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SOME VON NEUMANN-JORDAN TYPE AND GAO TYPE CONSTANTS RELATED TO MINIMUM IN BANACH SPACES

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Abstract

In this note, some relations between the von Neumann-Jordan constant $C_{NJ}(X)$ and the Schäffer constant $S(X)$, and the von Neumann-Jordan type constant $c'_{NJ}(X)$ are investigated. Then some Gao type constants are introduced and studied. Moreover, the relations of these constants to the geometric properties of the Banach space are also shown.

1. Introduction

The geometric constant has received widespread attention, which makes it easier for us to deal with some problems in Banach space, because it not only essentially reflects the geometric properties of a space

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X, but also enables us to study the space quantitatively. As a research tool, geometric constants have mathematical beauty, and there are countless relationships between different geometric constants.

We denote by X a Banach space with the norm $\|\cdot\|$ and the unit sphere S_X. Throughout this paper, we assume that the dimension of X is at least two.

Recall that X is called uniformly non-square, if there exist $\delta \in (0, 1)$ such that for each $x, y \in S_X$, either $\|x + y\| \leq 2(1 - \delta)$ or $\|x - y\| \leq 2(1 - \delta)$. In 1964, James introduced the concept of uniformly non-square [11]. Later, Schäffer [15] gave an equivalent definition of a uniformly non-square Banach space by stating that X is uniformly non-square, if there exists $\lambda > 1$ such that for any $x, y \in S_X$ either $\|x + y\| > \lambda$ or $\|x - y\| > \lambda$. In order to measure the degree of uniform non-squareness of X. In 1982, Gao [7] introduced the two constants

$$J(X) := \sup\{\min\{\|x + y\|, \|x - y\|\} : x, y \in S_X\},$$

$$S(X) := \inf\{\max\{\|x + y\|, \|x - y\|\} : x, y \in S_X\}.$$

usually called the James constant and Schäffer constant, respectively. Therefore, X is uniformly non-square in the sense of James (resp., Schäffer) if $J(X) < 2$ (resp., $S(X) > 1$)). Then, in 1986, Casini [3] proved that $J(X)S(X) = 2$, which implies the equivalence of the two definitions. More studied related to Schäffer constant, one can see [22].

The von Neumann-Jordan constant $C_{NJ}(X)$, in connection with the famous work of Jordan and von Neumann [10] concerning inner products, was introduced by Clarkson [4] as the smallest constant C for which

$$\frac{1}{C} \leq \frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C$$

holds for all $x, y \in X$ with $(x, y) \neq (0, 0)$. Indeed, it is not hard to see that

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X, (x, y) \neq (0, 0)\right\}.$$

It turns out that X is a Hilbert space if and only if $C_{NJ}(X) = 1$. Moreover, $J(X)$ and $C_{NJ}(X)$ can be mutually dominated as follows (see [13, 17, 19, 20]):

$$\frac{J(X)^2}{2} \leq C_{NJ}(X) \leq J(X), \tag{1.1}$$

from which we can also see that X is uniformly non-square if and only if $C_{NJ}(X) < 2$. For more results related to von Neumann-Jordan constant, one can refer to [23, 24].

The von Neumann-Jordan type constants essentially introduced by Gao [8] are defined as

$$C'_{NJ}(X) := \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{4} : x, y \in S_X\right\},$$

and

$$c'_{NJ}(X) := \inf\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{4} : x, y \in S_X\right\}.$$

The inequalities (1.1) hold with $C'_{NJ}(X)$ in place of $C_{NJ}(X)$ (see [1]), which implies that X is uniformly non-square if and only if $C'_{NJ}(X) < 2$. Moreover, $C'_{NJ}(X)$ and $c'_{NJ}(X)$ play an important role in [18], in which the famous Tingley's problem [16] is partially solved.

In [5], Cui et al. introduced the generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$, which is defined by

$$C_{NJ}^{(p)}(X) = \sup\left\{\frac{\|x+y\|^{p} + \|x-y\|^{p}}{2^{p-1}(\|x\|^{p} + \|y\|^{p})} : x,\ y \in X,\ (x,\ y) \neq (0,\ 0)\right\},$$

where $1 \leq p < \infty$. This motivates us to consider the constant $c_{NJ}^{(p)}(X)$, which is defined as

$$c_{NJ}^{(p)}(X) = \inf\left\{\frac{\|x+y\|^{p} + \|x-y\|^{p}}{2^{p}} : x,\ y \in S_X\right\},$$

where $1 \leq p < \infty$.

The constant $A_2(X)$ is given by

$$A_2(X) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} : x,\ y \in S_X\right\},$$

was first studied in Baronti et al. [2]. They give the value of this constant on some specific spaces.

This paper is organized as follows: In Section 2, we investigate the geometric constant $c'_{NJ}(X)$. First, we give the bounds of $c'_{NJ}(X)$. Then some relationships between $c'_{NJ}(X)$ and other well-known constants are given. Meanwhile, we state characterization of inner product space and uniformly non-square Banach space in terms of $c'_{NJ}(X)$. Finally, we consider the constant $c_{NJ}^{(p)}(X)$, which is a generalization of $c'_{NJ}(X)$.

In Section 3, we introduce the constant $f_H(X)$ and $e_H(X)$, which can be seen as Gao type constants. Some properties of them can be obtained. Estimates of two geometric constants in inner product space and in the Minkowski planes can also be shown. We also establish the relation between $e_H(X)$ and $c'_{NJ}(X)$ by means of the inequality. Finally, we characterize the consistent non-square space by $f_H(X)$.

2. The von Neumann-Jordan Type Constant Related to Minimum

In this section, we mainly pay attention to the geometric constant $c'_{NJ}(X)$. First, we give the bounds of the geometric constant $c_{NJ}^{(p)}(X)$ on Banach space in the following proposition:

Proposition 2.1. *Let* X *be a Banach space. Then* $\frac{1}{2^{p-1}} \leq c_{NJ}^{(p)}(X) \leq 1.$

Proof. For any $x, y \in S_X$, let $y = x$, then we have

$$c_{NJ}^{(p)}(X) \leq \frac{\|x + x\|^p + \|x - x\|^p}{2^p} = 1.$$

Actually, by the inequality $(a + b)^p \leq 2^{p-1}(a^p + b^p)$, where a, b are nonnegative scalars, then we have

$$\begin{aligned} \|2x\|^p &\leq (\|x + y\| + \|x - y\|)^p \\ &\leq 2^{p-1}(\|x + y\|^p + \|x - y\|^p). \end{aligned}$$

This gives $c_{NJ}^{(p)}(X) \geq 1/2^{p-1}$. □

Remark 2.2. In the proposition above, we can see that if $p = 1$, then $c_{NJ}^{(1)}(X) = 1$ for any Banach space X. If $p = 2$, then $\frac{1}{2} \leq c'_{NJ}(X) \leq 1$ for any Banach space X.

Now, we give the following example to show that the bounds given in the Proposition 2.1 are sharp.

Example 2.3. Let $X = (\mathbb{R}^2, \|\cdot\|_1)$, then $c_{NJ}^{(p)}(X) = \frac{1}{2^{p-1}}$.

Proof. Fix $x = \left(\frac{1}{2}, \frac{1}{2}\right)$, $y = \left(\frac{1}{2}, -\frac{1}{2}\right)$, then $\|x + y\| = 1$ and $\|x - y\| = 1$. We thus get

$$c_{NJ}^{(p)}(X) \leq \frac{\|x + y\|^{p} + \|x - y\|^{p}}{2^{p}} = \frac{1}{2^{p-1}}.$$

Therefore $c_{NJ}^{(p)}(X) = \frac{1}{2^{p-1}}$. □

It is natural to try to relate the geometric constant $c'_{NJ}(X)$ to inner product space. In order to complete the proof of our proposition, we give the following lemma:

Lemma 2.4 ([6])**.** *A normed space $(X, \|\cdot\|)$ is an inner product space if and only if*

$$\|x + y\|^2 + \|x - y\|^2 \sim 4,$$

for any $x, y \in S_X$, where $\sim$ stands for $=, \geq$ or $\leq$.

Proposition 2.5. *Let X be a Banach space. Then X is an inner product space if and only if $c'_{NJ}(X) = 1$.*

Proof. If X is an inner product space, then we get

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2),$$

for any $x, y \in S_X$. This gives $c'_{NJ}(X) = 1$.

Conversely, if $c'_{NJ}(X) = 1$, then we have

$$\|x + y\|^2 + \|x - y\|^2 \geq 4,$$

for each $x, y \in S_X$. It follows from Lemma 2.4 that X is an inner product space. □

The geometric constant $c'_{NJ}(X)$ is closely related to the other geometric constants. First, we give the relationship between $c'_{NJ}(X)$ and $S(X)$.

Theorem 2.6. *Let X be a Banach space. Then* $S(X)^2/4 \le c'_{NJ}(X) \le S(X)^2/2$.

Proof. For any $x, y \in S_X$, we have

$$\|x+y\|^2 + \|x-y\|^2 \le 2\max\{\|x+y\|^2, \|x-y\|^2\},$$

which yields $c'_{NJ}(X) \le S(X)^2/2$.

On the other hand, since

$$\|x+y\|^2 \le 4\,\frac{\|x+y\|^2 + \|x-y\|^2}{4},$$

and

$$\|x-y\|^2 \le 4\,\frac{\|x+y\|^2 + \|x-y\|^2}{4},$$

then we get

$$\max\{\|x+y\|^2, \|x-y\|^2\} \le 4\,\frac{\|x+y\|^2 + \|x-y\|^2}{4}.$$

From this, we obtain

$$\inf\{\max\{\|x+y\|^2, \|x-y\|^2\}\} \le 4\inf\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{4}\right\}.$$

This completes the proof. □

Below we will consider the connection between $c'_{NJ}(X)$ and $C_{NJ}(X)$.

Theorem 2.7. *Let X be a Banach space. Then* $C_{NJ}(X)^{-1} \leq c'_{NJ}(X) \leq 2C_{NJ}(X)^{-2}$.

Proof. Let $x, y \in S_X$, then we see

$$\frac{1}{c'_{NJ}(X)} \leq \sup\left\{\frac{2(\|x\|^2 + \|y\|^2)}{\|x+y\|^2 + \|x-y\|^2}\right\} \leq C_{NJ}(X).$$

By Theorem 2.6 and the fact that $J(X)S(X) = 2$, $J(X)^2/2 \leq C_{NJ}(X) \leq J(X)$, we obtain $c'_{NJ}(X) \leq 2C_{NJ}(X)^{-2}$. □

Proposition 2.8. *Let X be a Banach space*. Then $c'_{NJ}(X) \leq A_2(X)^2$.

Proof. For any $x, y \in S_X$, since

$$\frac{\|x+y\|^2 + \|x-y\|^2}{4} \leq \frac{(\|x+y\| + \|x-y\|)^2}{4},$$

which shows $c'_{NJ}(X) \leq A_2(X)^2$. □

The next theorem that follows was stated in [12]. Now we can see that it is a corollary of Theorem 2.6 or Theorem 2.7.

Theorem 2.9. *Let X be a Banach space. Then X is uniformly non-square if and only if* $c'_{NJ}(X) > 1/2$.

Remark 2.10. We know that uniformly non-square in the sense of James and in the sense of Schäffer are equivalence since $J(X)S(X) = 2$, and we also see that X is uniformly non-square if and only if $C'_{NJ}(X) < 2$ if and only if $c'_{NJ}(X) > 1/2$. However, $C'_{NJ}(X)c'_{NJ}(X) = 1$ can fail. See Remark 3.5 in [21].

3. The Gao Type Constants Related to Minimum

Now let us pay attention to Gao's parameter $f_t(X)$, which is defined by the formula

$$f_t(X) := \inf\{\|x + ty\|^2 + \|x - ty\|^2 : x, y \in S_X\},$$

where t is a nonnegative number.

Observe that $f_1(X) = 4c'_{NJ}(X)$. Moreover, Jiao and Pang [12] showed that, if $f_t(X) > [(1+t^2)^2 + 2t(1-t^2)](2+t^2 - t\sqrt{4+t^2})$ for some $t \in (0, 1]$, then X has uniform normal structure.

Inspired by the Gao's parameter, we introduce the following definitions:

Definition 3.1.

$$f_H(X) := \inf\{\|x + y\| + \|2x - y\| : \|x\| = \|y\| = \|x - y\| = 1\},$$

$$e_H(X) := \inf\{\|x + y\|^2 + \|2x - y\|^2 : \|x\| = \|y\| = \|x - y\| = 1\}.$$

Proposition 3.2. *Let X be a Banach space. Then* $9 \le f_H(X)^2 \le 2e_H(X) \le 16.$

Proof. For any $x, y \in S_X$, we see

$$\begin{aligned} \|3x\|^2 &\le (\|x + y\| + \|2x - y\|)^2 \\ &\le 2(\|x + y\|^2 + \|2x - y\|^2) \\ &\le 2(\|x\| + \|y\|)^2 + 2\|x\| + \|x - y\|)^2 = 16. \end{aligned}$$

This completes the proof. □

Calculating the exact values of $f_H(X)$ and $e_H(X)$ on some specific spaces is difficult, but we next give estimates for two geometric constants on inner product spaces.

Proposition 3.3. *If X is an inner product spaces space, then* $f_H(X) \leq 2\sqrt{3}$ *and* $e_H(X) \leq 6$.

Proof. It follows immediately from the observation that for any $x, y \in S_X$ satisfying $\|x - y\| = 1$,

$$\|x + y\|^2 = 2(\|x\|^2 + \|y\|^2) - \|x - y\|^2 = 3,$$

and

$$\|2x - y\|^2 = 2(\|x\|^2 + \|x - y\|^2) - \|x - (x - y)\|^2 = 3.$$

□

In [14], Martin et al. have stated that, in any Minkowski planes,

$$\inf\{\|x + y\| : \|x\| = \|y\| = \|x - y\| = 1\} \leq \sqrt{3}.$$

Thus we have the following theorem:

Theorem 3.4. *If X is a Minkowski planes, then* $f_H(X) \leq 2 + \sqrt{3}$ *and* $e_H(X) \leq 7$.

Now, we give the relationship between $c'_{NJ}(X)$ and $e_H(X)$.

Proposition 3.5. *Let X be a Banach space. Then* $e_H(X) \geq 4c'_{NJ}(X) - 1$.

Proof. For any $x, y \in S_X$ satisfying $\|x - y\| = 1$, we have

$$\begin{aligned}
\|x + y\|^2 + \|2x - y\|^2 &\geq \|x + y\|^2 + (\|x - y\| - \|x\|)^2 \\
&= \|x + y\|^2 + \|x - y\|^2 - 2\|x - y\| + 1 \\
&\geq 4c'_{NJ}(X) - 1,
\end{aligned}$$

which completes the proof. □

We give the connection between $f_H(X)$ and uniformly non-square in the following theorem:

Theorem 3.6. *A finite-dimensional Banach space X is uniformly non-square if* $f_H(X) \geq 3$.

Proof. We will show that $f_H(X) \leq 3$ if X is not uniformly non-square, i.e., if $S(X) = 1$. Since $S(X) = \inf\{\|x + y\| : x, y \in S_X, \|x + y\| = \|x - y\|\}$ (see Proposition 7.4 in [9]) and S_X is compact, there exists $x, y \in S_X$ such that $\|x + y\| = \|x - y\| = 1$. Hence

$$f_H(X) \leq \|x + y\| + \|2x - y\| \leq \|x + y\| + \|x - y\| + \|x\| = 3.$$

This completes the proof. □

4. Conclusion

In this paper, we consider von Neumann-Jordan type constant $c'_{NJ}(X)$ and generalized von Neumann-Jordan type constant $c_{NJ}^{(p)}(X)$. Meanwhile, we introduced the Gao type constants $f_H(X)$ and $e_H(X)$. Then, some properties of these constants are given. It is shown that characterization of uniformly non-square Banach space and inner product space in terms of $c'_{NJ}(X)$. We also state some relationships between $c'_{NJ}(X)$ and other well-known constants. Next, we give the relationship between $c'_{NJ}(X)$ and $e_H(X)$. In addition, we obtain estimates for $f_H(X)$ and $e_H(X)$ in inner product space and Minkowski plane. However, the values of these geometric constants on some specific spaces are not yet known, and whether the geometric constant $c_{NJ}^{(p)}(X)$ can be used to characterize the inner product space. These are also questions that we will investigate later.

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