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THE DYNAMICAL BEHAVIOUR OF SOLUTIONS FOR NONLINEAR SYSTEMS OF RATIONAL DIFFERENCE EQUATIONS

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Abstract

In this paper, we investigate the form of the solution of the following system of difference equations of second order:

$$x_{n+1} = \frac{ax_{n-1}y_{n-1}}{b + cy_{n-1}}, \quad y_{n+1} = \frac{dx_{n-1}y_{n-1}}{e + fx_{n-1}}, \quad n = 0, 1, \ldots,$$

where the parameters a, b, c, d, e, f and initial conditions x_{-1}, x_0, y_{-1}, y_0 are arbitrary positive real numbers.

1. Introduction

In this paper, we deal with the behaviour of the solution of the following system of difference equation:

$$x_{n+1} = \frac{ax_{n-1}y_{n-1}}{b + cy_{n-1}}, \quad y_{n+1} = \frac{dx_{n-1}y_{n-1}}{e + fx_{n-1}}, \quad n = 0, 1, \ldots, \tag{1}$$

where the initial conditions x_{-1}, x_0, y_{-1}, y_0 and a, b, c, d, e, f are positive real numbers.

The hypothesis of difference equations involves a focal position in applicable analysis. There is no uncertainty that the hypothesis of difference equations will keep on playing a vital part in science overall.

Nonlinear difference equations, of order more than one, are of principal significance in applications. Such equations likewise seem normally as discrete analogs and as numerical arrangements of differential equations which show several assorted wonders in science, biology, physics, physiology, engineering and economics, see [1]-[39].

As of late, there has been incredible enthusiasm for examining systems of difference equations. One reason for this is the need for a few strategies that can be utilized as part of investigating equations emerging in mathematical models. There are many papers on systems of difference equations.

Khan and Qureshi [6] investigated the qualitative behaviour of the following systems of second-order rational difference equations:

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta - \gamma y_n y_{n-1}}, \quad y_{n+1} = \frac{\alpha_1 y_{n-1}}{\beta_1 - \gamma_1 x_n x_{n-1}},$$

and

$$x_{n+1} = \frac{a y_{n-1}}{b - c x_n x_{n-1}}, \quad y_{n+1} = \frac{a_1 x_{n-1}}{b_1 - c_1 y_n y_{n-1}}.$$

The authors in [47] have obtained the form of the solutions of the following system of difference equations:

$$x_{n+1} = \frac{A x_n + y_n}{x_{n-p}}, \quad y_{n+1} = \frac{A + x_n}{y_{n-q}}.$$

Touafek et al. [51] investigated the periodic nature and gave the form of the solutions of the following systems of rational difference equations:

$$x_{n+1} = \frac{y_n}{x_{n-1}(\pm 1 \pm y_n)}, \quad y_{n+1} = \frac{x_n}{y_{n-1}(\pm 1 \pm x_n)}.$$

Din et al. [7] dealt with the behaviour of the solutions of the following fourth-order system of rational difference equations of the form:

$$x_{n+1} = \frac{\alpha x_{n-3}}{\beta + \gamma y_n y_{n-1} y_{n-2} y_{n-3}}, \quad y_{n+1} = \frac{\alpha_1 x_{n-3}}{\beta_1 - \gamma_1 x_n x_{n-1} x_{n-2} x_{n-3}}.$$

The persistence and the asymptotic behaviour of the positive solutions of the system of two difference equations of exponential form

$$x_{n+1} = a + b x_{n-1} e^{-y_n}, \quad y_{n+1} = c + d y_{n-1} e^{-x_n},$$

were studied by Papaschinopoulos et al. [48].

Yalçinkaya [58] obtained the sufficient conditions for the global asymptotic stability of the system of two nonlinear difference equations

$$x_{n+1} = \frac{x_n + y_{n-1}}{x_n y_{n-1} - 1}, \quad y_{n+1} = \frac{y_n + x_{n-1}}{y_n x_{n-1} - 1}.$$

Elsayed and El-Dessoky [31] investigated the behaviour of the rational difference equation

$$x_{n+1} = ax_n + \frac{bx_n x_{n-2}}{cx_{n-2} + dx_{n-3}}.$$

Yang et al. [60] studied the global behaviour of the system of the two nonlinear difference equations

$$x_{n+1} = \frac{Ax_n}{1 + y_n^p}, \quad y_{n+1} = \frac{By_n}{1 + x_n^p}.$$

Mnguni et al. [46] studied the Lie point symmetries of difference equations of the form

$$U_{n+4} = \frac{U_n}{A_n + B_n U_n U_{n+2}}.$$

In [35], Folly-Gbetoula and Nyirenda investigated the sixth-order recursive sequences of the form

$$x_{n+1} = \frac{x_n x_{n-5}}{x_{n-4}(a_n + b_n x_n x_{n-5})},$$

where a_n and b_n are sequences of real numbers.

Also, Folly-Gbetoula and Nyirenda [34] found the exact formulas for the solutions of the following system of $(k+1)$-th-order difference equations:

$$x_{n+1} = \frac{x_{n-k+1} y_{n-k}}{y_n (a_n + b_n x_{n-k+1} y_{n-k})}, \quad y_{n+1} = \frac{x_{n-k} y_{n-k+1}}{y_n (c_n + d_n x_{n-k} y_{n-k+1})}.$$

See also [40]-[62].

Let us consider a two-dimensional discrete dynamical system of the form

$$x_{n+1} = f(x_{n-1}, y_{n-1}), \; y_{n+1} = g(x_{n-1}, y_{n-1}), \quad n = 0, 1, \ldots, \tag{2}$$

where $f : I^2 \times J^2 \longrightarrow I$ and $g : I^2 \times J^2 \longrightarrow J$ are continuously differentiable functions and I, J are some intervals of real numbers.

Furthermore, a solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ of system (2) is uniquely determined by the initial conditions $(x_i, y_i) \in I \times J$ for $i \in \{-1, 0\}$.

Definition 1 ([50])**.** Let $(\bar{x}, \bar{y})$ be an equilibrium point of the system (2).

(i) An equilibrium point $(\bar{x}, \bar{y})$ is said to be *locally stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every initial condition (x_i, y_i), $i \in \{-1, 0\}$ with $\sum_{i=-1}^{0} |x_i - \bar{x}| < \delta$, $\sum_{i=-1}^{0} |y_i - \bar{y}| < \delta$, we have $|x_n - \bar{x}| < \varepsilon$, $|y_n - \bar{y}| < \varepsilon$ for all $n > 0$.

(ii) An equilibrium point $(\bar{x}, \bar{y})$ is said to be *unstable* if it is not stable.

(iii) An equilibrium point $(\bar{x}, \bar{y})$ is said to be *asymptotically stable* if there exists $\eta > 0$ such that $\sum_{i=-1}^{0} |x_i - \bar{x}| < \eta$, $\sum_{i=-1}^{0} |y_i - \bar{y}| < \eta$, and $\lim_{n\to\infty} x_n = \bar{x}$, $\lim_{n\to\infty} y_n = \bar{y}$.

(iv) An equilibrium point $(\bar{x}, \bar{y})$ is called a *global attractor* if $(x_n, y_n) \to (\bar{x}, \bar{y})$ as $n \to \infty$.

(v) An equilibrium point $(\bar{x}, \bar{y})$ is called a *globally asymptotically stable* if it is a global attractor and stable.

Definition 2. Let $(\bar{x}, \bar{y})$ be an equilibrium point of the map $F = (f, x_{n-1}, g, y_{n-1})$, where f and g are continuously differentiable functions at $(\bar{x}, \bar{y})$. The linearized system of (2) about the equilibrium point $(\bar{x}, \bar{y})$ is

$$X_{n+1} = F(X_n) = F_j X_n,$$

where $X_n = \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}$ and F_J is the Jacobian matrix of the system (2) about the equilibrium point $(\bar{x}, \bar{y})$.

Lemma 1 ([50]). *For the system* $X_{n+1} = F(X_n)$, $n = 0, 1, \ldots$ *of difference equations with* $\overline{X}$ *as a fixed point of* F. *If all eigenvalues of the Jacobian matrix* J_f *about* $\overline{X}$ *lie inside an open unit disk* $|\lambda| < 1$, *then* $\overline{X}$ *is locally asymptotically stable. If one of them has norm greater than one, then* $\overline{X}$ *is unstable.*

Definition 3 ([9] (Lyapunov function)). Let $V : \mathbb{R}^k \to \mathbb{R}$. The variation of V relative to

$$x(n+1) = f(x(n)), \tag{3}$$

where $f : G \to \mathbb{R}^k$, $G \subset \mathbb{R}^k$, is continuous. We assume that $\overline{x}$ is an equilibrium point of (3), that is, $f(\overline{x}) = \overline{x}$.

Let $V : \mathbb{R}^k \to \mathbb{R}$ be a real-valued function. The variation of V relative to (3) would then be defined as

$$\Delta V(x) = V(f(x)) - V(x),$$

and

$$\Delta V(x(n)) = V(f(x(n))) - V(x(n)) = V(x(n+1)) - V(x(n)).$$

Note that if $\Delta V(x) \le 0$, then V is non-increasing along solutions of (3). The function V is said to be a Lyapunov function on a subset H of $\mathbb{R}^K$ if:

(i) V is continuous on H, and

(ii) $\Delta V(x) \le 0$, whenever x and $f(x)$ belong to H.

Definition 4 ([9]). Let $B(x, \gamma)$ denote the open ball in $\mathbb{R}^k$ of radius γ and center x defined by $B(x, \gamma) = \{y \in \mathbb{R}^k \mid \|y - x\| < \gamma\}$. For the sake of brevity, $B(0, \gamma)$ will henceforth be denoted by $B(\gamma)$. We say that the real-valued function V is positive definite at $\overline{x}$ if:

(i) $V(\overline{x}) = 0$, and

(ii) $V(x) > 0$ for all $x \in B(\overline{x}, \gamma)$, $x \neq \overline{x}$, for some $\gamma > 0$.

Definition 5 ([9] (Lyapunov stability theorem))**.** If V is a Lyapunov function for (3) in a neighbourhood H of the equilibrium point $\overline{x}$, and V is positive definite with respect to $\overline{x}$, then $\overline{x}$ is stable. If, in addition, $\Delta V(x) < 0$ whenever $x, f(x) \in H$ and $x = \overline{x}$, then $\overline{x}$ is asymptotically stable.

Moreover, if $G = H = \mathbb{R}^k$ and $V(x) \to \infty$ as $x \to \infty$, then $\overline{x}$ is globally asymptotically stable.

2. Stability of System (1)

In this section, we investigate the local stability character of the solutions of system (1). System (1) has two equilibrium points and are given by

$$\overline{x} = \frac{a\overline{xy}}{b + c\overline{y}} \Rightarrow \overline{x}(b + c\overline{y} - a\overline{y}) = 0 \Rightarrow \overline{x} = 0 \text{ or } \overline{y} = \frac{b}{a - c},$$

$$\overline{y} = \frac{d\overline{xy}}{e + f\overline{x}} \Rightarrow \overline{y}(e + f\overline{x} - d\overline{x}) = 0 \Rightarrow \overline{y} = 0 \text{ or } \overline{x} = \frac{e}{d - f}.$$

Then, they are two equilibrium points $O \equiv (0, 0)$ and $E \equiv \left(\frac{e}{d - f}, \frac{b}{a - c}\right)$ if $d \neq f$ and $a \neq c$.

Let $f : I^2 \times J^2 \longrightarrow I$ and $g : I^2 \times J^2 \longrightarrow J$ be continuously differentiable functions and I, J some intervals of real numbers defined by

$$f(x, y) = \frac{axy}{b + cy} \text{ and } g(x, y) = \frac{dxy}{e + fx}.$$

Therefore, it follows that

$$f_x(x, y) = \frac{ay}{b + cy}, \quad f_y(x, y) = \frac{abx}{(b + cy)^2},$$

$$g_x(x, y) = \frac{edy}{(e + fx)^2}, \quad g_y(x, y) = \frac{dx}{e + fx}.$$

At $E \equiv \left(\frac{e}{d - f}, \frac{b}{a - c}\right)$, we get

$$f_x\left(\frac{e}{d - f}, \frac{b}{a - c}\right) = 1,$$

$$f_y\left(\frac{e}{d - f}, \frac{b}{a - c}\right) = \frac{ab(\frac{e}{d - f})}{(b + c(\frac{b}{a - c}))^2} = \frac{e(a - c)^2}{ab(d - f)},$$

$$g_x\left(\frac{e}{d - f}, \frac{b}{a - c}\right) = \frac{ed(\frac{b}{a - c})}{(e + f(\frac{e}{d - f}))^2} = \frac{b(d - f)^2}{ed(a - c)},$$

$$g_y\left(\frac{e}{d - f}, \frac{b}{a - c}\right) = 1.$$

Theorem 2.1. *The equilibrium point E is unstable.*

Proof. The linearized equation of system (1) about the equilibrium E is

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & \frac{e(a - c)^2}{ab(d - f)} \\ \frac{b(d - f)^2}{ed(a - c)} & 1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}.$$

The roots of the characteristic equation of the system (1) about E are given by $\lambda_{1,2} = 1 \pm \sqrt{\frac{(a - c)(d - f)}{ad}}$, and it turns out that $|\lambda_1| > 1$. So the equilibrium point E is unstable.

Theorem 2.2. *The equilibrium point* O *is locally stable if* $ad \leq 1$.

Proof. Let

$$u_1(n) = x_{n-1}, \quad u_2(n) = x_n,$$

$$v_1(n) = y_{n-1}, \quad v_2(n) = y_n.$$

Now, we can rewrite the system (1) as

$$u_1(n+1) = u_2(n) \quad u_2(n+1) = \frac{au_1(n)v_1(n)}{b + cv_1(n)},$$

$$v_1(n+1) = v_2(n) \quad v_2(n+1) = \frac{du_1(n)v_1(n)}{e + fu_1(n)}.$$

Our first choice of a Lyapunov function will be $V(u_1, v_1, u_2, v_2) = u_1^2 v_1^2 + u_2^2 v_2^2$. This is clearly continuous and positive definite on $\mathbb{R}^4$.

So

$$\Delta V(u_1(n), v_1(n), u_2(n), v_2(n)) = u_1^2(n+1)v_1^2(n+1) + u_2^2(n+1)v_2^2(n+1)$$

$$- u_1^2(n)v_1^2(n) - u_2^2(n)v_2^2(n)$$

$$= u_2^2(n)v_2^2(n) + \left(\frac{a^2 u_1^2(n)v_1^2(n)}{(b + cv_1(n))^2}\right)\left(\frac{d^2 u_1^2(n)v_1^2(n)}{(e + fu_1(n))^2}\right)$$

$$- u_1^2(n)v_1^2(n) - u_2^2(n)v_2^2(n)$$

$$= \left(\frac{a^2 d^2 u_1^2(n)v_1^2(n)}{(b + cv_1(n))^2(e + fu_1(n))^2} - 1\right) u_1^2(n)v_1^2(n)$$

$$\leq (a^2 d^2 - 1)u_1^2(n)v_1^2(n).$$

Then $\Delta V \leq 0$ if $(ad)^2 \leq 1 \Rightarrow ad \leq 1$, thus the equilibrium point O is locally stable if $ad \leq 1$.

3. Existence of Bounded Solutions of System (1)

In this section, we study the boundedness of the solution of system (1).

Theorem 3.1. *Every positive solution of system* (1) *is bounded if* $a < c$, $d < f$ *and* $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$.

Proof. It follows from Equation (1) that

$$x_{n+1} = \frac{ax_{n-1}y_{n-1}}{b + cy_{n-1}} \leq \frac{ax_{n-1}y_{n-1}}{cy_{n-1}} = \frac{a}{c} x_{n-1} < x_{n-1},$$

$$y_{n+1} = \frac{dx_{n-1}y_{n-1}}{e + fx_{n-1}} \leq \frac{dx_{n-1}y_{n-1}}{fx_{n-1}} = \frac{d}{f} y_{n-1} < y_{n-1}.$$

Then

$$x_{n+1} < x_{n-1}, \quad y_{n+1} \leq y_{n-1}.$$

This implies that $x_{2n+1} < x_{2n-1}$ and $x_{2n+3} < x_{2n+1}$. Hence, the subsequences $\{x_{2n+1}\}$, $\{x_{2n+2}\}$ are decreasing, i.e., the sequence $\{x_n\}$ is decreasing. Similarly, one has $y_{2n+1} < y_{2n-1}$ and $y_{2n+3} < y_{2n+1}$. Hence, the subsequences $\{y_{2n+1}\}$, $\{y_{2n+2}\}$ are decreasing, i.e., the sequence $\{y_n\}$ is decreasing. Hence, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$.

Lemma 3.2. *If* $ad \leq 1$, *then the equilibrium point* O *is globally asymptotically stable.*

Proof. From Theorem 2.2, the equilibrium point O is locally stable. In addition, from Theorem 3.1, we have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$. Thus the equilibrium point O is a global attractor. This means O is globally asymptotically stable.

4. Existence of Periodic Solutions of System (1)

In this section, we study the existence of periodic solutions with period two of system (1).

Theorem 4.1. *The system* (1) *has no prime period-two solutions.*

Proof. Assume that $(p_1, q_1), (p_2, q_2), (p_1, q_1), \ldots$ is a prime period-two solution of system (1) such that $p_1 \neq p_2$ and $q_1 \neq q_2$. Then, from system (1), we have

$$p_1 = \frac{ap_1q_1}{b + cq_1}, \qquad p_2 = \frac{ap_2q_2}{b + cq_2}, \tag{4}$$

and

$$q_1 = \frac{ap_1q_1}{e + fp_1}, \qquad q_2 = \frac{dp_2q_2}{e + fp_2}. \tag{5}$$

From (4), we see that

$$bp_1 + (c - a)p_1q_1 = 0, \tag{6}$$

$$bp_2 + (c - a)p_2q_2 = 0. \tag{7}$$

Multiply Equation (6) by (p_2q_2) and Equation (7) by (p_1q_1), we get

$$bp_1p_2q_2 + (c - a)p_1p_2q_1q_2 = 0,$$

$$bp_1p_2q_1 + (c - a)p_1p_2q_1q_2 = 0.$$

Now, we have

$$bp_1p_2(q_2 - q_1) = 0 \Rightarrow q_1 = q_2.$$

Similarly, from (5)

$$eq_1 + (f - c)p_1q_1 = 0, \tag{8}$$

$$eq_2 + (f - c)p_2q_2 = 0. \tag{9}$$

Multiply Equation (8) by $(p_2 q_2)$ and Equation (9) by $(p_1 q_1)$, we get

$$ep_2 q_1 q_2 + (f - d) p_1 p_2 q_1 q_2 = 0,$$

$$ep_1 q_1 q_2 + (f - d) p_1 p_2 q_1 q_2 = 0.$$

Now, we have

$$eq_1 q_2 (p_2 - p_1) = 0 \Rightarrow p_1 = p_2,$$

which is a contradiction. Hence, system (1) has no prime period-two solutions.

5. Numerical Examples

To confirm the results of this paper, we consider numerical examples which represent different types of solutions to system (1).

Example 5.1. Figure 1 shows the solution when $x_{-1} = 2$, $x_0 = 4$, $y_{-1} = 5$, $y_0 = 3$, $a = 2$, $b = 5$, $c = 3$, $d = 7$, $e = 2$, and $f = 3$.

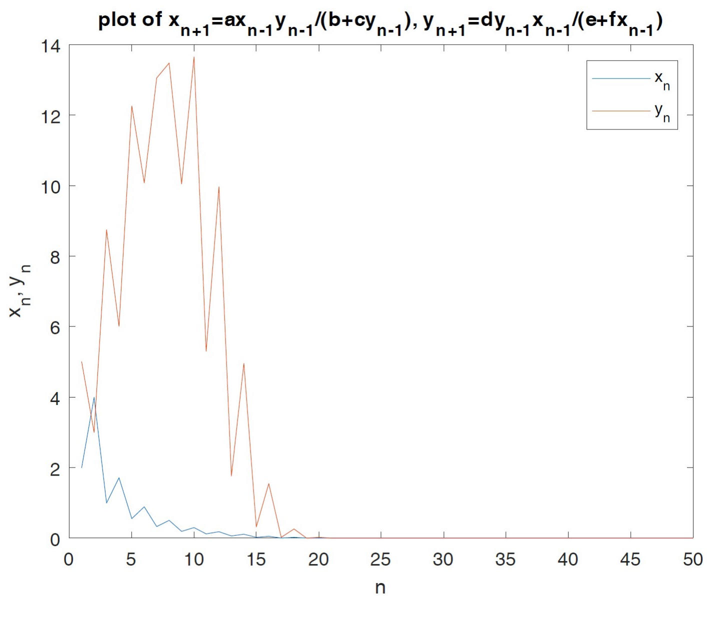

Figure 1.

Example 5.2. Figure 2 shows the solution when $x_{-1} = 12$, $x_0 = 0.8$, $y_{-1} = 1.1$, $y_0 = 9$, $a = 9$, $b = 0.5$, $c = 0.7$, $d = 2$, $e = 2$, and $f = 4$.

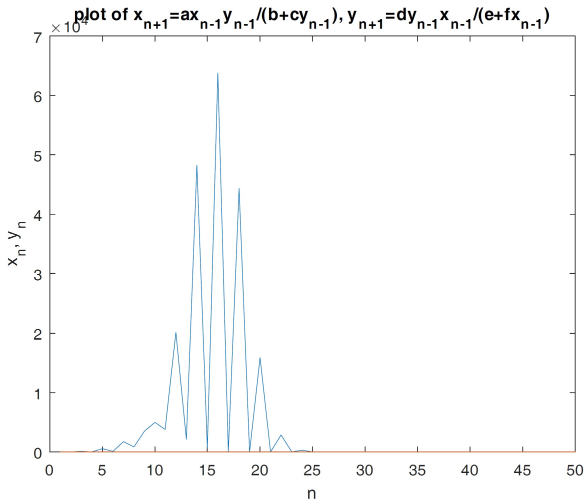

Figure 2.

Example 5.3. Figure 3 shows the behaviour of the solutions when we take $x_{-1} = 2$, $x_0 = 4$, $y_{-1} = 5$, $y_0 = 0.23$, $a = 12$, $b = 15$, $c = 3$, $d = 0.7$, $e = 2$, and $f = 0.3$.

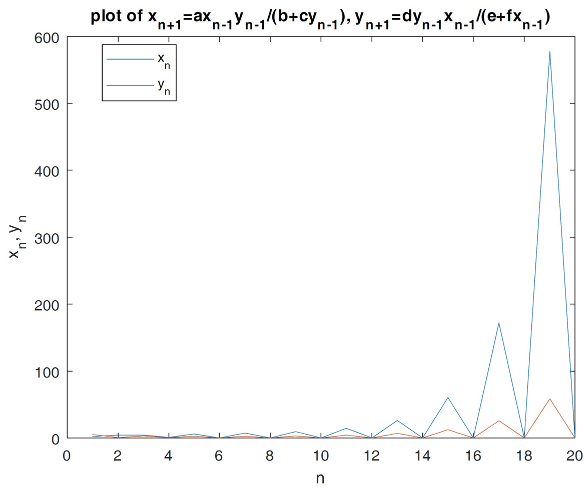

Figure 3.

Example 5.4. Figure 4 shows the dynamics of solutions of the system where $x_{-1} = 7$, $x_0 = 0.2$, $y_{-1} = 0.1$, $y_0 = 3$, $a = 1.2$, $b = 0.5$, $c = 0.3$, $d = 1.7$, $e = 12$, and $f = 0.3$.

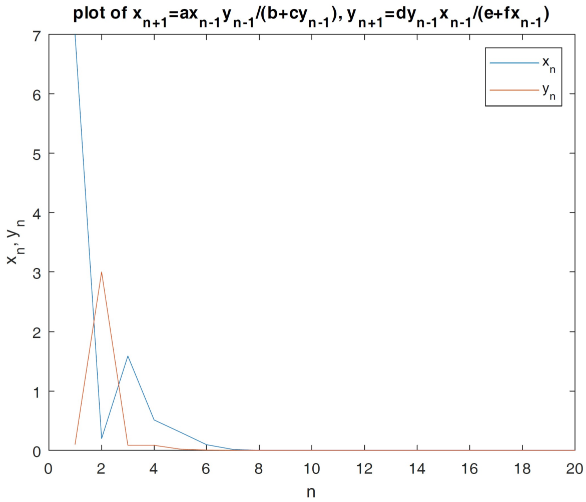

Figure 4.

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