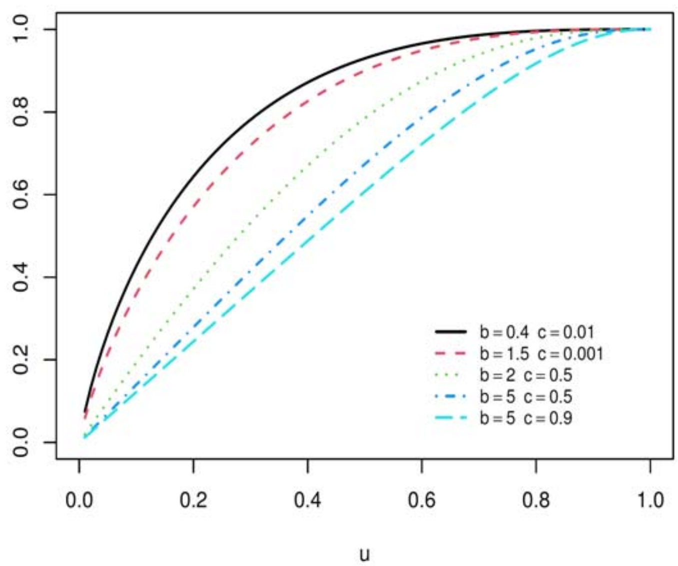


Figure 4. Graphics for the distribution function of Kendall of the GClay copula for various values of a, b, and c; the case $c \in (0, 1)$.

Last but not least, the GClay copula can serve as a two-dimensional distribution generator. Indeed, by considering two baseline uni-dimensional distribution functions, say $F(x)$ and $G(x)$, we introduce a new two-dimensional distribution function by

$$H(x, y) = C[F(x), G(y)]$$

$$= a^{1/b}\left\{1 + [(aF(x)^{-b} - 1)^c + (aG(y)^{-b} - 1)^c - (a-1)^c]^{1/c}\right\}^{-1/b},$$

$$(x, y) \in \mathbb{R}^2.$$

As a result, novel perspectives on two-dimensional modelling are offered for a variety of statistical applications. A particularly interesting direction is the use of lifetime (or survival) distributions as the baseline distributions (see [30]). Let us mention that other two-dimensional approaches can be considered, such as the structural one established in [19].

3.3. A new trigonometric copula

Copulas defined with trigonometric functions can be useful for circular or directional data. See [24], [14], [15], [29] and [6]. By their nature, they have different dependence modelling objectives than copulas defined with power functions, such as the GClay copula. We contribute in this direction by offering a new, flexible candidate. The strict Archimedean copula corresponding to the strict generator function described in Equation (4) has a closed-form expression, and it is determined in the next proposition.

Proposition 3.2. *The following function defines a valid strict Archimedean copula*:

$$C(x, y) = 1 - a^{-1/b} \times \left\{\arccos\left[\left\{\frac{1}{\cos[a(1-x)^b] - \cos(a)} + \frac{1}{\cos[a(1-y)^b] - \cos(a)} - \frac{1}{1-\cos(a)}\right\}^{-1} + \cos(a)\right]\right\}^{1/b},$$

$$(x, y) \in [0, 1]^2, \qquad (9)$$

for $a \in (0, \pi/2]$ *and* $b \geq 1$.

Proof. Let us consider the strict generator function $\eta(u)$ in Equation (4). Then the inverse function of $\eta(u)$ is obtained by developing the following equivalence: $\eta(u) = v \Leftrightarrow u = \eta^{-1}(v)$ for any $u \in [0, 1]$. This development is as follows:

$$\eta(u) = v \Leftrightarrow \frac{1}{\cos[a(1-u)^b] - \cos(a)} - \frac{1}{1-\cos(a)} = v$$

$$\Leftrightarrow \frac{1}{\cos[a(1-u)^b] - \cos(a)} = v + \frac{1}{1-\cos(a)}$$

$$\Leftrightarrow \cos[a(1-u)^b] = \frac{1-\cos(a)}{v[1-\cos(a)]+1} + \cos(a)$$

$$\Leftrightarrow (1-u)^b = a^{-1} \arccos\left\{\frac{1-\cos(a)}{v[1-\cos(a)]+1} + \cos(a)\right\}$$

$$\Leftrightarrow u = 1 - a^{-1/b}\left[\arccos\left\{\frac{1-\cos(a)}{v[1-\cos(a)]+1} + \cos(a)\right\}\right]^{1/b}.$$

Hence, we have

$$\eta^{-1}(v) = 1 - a^{-1/b}\left[\arccos\left\{\frac{1-\cos(a)}{v[1-\cos(a)]+1} + \cos(a)\right\}\right]^{1/b}, \quad v \geq 0.$$

Therefore, based on Definition 2, the corresponding strict Archimedean copula is obtained as

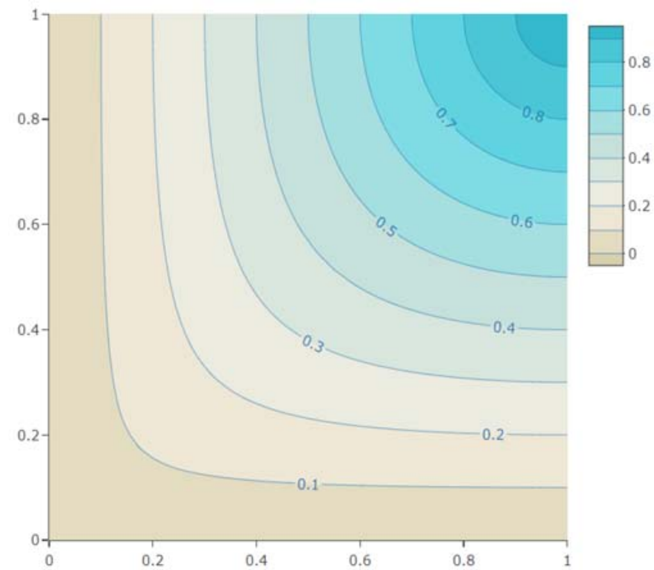
$$C(x, y) = \eta^{-1}[\eta(x) + \eta(y)] = 1 - a^{-1/b}$$

$$\times \left\{\arccos\left[\left\{\frac{1}{\cos[a(1-x)^b] - \cos(a)} + \frac{1}{\cos[a(1-y)^b] - \cos(a)} - \frac{1}{1-\cos(a)}\right\}^{-1} + \cos(a)\right]\right\}^{1/b}.$$

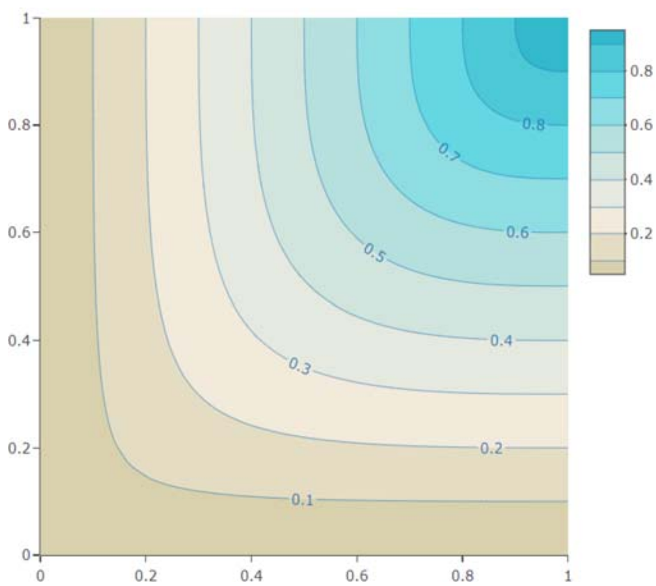
The desired copula is obtained, ending the proof. □

Let us call the copula in Equation (9) the generalized cosine (GCos) copula. When $a = \pi/2$ and $b = 1$, it becomes the cosecant copula as described in [24] with $\theta = 1$. To the best of our knowledge, it is a new two-parameter trigonometric copula in the literature. As for any (absolutely continuous) copula, one can prove that it satisfies $C(x, 0) = C(0, y) = 0$, $C(x, 1) = x$, $C(1, y) = y$, and $\partial^2 C(x, y)/(\partial x \partial y) \geq 0$.

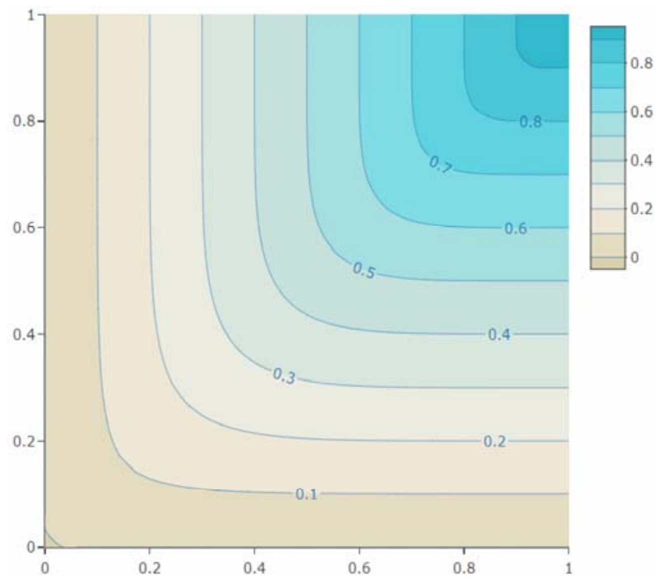
Figure 5 displays the contour shapes of the GCos copula for various values of a and b.



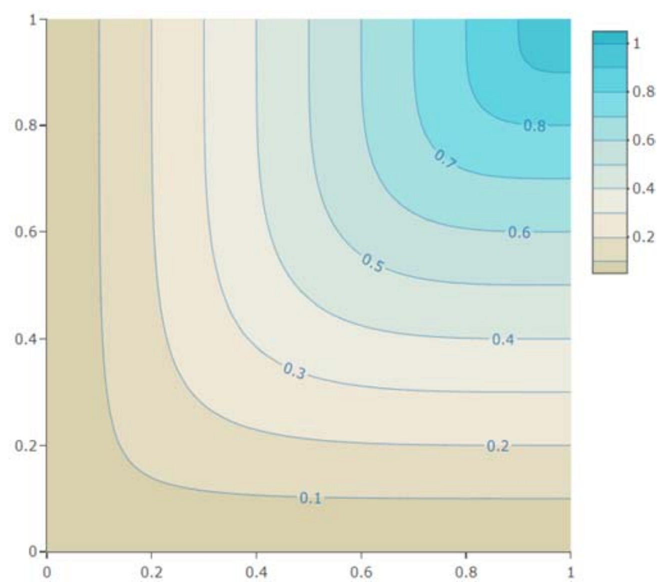
(a)



(b)



(c)



(d)

Figure 5. Plots of the contour shapes of the GCos copula for (a) $a = \pi/6, b = 1,$ (b) $a = \pi/3,\ b = 1.5,$ (c) $a = \pi/3,\ b = 3,$ and (d) $a = \pi/2,\ b = 2.$

Different contour shapes are seen, demonstrating how adaptable the GCos copula is. The GCos copula density can be expressed, but due to a large formula, it is omitted here. On the other hand, the survival GCos copula is obtained as

$$\hat{C}(x, y) = x + y - a^{-1/b} \times \left\{\arccos\left[\left\{\frac{1}{\cos(ax^b) - \cos(a)} + \frac{1}{\cos(ay^b) - \cos(a)} - \frac{1}{1 - \cos(a)}\right\}^{-1} + \cos(a)\right]\right\}^{1/b}, \quad (x, y) \in [0, 1]^2.$$

As the GCos copula, the survival GCos copula adds a new three-parameter copula to the body of knowledge.

Clearly, the GCos copula is diagonally symmetric. Since there exists (x, y) such that $\hat{C}(x, y) \neq C(x, y)$, the GCos copula is not radially symmetric.

The Fréchet-Hoeffding bounds can be applied. Hence, for any $(x, y) \in [0, 1]^2$, the GCos copula satisfies $\max(x + y - 1, 0) \leq C(x, y) \leq \min(x, y)$, i.e.,

$$\max(x + y - 1, 0) \leq 1 - a^{-1/b} \times \left\{\arccos\left[\left\{\frac{1}{\cos[a(1-x)^b] - \cos(a)} + \frac{1}{\cos[a(1-y)^b] - \cos(a)} - \frac{1}{1 - \cos(a)}\right\}^{-1} + \cos(a)\right]\right\}^{1/b} \leq \min(x, y).$$

This result can be used for other analytical purposes.

The lower tail dependence parameter of the GCos copula is calculated by using standard limit methods. We get

$$\lambda_L = \lim_{x\to 0}\frac{1}{x}\left[1 - a^{-1/b}\left\{\arccos\left[\left\{\frac{2}{\cos\left[a(1-x)^b\right] - \cos(a)} - \frac{1}{1-\cos(a)}\right\}^{-1} + \cos(a)\right]\right\}^{1/b}\right] = \frac{1}{2}.$$

Hence, the GCos copula has a rigid lower tail dependence, independent of a and b.

The upper tail dependence parameter is more technical to obtain; it follows as

$$\lambda_U = \lim_{x\to 1}\frac{1}{1-x}\times\left\{1 - 2x + \left[1 - a^{-1/b}\left\{\arccos\left[\left\{\frac{2}{\cos\left[a(1-x)^b\right] - \cos(a)} - \frac{1}{1-\cos(a)}\right\}^{-1} + \cos(a)\right]\right\}^{1/b}\right]\right\} = 2 - 2^{1/(2b)}.$$

As a result, the GCos copula has a flexible upper tail dependence depending on b. It is obviously a plus for upper tail dependence modelling.

The medial correlation of the GCos copula is

$$M = 3 - 4a^{-1/b}\left\{\arccos\left[\left\{\frac{2}{\cos\left(a2^{-b}\right) - \cos(a)} - \frac{1}{1-\cos(a)}\right\}^{-1} + \cos(a)\right]\right\}^{1/b}.$$

This expression is sophisticated but can be easily implemented for application purposes.

We have

$$\frac{\eta(u)}{\eta'(u)} = -\frac{1/\{\cos[a(1-u)^b] - \cos(a)\} - 1/[1-\cos(a)]}{ab(1-u)^{b-1}\sin[a(1-u)^b]/\{\cos[a(1-u)^b] - \cos(a)\}^2}.$$

The tau of Kendall of the GCos copula is thus expressed as

$$\tau = 1 - 4\int_0^1 \frac{1/\{\cos[a(1-u)^b] - \cos(a)\} - 1/[1-\cos(a)]}{ab(1-u)^{b-1}\sin[a(1-u)^b]/\{\cos[a(1-u)^b] - \cos(a)\}^2}du.$$

It cannot be developed more in the analytical sense because of its high level of complexity.

Table 3 gives some of its numerical values for various values of a and b satisfying $a \in (0, \pi/2]$ and $b \geq 1$.

Table 3. Values of the tau of Kendall of the GCos copula for selected values of a and b

$b = \rightarrow$	1	3	5	7	9	11	12	15	17	19	21
$a = 0.1$	0.5	0.75	0.8336	0.8764	0.903	0.9214	0.9349	0.9452	0.9532	0.9596	0.9647
$a = 1$	0.5004	0.7501	0.8335	0.8756	0.9016	0.9194	0.9326	0.9427	0.9506	0.957	0.9622
$a = \pi/2$	0.5025	0.7506	0.8338	0.876	0.9021	0.9201	0.9333	0.9435	0.9514	0.9578	0.963

From this table, we observe that $\tau \in [0.5, 0.96]$. The GCos copula seems suitable to model high positive dependence.

On the other hand, the distribution function of Kendall of the GCos copula is given by

$$K(u) = u + \frac{1/\{\cos[a(1-u)^b] - \cos(a)\} - 1/[1-\cos(a)]}{ab(1-u)^{b-1}\sin[a(1-u)^b]/\{\cos[a(1-u)^b] - \cos(a)\}^2}, \quad u \in [0, 1].$$

Figure 6 presents some of its plots for various values of a and b satisfying $a \in (0, \pi/2]$ and $b \geq 1$.

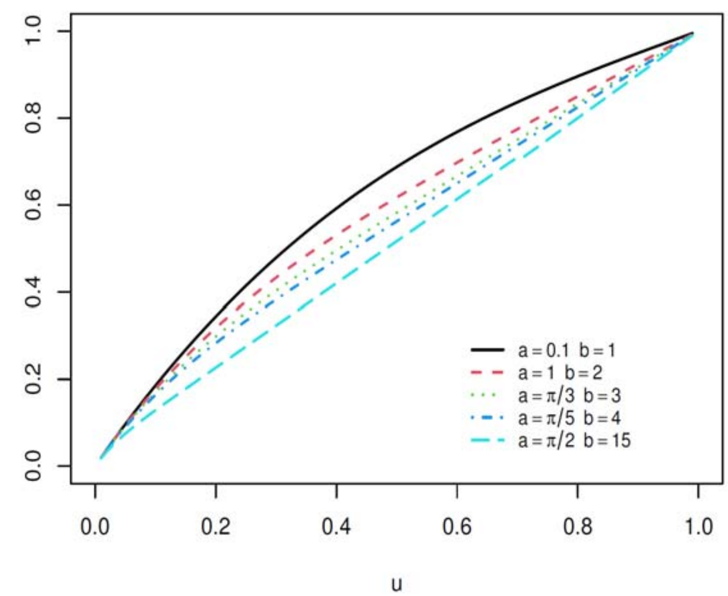


Figure 6. Graphics for the distribution function of Kendall of the GCos copula for various values of a and b.

Concave shapes of various sizes are seen, showing a certain degree of functional flexibility.

By taking into account the GCos copula and two uni-dimensional distribution functions, say $F(x)$ and $G(y)$, we introduce a new two-dimensional distribution function by

$$H(x,\ y) = C[F(x),\ G(y)]$$

$$= 1 - a^{-1/b} \times \left\{ \arccos\left[\left\{ \frac{1}{\cos\{a[1 - F(x)]^b\} - \cos(a)} + \frac{1}{\cos\{a[1 - G(y)]^b\} - \cos(a)} - \frac{1}{1 - \cos(a)} \right\}^{-1} + \cos(a) \right] \right\}^{1/b},$$

$$(x,\ y) \in \mathbb{R}^2.$$

As a result, novel perspectives on two-dimensional trigonometric modelling are offered for a variety of statistical applications.

4. Conclusion

In this article, we have developed four new strict Archimedean generator functions that generalize existing ones. The process of generalizations is based on a thorough addition of parameters or power functions. The main results determine the admissible values of the involved parameters, validating the strict Archimedean generator functions. These functions and parameter conditions are summarized in Table 4.

Table 4. Summary of the main strict Archimedean generator functions

	$\eta(u)$	Conditions
First	$(au^{-b}-1)^c-(a-1)^c$	$b>0,\ c>0,\ a\geq\max\left(1, \frac{b+1}{bc+1}\right)$
Second	$\log\left[\frac{1}{u^a}(2+bu)-(1+b)\right]$	$b\in[-1, 0],\ \frac{b^2}{(b+2)(b+1)}\leq a\leq 1$
Third	$\exp\left[\frac{1}{u}(a+bu^c)\right]-\exp(a+b)$	$a>0,\ b\leq 0,\ c\in[1, 2]$ or $a>0,\ b>0,\ c\geq 1$
Fourth	$\frac{1}{\cos[a(1-u)^b]-\cos(a)}-\frac{1}{1-\cos(a)}$	$a\in\left(0, \frac{\pi}{2}\right],\ b\geq 1$

Based on the first and fourth strict Archimedean generator functions, two new strict Archimedean copulas are established. One can be thought of as a generalized Clayton copula that depends on three parameters and allows a wide range of dependence, including negative dependence, and the other offers a new two-parameter trigonometric copula with tunable upper tail dependence.

The theoretical findings of this article lay the foundation for many applications to the analysis of two-dimensional data. In particular, in all applications where the Clayton copula offers an acceptable model, one can test the proposed generalized Clayton copula model. The directions for future work include the investigation of the multi-dimensional case, more copula developments based on the second and third strict Archimedean generator functions, real data analysis of modern two-dimensional phenomena, and the development of copula regression models. These complementary aspects need more investigation, which we leave for future research.

Acknowledgements

The author would like to thank the associate editor and reviewers for their constructive feedback.

References

[1] F. Aldhufairi and J. Sepanski, New families of bivariate copulas via unit Weibull distortion, Journal of Statistical Distributions and Applications 7(A8) (2020), 1-20.

DOI: https://doi.org/10.1186/s40488-020-00110-z

[2] M. I. Bhatti and H. Q. Do, Recent development in copula and its applications to the energy, forestry and environmental sciences, International Journal of Hydrogen Energy 44(36) (2019), 19453-19473.

DOI: https://doi.org/10.1016/j.ijhydene.2019.06.015

[3] H. Bekrizadeh and B. Jamshidi, A new class of bivariate copulas: Dependence measures and properties, Metron 75 (2017), 31-50.

DOI: https://doi.org/10.1007/s40300-017-0107-1

[4] J. F. Carriere, Bivariate survival models for coupled lives, Scandinavian Actuarial Journal 2000(1) (2000), 17-32.

DOI: https://doi.org/10.1080/034612300750066700

[5] C. Chesneau, On new three- and two-dimensional ratio-power copulas, Computational Journal of Mathematical and Statistical Sciences 2(1) (2023), 106-122.

DOI: https://doi.org/10.21608/CJMSS.2023.195134.1005

[6] C. Chesneau, A collection of new trigonometric- and hyperbolic-FGM-type copulas, AppliedMath 3(1) (2023), 147-174.

DOI: https://doi.org/10.3390/appliedmath3010010

[7] C. Chesneau, Extensions of two bivariate strict Archimedean copulas, Computational Journal of Mathematical and Statistical Sciences 2(2) (2023), 159-180.

DOI: https://doi.org/10.21608/CJMSS.2023.205330.1007

[8] C. Chesneau, Theoretical advancements on a few new dependence models based on copulas with an original ratio form, Modelling 4(2) (2023), 102-132.

DOI: https://doi.org/10.3390/modelling4020008

[9] K. Cooray, Strictly Archimedean copulas with complete association for multivariate dependence based on the Clayton family, Dependence Modeling 6(1) (2018), 1-18.

DOI: https://doi.org/10.1515/demo-2018-0001

[10] W. Diaz and C. M. Cuadras, An extension of the Gumbel-Barnett family of copulas, Metrika 85(7) (2022), 913-926.

DOI: https://doi.org/10.1007/s00184-022-00859-0

[11] F. El Ktaibi, R. Bentoumi, N. Sottocornola and M. Mesfioui, Bivariate copulas based on counter-monotonic shock method, Risks 10(11) (2022); Article 202.

DOI: https://doi.org/10.3390/risks10110202

[12] P. Gaillardetz and X. S. Lin, Valuation of equity-linked insurance and annuity products with binomial models, North American Actuarial Journal 10(4) (2006), 117-144.

DOI: https://doi.org/10.1080/10920277.2006.10597417

[13] C. Gourieroux and Y. Lu, Love and death: A freund model with frailty, Insurance: Mathematics and Economics 63 (2015), 191-203.

DOI: https://doi.org/10.1016/j.insmatheco.2015.03.016

[14] F. H. Hodel and J. R. Fieberg, Circular-linear copulae for animal movement data, Methods in Ecology and Evolution 13(5) (2022), 1001-1013, bioRxiv.

DOI: https://doi.org/10.1101/2021.07.14.452404

[15] F. H. Hodel and J. R. Fieberg, Cylcop: An R Package for Circular-Linear Copulae with Angular Symmetry (2021), bioRxiv.

DOI: https://doi.org/10.1101/2021.07.14.452253

[16] N. Kolev, U. dos Anjos and B. V. de M. Mendes, Copulas: A review and recent developments, Stochastic Models 22(4) (2006), 617-660.

DOI: https://doi.org/10.1080/15326340600878206

[17] T. Kularatne, J. Li and D. Pitt, On the use of Archimedean copulas for insurance modelling, Annals of Actuarial Science 15(1) (2021), 57-81.

DOI: https://doi.org/10.1017/S1748499520000147

[18] M. Manstavičius and G. Bagdonas, A class of bivariate independence copula transformations, Fuzzy Sets and Systems 428 (2022), 58-79.

DOI: https://doi.org/10.1016/j.fss.2020.12.022

[19] S. Mondal and D. Kundu, A bivariate inverse Weibull distribution and its application in complementary risks model, Journal of Applied Statistics 47(6) (2020), 1084-1108.

DOI: https://doi.org/10.1080/02664763.2019.1669542

[20] S. Nadarajah, E. Afuecheta and S. Chan, A compendium of copulas, Statistica 77(4) (2017), 279-328.

DOI: https://doi.org/10.6092/issn.1973-2201/7202

[21] R. B. Nelsen, Dependence Modeling with Archimedean Copulas, Proceeding of the Second Brazilian Conference on Statistical Modelling in Insurance and Finance, Institute of Mathematics and Statistics, University of Sao Paulo, 2005.

[22] R. B. Nelsen, An Introduction to Copulas, Springer Science+Business Media, Inc., Second Edition, 2006.

DOI: https://doi.org/10.1007/0-387-28678-0

[23] A. J. Patton, A review of copula models for economic time series, Journal of Multivariate Analysis 110 (2012), 4-18.

DOI: https://doi.org/10.1016/j.jmva.2012.02.021

[24] A. Pirmoradian, A new one parameter family of Archimedean copula and its extensions, PhD thesis, Institute of Mathematical Sciences - Universiti Malaya, 2013.

[25] R Core Team, R: A Language and Environment for Statistical Computing, Vienna, Austria, 2016.

Available at: https://www.R-project.org/

[26] D. J. Roberts and T. Zewotir, Copula geoadditive modelling of anaemia and malaria in young children in Kenya, Malawi, Tanzania and Uganda, Journal of Health, Population and Nutrition 39 (2020); Article 8.

DOI: https://doi.org/10.1186/s41043-020-00217-8

[27] M. M. Seyam and S. M. Elsobky, New bivariate MS copula via Rüschendorf method, Information Sciences Letters 11(4) (2022), 1087-1092.

[28] A. Sklar, Fonctions de répartition à n dimensions et leurs marges, Publications de l'Institut Statistique de l'Université de Paris 8 (1959), 229-231.

[29] S. O. Susam, A new family of Archimedean copula via trigonometric generator function, Gazi University Journal of Science 33(3) (2020), 806-813.

DOI: https://doi.org/10.35378/gujs.635032

[30] N. Taketomi, K. Yamamoto, C. Chesneau and T. Emura, Parametric distributions for survival and reliability analyses: A review and historical sketch, Mathematics 10(20) (2022); Article 3907.

DOI: https://doi.org/10.3390/math10203907

[31] Q. Yang, M. Xu, X. Lei, X. Zhou and X. Lu, A methodological study on AMH copula-based joint exceedance probabilities and applications for assessing tropical cyclone impacts and disaster risks (Part I), Tropical Cyclone Research and Review 3(1) (2014), 53-62.

DOI: https://doi.org/10.6057/2014TCRR01.05

[32] X. Zhao and X. Zhou, Copula models for insurance claim numbers with excess zeros and time-dependence, Insurance: Mathematics and Economics 50(1) (2012), 191-199.

DOI: https://doi.org/10.1016/j.insmatheco.2011.11.004

■