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NON RELATIVISTIC LIMIT OF THE CLOSURE OF A RECENT RELATIVISTIC MODEL FOR POLYATOMIC GASES

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Abstract

A recent 15 moments relativistic model of extended thermodynamics for polyatomic gases is here considered; it is based on a new hierarchy of moments that takes into account the total energy, i.e., the rest energy and the energy of the molecular internal mode. The classical limit was there studied but only for the equations in integral form. However, after that, these equations were manipulated to obtain the closure up to first order with respect to equilibrium but expressed in terms of physical variables. Therefore, it is necessary to perform the non relativistic limit also of the resulting final closure and compare it with the known results concerning the classical case. This goal is realized in the present article. Another result here obtained is the expansions of the scalar coefficients appearing in the closure as finite polynomials in $\frac{1}{c^2}$,

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where c is the light speed. The characteristic equations for determining the wave velocities are here deducted also for the subsystems with 5 moments (Euler equations), and with 6 or 14 moments. All the present results holds in the non polytropic case; the polytropic case is obtained only as a subcase.

1. Introduction

To understand the scientific framework in which the present article is inserted, let us recall that Extended Thermodynamics was formulated to eliminate some drawbacks of Ordinary Thermodynamics, for example by obtaining a set of field equations which is hyperbolic (in armony with the cause and effect principle) and predicts finite wave speeds of propagation (in armony with the Einstenian Principle according to which nothing can propagate with a velocity greater than that of light).

An important article in this context was [1] concerning the classical case, followed by [2] which implemented it for the relativistic case. However, the resulting field equations contained an expression for the energy e and the pressure p which restricted the same results only to the case of monoatomic gases. Subsequently, Prof. Ruggeri and coworkers found the way to implement them, for the classical case, to polyatomic gases and produced the article [3]. One of the outstanding ideas of this and following articles was that there exist two blocks of field equations, namely the so-called mass block and a new one called the energy block. The relativistic version of this work was realized in [4]. The starting point was the Boltzmann-Chernikov equation (see Equation (4) of [4]), where the distribution function f is supposed to depend not only on the 4-dimensional position x^{α} and on the four-momentum p^{α}, but also on the internal energy $\mathcal{I}$. Moreover, Q is the collisional term. From the Boltzmann-Chernikov equation, the following set of field equations were obtained:

$$\partial_{\alpha} V^{\alpha} = 0, \quad \partial_{\alpha} T^{\alpha\beta} = 0, \quad \partial_{\alpha} A^{\alpha\beta\gamma} = I^{\beta\gamma}. \tag{1}$$

But these equations have 15 independent components; so to recover the classical known case, it was decided to take only the traceless part of Equation $(1)_3$, i.e., $\partial_\alpha A^{\alpha<\beta\gamma>} = I^{<\beta\gamma>}$ instead of $\partial_\alpha A^{\alpha\beta\gamma} = I^{\beta\gamma}$.

However, the fact to simply put out the trace of Equation $(1)_3$ is someway unnatural and this was confirmed in the subsequent article [6] where more moments were considered and it was seen that in the non relativistic limit the relativistic model predicts for the classical case field equations with more than 2 blocks but, in any way, there is a particular structure with which these blocks can be assembled.

Following these guidelines, the article [7] was produced in the framework of classical thermodynamics. But, in [8], it was noted that the factor $1 + j\frac{\mathcal{I}}{mc^2}$ (introduced in the definitions of [4] for $T^{\alpha\beta}$, $A^{\alpha\beta\gamma}$ and $I^{\beta\gamma}$) is nothing more than the first two terms of the binomial Newtonian formula for $\left(1 + \frac{\mathcal{I}}{mc^2}\right)^j$; moreover, $1 + \frac{\mathcal{I}}{mc^2}$ has more physical meaning because it is the total energy (i.e., $\mathcal{I}$ plus the rest mass energy mc^2) normalized by dividing it by mc^2. So it is more convenient to replace $1 + j\frac{\mathcal{I}}{mc^2}$ with $(1 + \frac{\mathcal{I}}{mc^2})^j$ in the definitions of $A^{\alpha\alpha_1\ldots\alpha_j}$ and $I^{\alpha_1\cdots\alpha_j}$. For this reasons the article [7] was updated in [9]. The relativistic counterpart of this article has been published in [10]. In its Equation (8) we find the new expressions of $A^{\alpha\alpha_1\ldots\alpha_j}$ and $I^{\alpha_1\cdots\alpha_j}$. As a bonus, we obtain that all the closure can be determined in terms of the energy e and its derivatives with respect to the absolute temperature T or, equivalently of $\gamma = \frac{mc^2}{k_B T}$ with k_B the Boltzmann constant.

However, for length reasons some interesting further investigations are missing in this article and also in the subsequent one [11]. So they are here reported. They are too important; for example, in [10], the classical limit was studied but only for the equations in integral form. However, after that, these equations were manipulated to obtain the closure up to first order with respect to equilibrium but expressed in terms of physical variables. Therefore, it is necessary to perform the non relativistic limit also of the resulting final closure and compare it with the known results concerning the classical case. This was done in [6]; in fact, on pages 432 and 433 the classical limit was studied in the integral form and, after that, on pages 434 and 435 it was performed for the final set of equations by using also the equations there found at the end of page 435. The importance of this aspect is outlined by the fact that the same calculations are reported with more particulars in the second set of equations on page 509 of [12] which concerned always the old approach of [4]. Now that [4] has been updated in [10], the same considerations need to be done for the new approach. Moreover, the new structure of [10] allows to perform similar calculations but in an easier and more elegant way. The new counterparts of the second set of equations on page 509 of [12] are here reported in (13).

In Section 4, the particular case of a polytropic case is exploited, while in sect. 5 the wave speeds are investigated for the subsystems with 14, 6 or 5 moments. In this way we can see how they evolve in the transition from a subsystem to the other with an increasing number of moments, In particular, the 6 moments model is the smallest one with dissipative effects and can be useful to study the evolution of the universe in its first stage when dynamic pressure dominates over all the other dissipative components. So in the present article there are the following new original parts:

• The expansions of the scalar coefficients appearing in the closure as finite polynomials in $\frac{1}{c^2}$, both for the non polytropic case and for the polytropic case. This is necessary for the following point and can be useful for forthcoming articles.

• The classical limit of the closed system of 15 balance equations, showing that it coincides with what is reported in the paper [9].

• The calculation of the wave velocities for the subsystems with 14, 6, and 5 moments.

2. The New Closed Field of Equations and the Expansions of the Coefficients as Powers of $\frac{1}{c^2}$

The closure found in [10] for the field equations (1) is the following one:

$$V^{\alpha} = \rho U^{\alpha}, \quad T^{\alpha\beta} = eU^{\alpha}U^{\beta} + (p + \Pi)h^{\alpha\beta} + 2U^{(\alpha}q^{\beta)} + t^{<\alpha\beta>},$$

$$A^{\alpha\beta\gamma} = \left(\rho\theta_{02} + \frac{1}{4c^4}\Delta\right)U^{\alpha}U^{\beta}U^{\gamma} + \left(\rho c^2\theta_{12} - \frac{3}{4c^2}\frac{N^{\Delta}}{D_4}\Delta - 3\frac{N^{\pi}}{D_4}\Pi\right)h^{(\alpha\beta}U^{\gamma)}$$

$$+ \frac{3}{c^2}\frac{N_3}{D_3}q^{(\alpha}U^{\beta}U^{\gamma)} + \frac{3}{5}\frac{N_{31}}{D_3}h^{(\alpha\beta}q^{\gamma)} + 3C_5 t^{(<\alpha\beta>}U^{\gamma)},$$

$$I^{\beta\gamma} = -\frac{1}{4c^4\tau}\Delta U^{\beta}U^{\gamma} + \left(\frac{1}{4c^2\tau}\frac{N^{\Delta}}{D_4}\Delta + \frac{N^{\pi}}{D_4}\frac{1}{\tau}\Pi\right)h^{\beta\gamma}$$

$$+ \left(-\frac{2}{c^2\tau}\frac{N_3}{D_3} + \frac{\theta_{1,3}}{\theta_{1,2}}\frac{1}{c^2\tau}\right)q^{(\beta}U^{\gamma)} - \frac{1}{\tau}C_5 t^{<\beta\gamma>_3}, \tag{2}$$

where the independent variables are ρ (the mass density), U^α (the 4-velocity constrained by $U^\alpha U_\alpha = c^2$), γ (related to the absolute temperature T by $\gamma = \frac{mc^2}{k_B T}$, with k_B the Boltzmann constant), Π (the dynamic pressure), q^α (the heat flux constrained by $q^\alpha U_\alpha = 0$), $t^{<\alpha\beta>}$ (the deviatoric shear viscous stress tensor constrained by $t^{<\alpha\beta>} U_\alpha = 0$, $t^{<\alpha\beta>} g_{\alpha\beta} = 0$) and Δ as 15-th variable (from $(2)_3$ it is evident that $\Delta = \frac{4}{c^2}\left(A^{\alpha\beta\gamma} - A^{\alpha\beta\gamma}_{eq.}\right)$). Moreover, we have that $h^{\alpha\beta} = -g^{\alpha\beta} + \frac{1}{c^2} U^\alpha U^\beta$, the pressure p and the energy are given by

$$p = \frac{nmc^2}{\gamma} = nk_B T, \qquad e = nmc^2 \frac{\int_0^{+\infty} J^*_{2,2}\left(1 + \frac{\mathcal{I}}{mc^2}\right)\phi(\mathcal{I})\, d\mathcal{I}}{\int_0^{+\infty} J^*_{2,1}\phi(\mathcal{I})\, d\mathcal{I}}, \tag{3}$$

with $J^*_{m,n} = J_{m,n}(\gamma^*)$ and $\gamma^* = \gamma \cdot \left(1 + \frac{\mathcal{I}}{mc^2}\right)$.

The matrices D_4, N^Π, N^Δ, D_3, N_3, N_{31} and the scalars C_5, θ_{ij} are reported in Equations $(29)_{2-4}$, $(32)_{2-4}$, $(34)_2$ and (16) of [10]. Here we define

$$\epsilon = \frac{e}{\rho} - c^2 \text{ (In view of the fact that } \lim_{c\to+\infty} \frac{e - \rho c^2}{\rho} = \epsilon_{classic}),$$

$$f_1 = \varepsilon^2 + \frac{k_B T^2}{m}\frac{\partial \varepsilon}{\partial T}, \quad f_2 = \varepsilon f_1 + \frac{k_B T^2}{m}\frac{\partial f_1}{\partial T},$$

$$f_3 = \varepsilon f_2 + \frac{k_B T^2}{m}\frac{\partial f_2}{\partial T}, \quad g_1 = \varepsilon + \frac{k_B T}{m},$$

$$g_2 = f_1 + 2\frac{k_B T}{m} g_1, \quad g_3 = f_2 + 3\frac{k_B T}{m} g_2. \tag{4}$$

It is interesting that these quantities have finite limits for c going to $+\infty$ and they hold both in the polytropic case and in the non polytropic case. They allow also to rewrite the coefficients ϑ_{ij} as finite polynomials in $\frac{1}{c^2}$, i.e.,

$$\vartheta_{0,0} = 1, \quad \vartheta_{0,1} = 1 + \frac{\varepsilon}{c^2}, \quad \vartheta_{0,2} = 1 + 2\frac{\varepsilon}{c^2} + f_1 \frac{1}{c^4},$$

$$\vartheta_{0,3} = 1 + 3\frac{\varepsilon}{c^2} + 3f_1 \frac{1}{c^4} + f_2 \frac{1}{c^6},$$

$$\vartheta_{0,4} = 1 + 4\frac{\varepsilon}{c^2} + 6f_1 \frac{1}{c^4} + 4f_2 \frac{1}{c^6} + f_3 \frac{1}{c^8},$$

$$\vartheta_{1,1} = \frac{k_B T}{m}\frac{1}{c^2}, \quad \vartheta_{1,2} = 3\frac{k_B T}{m}\frac{1}{c^2} + 3\frac{k_B T}{m} g_1 \frac{1}{c^4},$$

$$\vartheta_{1,3} = 6\frac{k_B T}{m}\frac{1}{c^2} + 12\frac{k_B T}{m} g_1 \frac{1}{c^4} + 6\frac{k_B T}{m} g_2 \frac{1}{c^6},$$

$$\vartheta_{2,3} = 3\left(\frac{k_B T}{m}\right)^2 \frac{1}{c^4} + 3\left(\frac{k_B T}{m}\right)^2 g_1 \frac{1}{c^6},$$

$$\vartheta_{1,4} = 10\frac{k_B T}{m}\frac{1}{c^2} + 30\frac{k_B T}{m} g_1 \frac{1}{c^4} + 30\frac{k_B T}{m} g_2 \frac{1}{c^6} + 10\frac{k_B T}{m} g_3 \frac{1}{c^8},$$

$$\vartheta_{2,4} = 15\left(\frac{p}{\rho}\right)^2 \frac{1}{c^4} + 15\left(\frac{p}{\rho}\right)^2\left(\frac{p}{\rho} + 2g_1\right)\frac{1}{c^6} + 15\left(\frac{p}{\rho}\right)^2\left(\frac{p}{\rho} g_1 + g_2\right)\frac{1}{c^8}. \tag{5}$$

From these relations, it follows

$$\vartheta_{0,1} - \vartheta_{0,0} = \frac{\varepsilon}{c^2},$$

$$\vartheta_{0,2} - 2\vartheta_{0,1} + \vartheta_{0,0} = f_1 \frac{1}{c^4},$$

$$\vartheta_{0,3} - 3\vartheta_{0,2} + 3\vartheta_{0,1} - \vartheta_{0,0} = f_2 \frac{1}{c^6},$$

$$\vartheta_{0,4} - 4\vartheta_{0,3} + 6\vartheta_{0,2} - 4\vartheta_{0,1} + \vartheta_{0,0} = f_3 \frac{1}{c^8},$$

$$\frac{\vartheta_{1,2}}{3} - \vartheta_{1,1} = \frac{k_B T}{m} g_1 \frac{1}{c^4},$$

$$\frac{\vartheta_{1,3}}{6} - 2\frac{\vartheta_{1,2}}{3} + \vartheta_{1,1} = \frac{k_B T}{m} g_2 \frac{1}{c^6},$$

$$\frac{\vartheta_{1,4}}{10} - \frac{\vartheta_{1,3}}{2} + \vartheta_{1,2} - \vartheta_{1,1} = \frac{p}{\rho} g_3 \frac{1}{c^8},$$

$$\vartheta_{2,4} - 5\vartheta_{2,3} = 15\left(\frac{p}{\rho}\right)^2\left(\frac{p}{\rho} + g_1\right)\frac{1}{c^6} + 15\left(\frac{p}{\rho}\right)^2\left(\frac{p}{\rho} g_1 + g_2\right)\frac{1}{c^8}. \tag{6}$$

By using these expressions we will prove that the scalar functions appearing in the closure be written as

$$e = \rho\epsilon + c^2, \quad p = \frac{k_B}{m}\rho T, \quad \vartheta_{0,2} = 1 + 2\frac{\varepsilon}{c^2} + f_1\frac{1}{c^4},$$

$$\vartheta_{1,2} = 3\frac{k_B T}{m}\frac{1}{c^2} + 3\frac{k_B T}{m} g_1 \frac{1}{c^4},$$

$$\vartheta_{1,3} = 6\frac{k_B T}{m}\frac{1}{c^2} + 12\frac{k_B T}{m} g_1 \frac{1}{c^4} + 6\frac{k_B T}{m} g_2 \frac{1}{c^6}, \tag{7}$$

$$D_4 = \begin{vmatrix} 1 & \varepsilon\frac{1}{c^2} & f_1\frac{1}{c^4} & \frac{p}{\rho}\frac{1}{c^2} + \frac{p}{\rho}g_1\frac{1}{c^4} \\ \varepsilon\frac{1}{c^2} & f_1\frac{1}{c^4} & f_2\frac{1}{c^6} & \frac{p}{\rho}g_1\frac{1}{c^4} + \frac{p}{\rho}g_2\frac{1}{c^6} \\ f_1\frac{1}{c^4} & f_2\frac{1}{c^6} & f_3\frac{1}{c^8} & \frac{p}{\rho}g_2\frac{1}{c^6} + \frac{p}{\rho}g_3\frac{1}{c^8} \\ \frac{p}{\rho}\frac{1}{c^2} & \frac{p}{\rho}g_1\frac{1}{c^4} & \frac{p}{\rho}g_2\frac{1}{c^6} & \frac{5}{3}\left(\frac{p}{\rho}\right)^2\frac{1}{c^4} + \frac{5}{3}\left(\frac{p}{\rho}\right)^2 g_1\frac{1}{c^6} \end{vmatrix}$$

$$= \left(\frac{p}{\rho}\right)^2\frac{1}{c^{16}}\left(D_4' + \frac{1}{c^2}D_4''\right), \tag{8}$$

with

$$D_4' = \begin{vmatrix} 1 & \varepsilon & f_1 & 1 \\ \varepsilon & f_1 & f_2 & g_1 \\ f_1 & f_2 & f_3 & g_2 \\ 1 & g_1 & g_2 & \frac{5}{3} \end{vmatrix}, \quad D_4'' = \begin{vmatrix} 1 & \varepsilon & f_1 & g_1 \\ \varepsilon & f_1 & f_2 & g_2 \\ f_1 & f_2 & f_3 & g_3 \\ 1 & g_1 & g_2 & \frac{5}{3} g_1 \end{vmatrix}.$$

$$N^\pi + D_4 = \begin{vmatrix} 1 & \varepsilon \frac{1}{c^2} & f_1 \frac{1}{c^4} & \frac{1}{c^2} + g_1 \frac{1}{c^4} \\ \varepsilon \frac{1}{c^2} & f_1 \frac{1}{c^4} & f_2 \frac{1}{c^6} & g_1 \frac{1}{c^4} + g_2 \frac{1}{c^6} \\ f_1 \frac{1}{c^4} & f_2 \frac{1}{c^6} & f_3 \frac{1}{c^8} & g_2 \frac{1}{c^6} + g_3 \frac{1}{c^8} \\ -g_1 \frac{1}{c^4} & -g_2 \frac{1}{c^6} & -g_3 \frac{1}{c^8} & -\frac{5}{3}\left(\frac{p}{\rho} + g_1\right)\frac{1}{c^6} - \frac{5}{9}\left(\frac{p}{\rho} g_1 + g_2\right)\frac{1}{c^8} \end{vmatrix}$$

$$\left(\frac{p}{\rho}\right)^2 = \frac{1}{c^{18}}\left(\frac{p}{\rho}\right)^2 N_1^\pi + \frac{1}{c^{20}}\left(\frac{p}{\rho}\right)^2 N_2^\pi, \tag{9}$$

where

$$N_1^\pi = \begin{vmatrix} 1 & \varepsilon & f_1 & 1 \\ \varepsilon & f_1 & f_2 & g_1 \\ f_1 & f_2 & f_3 & g_2 \\ -g_1 & -g_2 & -g_3 & -\frac{5}{3}\left(\frac{p}{\rho} + g_1\right) \end{vmatrix},$$

$$N_2^\pi = \begin{vmatrix} 1 & \varepsilon & f_1 & g_1 \\ \varepsilon & f_1 & f_2 & g_2 \\ f_1 & f_2 & f_3 & g_3 \\ -g_1 & -g_2 & -g_3 & -\frac{5}{9}\left(\frac{p}{\rho} g_1 + g_2\right) \end{vmatrix},$$

$$N^{\Delta} = \begin{vmatrix} 1 & \varepsilon \frac{1}{c^2} & f_1 \frac{1}{c^4} & \frac{p}{\rho}\frac{1}{c^2} + \frac{p}{\rho} g_1 \frac{1}{c^4} \\ \varepsilon \frac{1}{c^2} & f_1 \frac{1}{c^4} & f_2 \frac{1}{c^6} & \frac{p}{\rho} g_1 \frac{1}{c^4} + \frac{p}{\rho} g_2 \frac{1}{c^6} \\ \frac{p}{\rho}\frac{1}{c^2} & \frac{p}{\rho} g_1 \frac{1}{c^4} & \frac{p}{\rho} g_2 \frac{1}{c^6} & \frac{5}{3}\left(\frac{p}{\rho}\right)^2 \frac{1}{c^4} + \frac{5}{3}\left(\frac{p}{\rho}\right)^2 g_1 \frac{1}{c^6} \\ \frac{p}{\rho} g_1 \frac{1}{c^4} & \frac{p}{\rho} g_2 \frac{1}{c^6} & \frac{p}{\rho} g_3 \frac{1}{c^8} & \frac{5}{3}\left(\frac{p}{\rho}\right)^2 \left[\left(\frac{p}{\rho} + g_1\right)\frac{1}{c^6} + \left(\frac{p}{\rho} g_1 + g_2\right)\frac{1}{c^8}\right] \end{vmatrix}$$

$$= \frac{1}{c^{16}}\left(\frac{p}{\rho}\right)^3 N_1^{\Delta} + \frac{1}{c^{18}}\left(\frac{p}{\rho}\right)^3 N_2^{\Delta}, \qquad (10)$$

where

$$N_1^{\Delta} = \begin{vmatrix} 1 & \varepsilon & f_1 & 1 \\ \varepsilon & f_1 & f_2 & g_1 \\ 1 & g_1 & g_2 & \frac{5}{3} \\ g_1 & g_2 & g_3 & \frac{5}{3}\left(\frac{p}{\rho} + g_1\right) \end{vmatrix}, \quad N_2^{\Delta} = \begin{vmatrix} 1 & \varepsilon & f_1 & g_1 \\ \varepsilon & f_1 & f_2 & g_2 \\ 1 & g_1 & g_2 & \frac{5}{3} g_1 \\ g_1 & g_2 & g_3 & \frac{5}{3}\left(\frac{p}{\rho} g_1 + g_2\right) \end{vmatrix},$$

$$D_3 = 9\left(\frac{k_B T}{m}\right)^2 \left[g_2 - (g_1)^2\right]\frac{1}{c^8}, \quad N_3 - 2D_3 = 9\left(\frac{k_B T}{m}\right)^2 (g_3 - g_1 g_2)\frac{1}{c^{10}}, \qquad (11)$$

$$N_{31} = \frac{5}{\gamma} D_3 + 45\left(\frac{k_B T}{m}\right)^4 \frac{1}{c^8}\left(1 + g_1 \frac{1}{c^2}\right),$$

$$C_5 = \frac{1 + \left(\frac{k_B T}{m} + 2g_1\right)\frac{1}{c^4} + \left(\frac{k_B T}{m} g_1 + g_2\right)\frac{1}{c^4}}{1 + g_1 \frac{1}{c^2}}. \qquad (12)$$

2.1. Proof of (7)-(12)

The expressions (7), (12) are easy consequences of (5).

• To make easier the proof of Equation (8), let us introduce the matrix

$$M = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ from which it follows}$$

$$D_4 = |M^T| D_4 |M|$$

$$= \begin{vmatrix} \vartheta_{0,0} & \vartheta_{0,1} - \vartheta_{0,0} & \vartheta_{0,2} - 2\vartheta_{0,1} + \vartheta_{0,0} & \frac{1}{3}\vartheta_{1,2} \\ \vartheta_{0,1} - \vartheta_{0,0} & \vartheta_{0,2} - 2\vartheta_{0,1} + \vartheta_{0,0} & \begin{matrix}\vartheta_{0,3} - 3\vartheta_{0,2} \\ +3\vartheta_{0,1} - \vartheta_{0,0}\end{matrix} & \frac{1}{6}(\vartheta_{1,3} - 2\vartheta_{1,2}) \\ \vartheta_{0,2} - 2\vartheta_{0,1} + \vartheta_{0,0} & \begin{matrix}\vartheta_{0,3} - 3\vartheta_{0,2} \\ +3\vartheta_{0,1} - \vartheta_{0,0}\end{matrix} & \begin{matrix}\vartheta_{0,4} - 4\vartheta_{0,3} + 6\vartheta_{0,2} \\ -4\vartheta_{0,1} + \vartheta_{0,0}\end{matrix} & \frac{\vartheta_{1,4}}{10} - \frac{\vartheta_{1,3}}{3} + \frac{\vartheta_{1,2}}{3} \\ \vartheta_{1,1} & \frac{\vartheta_{1,2}}{3} - \vartheta_{1,1} & \frac{\vartheta_{1,3}}{6} - 2\frac{\vartheta_{1,2}}{3} + \vartheta_{1,1} & 5\frac{\vartheta_{2,3}}{9} \end{vmatrix}.$$

By substituting here $(5)_{1,6,7,9}$ and $(6)_{1-7}$ we obtain (8).

• Let us consider now N^π. In particular, we have

$$N^\pi + D_4 = \begin{vmatrix} \vartheta_{0,0} & \vartheta_{0,1} & \vartheta_{0,2} & \frac{1}{3}\vartheta_{1,2} \\ \vartheta_{0,1} & \vartheta_{0,2} & \vartheta_{0,3} & \frac{1}{6}\vartheta_{1,3} \\ \vartheta_{0,2} & \vartheta_{0,3} & \vartheta_{0,4} & \frac{\vartheta_{1,4}}{10} \\ \vartheta_{1,1} - \frac{\vartheta_{1,2}}{3} & \frac{\vartheta_{1,2}}{3} - \frac{\vartheta_{1,3}}{6} & \frac{\vartheta_{1,3}}{6} - \frac{\vartheta_{1,4}}{10} & \frac{5\vartheta_{2,3} - \vartheta_{2,4}}{9} \end{vmatrix}.$$

With the above used matrix M, this determinant becomes

$$N^{\pi} + D_4 = |M^T|(N^{\pi} + D_4)|M|$$

$$= \begin{vmatrix} \vartheta_{0,0} & \vartheta_{0,1} - \vartheta_{0,0} & \vartheta_{0,2} - 2\vartheta_{0,1} + \vartheta_{0,0} & \frac{1}{3}\vartheta_{1,2} \\ \vartheta_{0,1} - \vartheta_{0,0} & \vartheta_{0,2} - 2\vartheta_{0,1} + \vartheta_{0,0} & \begin{matrix}\vartheta_{0,3} - 3\vartheta_{0,2} \\ +3\vartheta_{0,1} - \vartheta_{0,0}\end{matrix} & \frac{1}{6}\left(\vartheta_{1,3} - 2\vartheta_{1,2}\right) \\ \vartheta_{0,2} - 2\vartheta_{0,1} + \vartheta_{0,0} & \begin{matrix}\vartheta_{0,3} - 3\vartheta_{0,2} \\ +3\vartheta_{0,1} - \vartheta_{0,0}\end{matrix} & \begin{matrix}\vartheta_{0,4} - 4\vartheta_{0,3} + 6\vartheta_{0,2} \\ -4\vartheta_{0,1} + \vartheta_{0,0}\end{matrix} & \frac{\vartheta_{1,4}}{10} - \frac{\vartheta_{1,3}}{3} + \frac{\vartheta_{1,2}}{3} \\ \vartheta_{1,1} - \frac{\vartheta_{1,2}}{3} & \frac{2\vartheta_{1,2}}{3} - \frac{\vartheta_{1,3}}{6} - \vartheta_{1,1} & \frac{\vartheta_{1,3}}{2} - \frac{\vartheta_{1,4}}{10} - \vartheta_{1,2} + \vartheta_{1,1} & 5\frac{\vartheta_{2,3} - \vartheta_{2,4}}{9} \end{vmatrix}.$$

By using here the decompositions (5) and (6) we obtain (9).

• Let us consider now N^{Δ} by using the above matrix M and we see that

$$N^{\Delta} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{vmatrix} N^{\Delta}|M|$$

$$= \begin{vmatrix} \vartheta_{0,0} & \vartheta_{0,1} - \vartheta_{0,0} & \vartheta_{0,2} - 2\vartheta_{0,1} + \vartheta_{0,0} & \frac{1}{3}\vartheta_{1,2} \\ \vartheta_{0,1} - \vartheta_{0,0} & \vartheta_{0,2} - 2\vartheta_{0,1} + \vartheta_{0,0} & \vartheta_{0,3} - 3\vartheta_{0,2} + 3\vartheta_{0,1} - \vartheta_{0,0} & \frac{1}{6}\left(\vartheta_{1,3} - 2\vartheta_{1,2}\right) \\ \vartheta_{1,1} & \frac{\vartheta_{1,2}}{3} - \vartheta_{1,1} & \frac{\vartheta_{1,3}}{6} - 2\frac{\vartheta_{1,2}}{3} + \vartheta_{1,1} & 5\frac{\vartheta_{2,3}}{9} \\ \frac{\vartheta_{1,2}}{3} - \vartheta_{1,1} & \frac{\vartheta_{1,3}}{6} - 2\frac{\vartheta_{1,2}}{3} + \vartheta_{1,1} & \frac{1}{10}\vartheta_{1,4} - \frac{1}{2}\vartheta_{1,3} + \vartheta_{1,2} - \vartheta_{1,1} & \frac{\vartheta_{2,4} - 5\vartheta_{2,3}}{9} \end{vmatrix}.$$

By using here the decompositions (5) and (6) we obtain (10).

• Let us continue with D_3. We see that

$$D_3 = \begin{vmatrix} 1 & 0 \\ -3 & 1 \end{vmatrix} D_3 \begin{vmatrix} 1 & -3 \\ 0 & 1 \end{vmatrix} = \begin{vmatrix} \theta_{1,1} & \theta_{1,2} - 3\theta_{1,1} \\ \theta_{1,2} - 3\theta_{1,1} & \frac{3}{2}\theta_{1,3} - 6\,\theta_{1,2} + 9\,\theta_{1,1} \end{vmatrix}.$$

By using here $(5)_6$ and $(6)_{5,6}$ we obtain $(11)_1$.

• Let us consider now $N_3 - 2\,D_3$. We see that

$$N_3 - 2\,D_3 = \begin{vmatrix} 1 & 0 \\ -3 & 1 \end{vmatrix} \begin{vmatrix} \theta_{1,1} & \theta_{1,2} \\ \frac{1}{2}\theta_{1,3} - 2\,\theta_{1,2} & \frac{9}{10}\theta_{1,4} - 3\,\theta_{1,3} \end{vmatrix} \begin{vmatrix} 1 & -3 \\ 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \theta_{1,1} & \theta_{1,2} - 3\,\theta_{1,1} \\ \frac{1}{2}\theta_{1,3} - 2\,\theta_{1,2} + 32\,\theta_{1,1} & \frac{9}{10}\theta_{1,4} - \frac{9}{2}\theta_{1,3} + 9\,\theta_{1,2} - 9\,\theta_{1,1} \end{vmatrix}.$$

By using here $(5)_6$ and $(6)_{5,6,7}$ we obtain $(11)_2$.

• Let us finish with N_{31}

$$N_{31} - \frac{5}{\gamma} D_3 = \begin{vmatrix} \theta_{1,1} & \theta_{1,2} \\ 5\,\theta_{2,3} - 5\,\frac{k_B T}{m}\theta_{1,2}\,\frac{1}{c^2} & 3\,\theta_{2,4} - \frac{15}{2}\,\frac{k_B T}{m}\theta_{1,3}\,\frac{1}{c^2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{k_B T}{m}\,\frac{1}{c^2} & \theta_{1,2} \\ 0 & 45\left(\frac{k_B T}{m}\right)^3 \frac{1}{c^6}\left(1 + g_1\,\frac{1}{c^8}\right) \end{vmatrix},$$

where in the last step we have used Equations (5); the result proves $(11)_3$.

3. The Non Relativistic Limit of the Final Equations

In [10, 11], the classical limit of the field equations was considered but only for the equations in the integral form. However, after that, these equations were manipulated to obtain the closure up to first order with respect to equilibrium but expressed in terms of physical variables. Therefore, it is necessary to perform the non relativistic limit also of the resulting final closure and compare it with the known results concerning the classical case. This was done in [6] which concerned the old set of balance equations; in fact, on pages 432 and 433 the classical limit was studied in the integral form and, after that, on pages 434 and 435 it was performed for the final set of equations by using also the equations there found at the end of page 435. The importance of this aspect is outlined by the fact that the same calculations are reported with more particulars in the second set of equations on page 509 of [12] which concerned always the old approach of [4]. Now that [4] has been updated in [10], the same considerations need to be done for the new approach and this is the subject of the present section. In the present new model this non relativistic limit becomes easier, thanks to the results of the previous section.

To do this non relativistic limit we need some ingredients, i.e., the following particular limits (here we use $A_1^0 = \rho\theta_{02}$, $3\,A_{11}^0 = \rho\, c^2\theta_{12}$ in order to have the same notation of [4])

$$\lim_{c\to+\infty} A_1^0 = \rho,\ \lim_{c\to+\infty}(A_1^0-\rho)\,c^2 = 2\,\rho\,\epsilon,\ A_1^0c^4+\rho\,c^4-2e\,c^2 = \rho f_1,$$

$$\lim_{c\to+\infty} A_{11}^0 = p,\ (A_{11}^0-p)\,c^2 = p\left(\epsilon+\frac{p}{\rho}\right),\ \lim_{c\to+\infty}\frac{N_3}{D_3} = 2,\ \left(\frac{N_3}{D_3}-2\right)c^2$$

$$= \frac{g_3-g_1g_2}{g_2-(g_1)^2},\quad \lim_{c\to+\infty}\frac{N_{31}}{D_3} = \frac{5}{g_2-(g_1)^2}\left(\frac{p}{\rho}\right)^2,\ \lim_{c\to+\infty} C_5 = 1,$$

$$\lim_{c\to+\infty}(C_5-1)c^2 = \varepsilon+\frac{2p}{\rho},\ \lim_{c\to\infty}\frac{N^\Delta}{D_4} = \frac{N_1^\Delta}{D_4^1},\ \lim_{c\to+\infty}\frac{N^\pi}{D_4} = -1,$$

$$\lim_{c\to+\infty}\left(\frac{N^\pi}{D_4}+1\right)c^2 = \frac{N_1^\pi}{D_4^1}. \tag{13}$$

Let us begin noting that $(13)_3$ is a direct consequence of $(4)_1$ and $(5)_3$; after that, $(13)_{1,2}$ are its consequences. Similarly, $(13)_5$ is a direct consequence of $(4)_5$ if we take into account that $p = \rho c^2 \theta_{11}$ according to $(5)_6$; after that, $(13)_4$ is its consequence. The Equation $(13)_7$ comes from $(11)_{1,2}$ and $(13)_6$ is its consequence. The Equation $(13)_8$ is a consequence of $(11)_{1,3}$. The Equation $(13)_{10}$ is a consequence of (12) and $(4)_5$. After that, it implies $(13)_9$.

From (8)-(10), it follows that

$$\lim_{c\to+\infty} c^{16} D_4 = \left(\frac{p}{\rho}\right)^2 D_4', \quad \lim_{c\to+\infty} c^{18}(N^\pi + D_4) = \left(\frac{p}{\rho}\right)^2 N_1^\pi,$$

$$\lim_{c\to+\infty} c^{16} N^\Delta = \left(\frac{p}{\rho}\right)^3 N_1^\Delta,$$

and these results imply $(13)_{10,12}$. Finally, $(13)_{11}$ is a consequence of $(13)_{12}$.

Now we use these ingredients to calculate the non relativistic limit of the field equations (47) of [10] which are expressed in terms of the material derivative (we don't report them here for the sake of brevity). In particular, we calculate the non relativistic limits of $(47)_1$, $(47)_2$ with $\delta = i$, $(47)_3$ minus $(47)_1$ multiplied by c^2, $(47)_4$ with $\delta\theta = ij$, $(47)_5$ minus $(47)_2$ multiplied by c^2 (with $\delta = i$), the sum of $(47)_6$ plus $(47)_3$ multiplied by $-2c^2$ and $(47)_1$ multiplied by c^4; by using the above particular limits, we find that the limits of these linear combination of the field equations are

$$\dot{\rho} + \rho \partial_k v^k = 0,$$

$$\rho \dot{v}^i + \partial_i (p + \pi) + \partial_k t^{<ki>} = 0,$$

$$\frac{d}{dt}(\rho \epsilon) + (\rho \epsilon + p + \pi) \partial_k v^k + \partial_k q^k + t^{<ki>} \partial_i v^k = 0,$$

$$(\dot{p} + \dot{\pi}) \delta_{ij} + t^{<ij>} + (p + \pi) \left(\delta_{ij} \partial_k v^k + 2 \partial_{(i} v_{j)} \right)$$

$$+ \frac{1}{g_2 - (g_1)^2} \left(\frac{p}{\rho} \right)^2 \left(\delta_{ij} \partial_k q^k + 2 \partial_{(i} q_{j)} \right) + t_{<ij>} \partial_k v^k + 2 \partial_k v_{(i} t_{<j)k>}$$

$$= - \frac{1}{\tau} \left(- \frac{N_1^\Delta}{4 D_4^1} \frac{\rho}{p} \Delta + \pi \right) \delta_{ij} - \frac{1}{\tau} t_{<ij>},$$

$$\dot{v}^i (\rho \epsilon + p + \pi) + \dot{q}^i + t_{<il>} \dot{v}^l + \partial_i \left[p \left(\epsilon + \frac{p}{\rho} \right) - \frac{N_1^\Delta}{4 D_4^1} \frac{\rho}{p} \Delta - \frac{N_1^\pi}{D_4^1} \pi \right]$$

$$+ \partial_k \left[\left(\varepsilon + \frac{2p}{\rho} \right) t^{<ki>} \right] + \left(1 + \frac{1}{g_2 - (g_1)^2} \left(\frac{p}{\rho} \right)^2 \right) (q^k \partial_k v^i + q^i \partial_k v^k)$$

$$+ \frac{1}{g_2 - (g_1)^2} \left(\frac{p}{\rho} \right)^2 q_k \partial_i v^k = - \frac{1}{\tau} q_i,$$

$$\frac{d}{dt}(\rho f_1) + \frac{1}{4} \dot{\Delta} - 2 q^l \dot{v}_l + \left(\partial_k v^k \right) \left[\rho f_1 + \frac{1}{4} \left(1 - 2 \frac{N_1^\Delta}{D_4^1} \frac{\rho}{p} \right) \Delta - 2 \frac{N_1^\pi}{D_4^1} \pi \right]$$

$$+ \partial_k \left(\frac{g_3 - g_1 \, g_2}{g_2 - (g_1)^2} q^k \right) + 2 \left(\varepsilon + \frac{2p}{\rho} \right) t^{<kl>} \partial_k v^l = - \frac{1}{4\tau} \Delta. \tag{14}$$

Obviously, we can substitute here f_1, g_1, g_2, g_3 from Equation (4) and D_4^1, N_1^Δ, N_1^π from Equations $(8)_2$, $(9)_2$ and $(10)_2$; the Equations (14) are equivalent to (53) of [9], where there is a change of notation which here I don't consider necessary because it makes the results apparently more complicated, without gaining elegance.

In (14), $\dot{f}$ or $\frac{df}{dt}$ denote the material derivative of f, i.e., $\dot{f} = \frac{df}{dt} = \partial_t f + v^i \partial_i f$.

4. The Polytropic Case

In this case the measure $\phi(\mathcal{I})$ has the form $\phi(\mathcal{I}) = \mathcal{I}^a$ with $a = \frac{D-5}{2}$ (see Equation (12) of [4]), for example, $D = 5$ for diatomic gases, while the monoatomic gases can be obtained in the limit for D going to 3.

So in the polytropic case we have $\varepsilon = \frac{D}{2}\frac{k_B T}{m}$ (see Equation (9) of [4] which doesn't change in the new model because the expression of $T^{\alpha\beta}_{eq.}$ remains the same). So (4) becomes

$$\varepsilon = \frac{D}{2}\frac{k_B T}{m} = \frac{D}{2}\frac{p}{\rho},$$

$$f_1 = \left(\frac{p}{\rho}\right)^2 \frac{D(D+2)}{4}, \; f_2 = \left(\frac{p}{\rho}\right)^3 \frac{D(D+2)(D+4)}{8},$$

$$f_3 = \left(\frac{p}{\rho}\right)^4 \frac{D(D+2)(D+4)(D+6)}{16}, \; g_1 = \frac{p}{\rho}\frac{D+2}{2},$$

$$g_2 = \left(\frac{p}{\rho}\right)^2 \frac{(D+2)(D+4)}{4}, \quad g_3 = \left(\frac{p}{\rho}\right)^3 \frac{(D+2)(D+4)(D+6)}{8},$$

and the determinants in Equations (8)-(12) become

$$D_4' = \begin{vmatrix} 1 & \frac{D}{2} & \frac{D(D+2)}{4} & 1 \\ \frac{D}{2} & \frac{D(D+2)}{4} & \frac{D(D+2)(D+4)}{8} & \frac{D+2}{2} \\ \frac{D(D+2)}{4} & \frac{D(D+2)(D+4)}{8} & \frac{D(D+2)(D+4)(D+6)}{16} & \frac{(D+2)(D+4)}{4} \\ 1 & \frac{D+2}{2} & \frac{(D+2)(D+4)}{4} & \frac{5}{3} \end{vmatrix} \times$$

$$\times \left(\frac{p}{\rho}\right)^6 = \left(\frac{p}{\rho}\right)^6 \frac{D(D+2)(D-3)}{6},$$

$$D_4'' = \begin{vmatrix} 1 & \frac{D}{2} & \frac{D(D+2)}{4} & \frac{D+2}{2} \\ \frac{D}{2} & \frac{D(D+2)}{4} & \frac{D(D+2)(D+4)}{8} & \frac{(D+2)(D+4)}{4} \\ \frac{D(D+2)}{4} & \frac{D(D+2)(D+4)}{8} & \frac{D(D+2)(D+4)(D+6)}{16} & \frac{(D+2)(D+4)(D+6)}{8} \\ 1 & \frac{D+2}{2} & \frac{(D+2)(D+4)}{4} & \frac{5}{3}\frac{D+2}{2} \end{vmatrix} \times$$

$$\times \left(\frac{p}{\rho}\right)^7.$$

$$N_1^{\pi} = \begin{vmatrix} 1 & \frac{D}{2} & \frac{D(D+2)}{4} & 1 \\ \frac{D}{2} & \frac{D(D+2)}{4} & \frac{D(D+2)(D+4)}{8} & \frac{D+2}{2} \\ \frac{D(D+2)}{4} & \frac{D(D+2)(D+4)}{8} & \frac{D(D+2)(D+4)(D+6)}{16} & \frac{(D+2)(D+4)}{4} \\ -\frac{D+2}{2} & -\frac{(D+2)(D+4)}{4} & -\frac{(D+2)(D+4)(D+6)}{8} & -\frac{5}{6}(D+4) \end{vmatrix} \times$$

$$\times \left(\frac{p}{\rho}\right)^7 = -\frac{D(D+2)(D-3)(D+4)}{12}\left(\frac{p}{\rho}\right)^7,$$

$$N_2^{\pi} = \begin{vmatrix} 1 & \frac{D}{2} & \frac{D(D+2)}{4} & \frac{D+2}{2} \\ \frac{D}{2} & \frac{D(D+2)}{4} & \frac{D(D+2)(D+4)}{8} & \frac{(D+2)(D+4)}{4} \\ \frac{D(D+2)}{4} & \frac{D(D+2)(D+4)}{8} & \frac{D(D+2)(D+4)(D+6)}{16} & \frac{(D+2)(D+4)(D+6)}{8} \\ -\frac{D+2}{2} & -\frac{(D+2)(D+4)}{4} & -\frac{(D+2)(D+4)(D+6)}{8} & -\frac{5}{9}\frac{(D+2)(D+6)}{4} \end{vmatrix} \times$$

$$\times \left(\frac{p}{\rho}\right)^8 = \frac{1}{36} D(D+2)^2 (D+6)(D+9)\left(\frac{p}{\rho}\right)^8,$$

$$N_1^{\Delta} = \begin{vmatrix} 1 & \frac{D}{2} & \frac{D(D+2)}{4} & 1 \\ \frac{D}{2} & \frac{D(D+2)}{4} & \frac{D(D+2)(D+4)}{8} & \frac{D+2}{2} \\ 1 & \frac{D+2}{2} & \frac{(D+2)(D+4)}{4} & \frac{5}{3} \\ \frac{D+2}{2} & \frac{(D+2)(D+4)}{4} & \frac{(D+2)(D+4)(D+6)}{8} & \frac{5}{6}(D+4) \end{vmatrix} \times$$

$$\times \left(\frac{p}{\rho}\right)^5 = -\frac{(D+2)(D-3)}{3}\left(\frac{p}{\rho}\right)^5.$$

$$N_2^{\Delta} = \begin{vmatrix} 1 & \frac{D}{2} & \frac{D(D+2)}{4} & \frac{D+2}{2} \\ \frac{D}{2} & \frac{D(D+2)}{4} & \frac{D(D+2)(D+4)}{8} & D+2 \\ 1 & \frac{D+2}{2} & \frac{(D+2)(D+4)}{4} & \frac{5}{6}(D+2) \\ \frac{D+2}{2} & \frac{(D+2)(D+4)}{4} & \frac{(D+2)(D+4)(D+6)}{8} & \frac{5}{6}(D+2)(D+6) \end{vmatrix} \times$$

$$\times \left(\frac{p}{\rho}\right)^6 = -\frac{1}{12}(D+2)^2(5D-12)\left(\frac{p}{\rho}\right)^6.$$

$$D_3 = \frac{9}{2}(D+2)\left(\frac{p}{\rho}\right)^4 \frac{1}{c^8}, \quad N_3 - 2D_3 = \frac{9}{2}(D+2)(D+4)\left(\frac{p}{\rho}\right)^5 \frac{1}{c^{10}},$$

$$N_{31} = 45\left(\frac{p}{\rho}\right)^4 \frac{1}{c^8}\left[1 + (D+2)\frac{p}{\rho}\frac{1}{c^2}\right],$$

$$C_5 - 1 = \frac{1}{c^2} \frac{\frac{D+4}{2}\frac{p}{\rho} + \frac{(D+2)(D+6)}{4}\left(\frac{p}{\rho}\right)^2 \frac{1}{c^2}}{1 + \frac{D+2}{2}\frac{p}{\rho}\frac{1}{c^2}}.$$

The above results are useful to have the closure of the field equations for these particular gases. Regarding the limits in Equation (13), they become

$$\lim_{c\to+\infty} A_1^0 = \rho, \ \lim_{c\to+\infty}(A_1^0 - \rho)c^2 = 2\rho\,\epsilon,$$

$$A_1^0 c^4 + \rho c^4 - 2ec^2 = \frac{p^2}{\rho}\frac{D(D+2)}{4},$$

$$\lim_{c\to+\infty} A_{11}^0 = p, \ (A_{11}^0 - p)c^2 = \frac{p^2}{\rho}\frac{D+2}{2}, \ \lim_{c\to+\infty}\frac{N_3}{D_3} = 2,$$

$$\left(\frac{N_3}{D_3} - 2\right)c^2 = (D+4)\frac{p}{\rho},$$

$$\lim_{c\to+\infty}\frac{N_{31}}{D_3} = \frac{10}{D+2}, \ \lim_{c\to+\infty} C_5 = 1, \ \lim_{c\to+\infty}(C_5 - 1)c^2 = \frac{p}{2\rho}(D+4),$$

$$\lim_{c\to\infty}\frac{N^\Delta}{D_4} = -\frac{2}{D}, \ \lim_{c\to+\infty}\frac{N^\pi}{D_4} = -1, \ \lim_{c\to+\infty}\left(\frac{N^\pi}{D_4} + 1\right)c^2 = -\frac{p}{\rho}\frac{D+4}{2}. \quad (15)$$

We can evaluate also the non relativistic limit of the field equations, i.e., (14) for this particular case of polytropic gases; I avoid to write them for the sake of brevity, but we note that their subsystem with 14 fields gives Equations (68) of [4].

5. Wave Equations for the Subsystems

In [11], the wave equations for the 15-moment model was written and studied. It was found that the corresponding characteristic equations can be written by simply taking the derivatives with respect to $\delta\lambda_A$ of the quantity

$$\delta K_E = -\frac{m}{k_B}\varphi_\alpha[V_E^\alpha(\delta\lambda)^2 + 2T_E^{\alpha\mu}\delta\lambda\delta\lambda_\mu + 2A_E^{\alpha\mu\nu}\delta\lambda\delta\lambda_{\mu\nu}$$

$$+ A_E^{\alpha\beta\delta}\delta\lambda_\beta\delta\lambda_\delta + 2A_E^{\alpha\beta\mu\nu}\delta\lambda_\beta\delta\lambda_{\mu\nu} + A_E^{\alpha\beta\gamma\mu\nu}\delta\lambda_{\beta\gamma}\delta\lambda_{\mu\nu}], \qquad (16)$$

and equating them to zero. The expressions of the tensors in the right hand side are reported in [9]; moreover,

$$\varphi_\alpha = \frac{u}{c}\xi_\alpha + \eta_\alpha \text{ with } \xi_\alpha\xi^\alpha = 1,\ \xi_\alpha\eta^\alpha = 0,\ \eta_\alpha\eta^\alpha = -1. \qquad (17)$$

In this way the speeds of wave propagation u are found; someone may not like that the eigenvectors are written in terms of the Lagrange multipliers $\delta\lambda_A$. But this isn't a problem because, after having written the characteristic equations, we can do the following change of variables:

$$\delta\lambda = \delta\lambda_E + (\delta\lambda - \delta\lambda_E),\ \delta\lambda_\beta = \delta\lambda_\beta^E + (\delta\lambda_\beta - \delta\lambda_\beta^E),$$

$$\text{with } \delta\lambda_E = -\frac{k_B}{m\rho}\delta\rho + \frac{e}{\rho T^2}\delta T,\ \delta\lambda_\beta^E = \delta\left(\frac{U_\beta}{T}\right), \qquad (18)$$

(Equations (26) and (40) of [4] have been used), while $\delta\lambda - \delta\lambda_E$, $\delta\lambda_\beta - \delta\lambda_\beta^E$, $\delta\lambda_{\mu\nu}$ can be deduced from Equation (36) of [10] and they read

$$\delta\lambda - \delta\lambda_E = a_1\delta_\pi + a_2\delta\Delta,$$

$$\delta\lambda_\beta - \delta\lambda_\beta^E = (b_1\delta_\pi + b_2\delta\Delta)U_\beta + b_3\delta_{q\beta},$$

$$\delta\lambda_{\beta\gamma} = (\alpha_1\delta_\pi + \beta_1\delta\Delta)U_\beta U_\gamma + (\alpha_2\delta_\pi + \beta_2\delta\Delta)h_{\beta\gamma} + 2\alpha_3 U_{(\beta}\delta q_{\gamma)}$$

$$+ 2\alpha_4\delta t_{<\beta\gamma>}. \qquad (19)$$

In this way the characteristic equations transform in those we would obtain by starting directly with the field equations (1), (2) expressed in terms of the physical variables, i.e.,

$$\varphi_\alpha \delta(\rho U^\alpha) = 0,\ \varphi_\alpha \delta\left[\frac{e}{c^2} U^\alpha U^\beta + p h^{\alpha\beta}\right] + \varphi_\alpha \left[h^{\alpha\beta}\delta\pi + \frac{2}{c^2} U^{(\alpha}\delta q^{\beta)} + \delta t^{<\alpha\beta>}\right] = 0,$$

$$\varphi_\alpha \delta\left[\rho\theta_{02} U^\alpha U^\beta U^\gamma + \rho c^2 \theta_{12} h^{(\alpha\beta}U^{\gamma)}\right]$$

$$+ \varphi_\alpha \delta\left[\frac{1}{4c^4} U^\alpha U^\beta U^\gamma \delta\Delta + \left(-\frac{3}{4c^2}\frac{N^\Delta}{D_4}\delta\Delta - 3\frac{N^\pi}{D_4}\delta\pi\right) h^{(\alpha\beta}U^{\gamma)}\right.$$

$$\left. + \frac{3}{c^2}\frac{N_3}{D_3} U^{(\alpha}U^\beta \delta q^{\gamma)} + \frac{3}{5}\frac{N_{31}}{D_3} h^{(\alpha\beta}\delta q^{\gamma)} + 3C_5 U^{(\alpha}\delta t^{(<\beta\gamma>_3)}\right] = 0,$$

$$U_\alpha \delta U^\alpha = 0,\ U_\alpha \delta q^\alpha = 0,\ U_\alpha \delta t^{<\alpha\beta>_3} = 0,\ h_{\alpha\beta}\delta t^{<\alpha\beta>_3} = 0, \tag{20}$$

where the last 5 equations are present because we are dealing with variables constrained by

$$U_\alpha U^\alpha = c^2, \quad U_\alpha q^\alpha = 0, \quad U_\alpha t^{<\alpha\beta>_3} = 0, \quad h_{\alpha\beta} t^{<\alpha\beta>_3} = 0.$$

Now the matrix of the coefficients of the differentials must have a determinant equal to zero and this equation gives the wave speeds. What we have said lets us understand that the determinant obtained by starting from Equations (20) is equal to that obtained in terms of the Lagrange multipliers multiplied on the right by the determinant of the linear transformation (19) and on the left by its transposite.

But now we want to find the wave speeds for the subsystems with 14, 6 and 5 moments which were not treated in [11].

5.1. Wave speeds for the subsystem with 14 moments

This study is easy in terms of the Lagrange multipliers. In fact, this subsystem can be obtained from that with 15 moments by putting $\lambda_{\mu\nu} = \Sigma_{\mu\nu}$, where $\Sigma_{\mu\nu}$ is a traceless tensor. So Equation (17) becomes

$$\delta K_E = -\frac{m}{k_B}\varphi_\alpha\left[V_E^\alpha(\delta\lambda)^2 + 2T_E^{\alpha\mu}\delta\lambda\delta\lambda_\mu + 2A_E^{\alpha<\mu\nu>}\delta\lambda\delta\Sigma_{\mu\nu}\right.$$

$$\left. + A_E^{\alpha\beta\delta}\delta\lambda_\beta\delta\lambda_\delta + 2A_E^{\alpha\beta<\mu\nu>}\delta\lambda_\beta\delta\Sigma_{\mu\nu} + A_E^{\alpha<\beta\gamma><\mu\nu>}\delta\Sigma_{\beta\gamma}\delta\Sigma_{\mu\nu}\right], \tag{21}$$

and the characteristic equations are

$$\varphi_\alpha\left[V_E^\alpha\delta\lambda + T_E^{\alpha\mu}\delta\lambda_\mu + A_E^{\alpha<\mu\nu>}\delta\sum\nolimits_{\mu\nu}\right] = 0,$$

$$\varphi_\alpha\left[T_E^{\alpha\delta}\delta\lambda + A_E^{\alpha\beta\delta}\delta\lambda_\beta + A_E^{\alpha\delta<\mu\nu>}\delta\sum\nolimits_{\mu\nu}\right] = 0,$$

$$\varphi_\alpha\left[A_E^{\alpha<\mu\nu>}\delta\lambda + A_E^{\alpha\beta<\mu\nu>}\delta\lambda_\beta + A_E^{\alpha<\beta\gamma><\mu\nu>}\delta\sum\nolimits_{\mu\nu}\right] = 0. \qquad (22)$$

Here the expressions of the tensors in the left hand sides are reported in [10]. For the sake of simplicity, we calculate also the coefficients of the differentials in the reference frame where U^α and φ^α have the components $U^\alpha \equiv (c, 0, 0, 0)$ and $\varphi_\alpha \equiv (\varphi_0, \varphi_1, 0, 0)$; in any case, we can at the end express again all the results in covariant form replacing φ_0 and $(\varphi_1)^2$ with $\varphi_0 = \frac{1}{c}\varphi^\alpha U_\alpha$ and $(\varphi_1)^2 = \varphi_\alpha\varphi_\beta h^{\alpha\beta}$.

- We note that a first eigenvalue is

$$\varphi_0 = 0, \text{ i.e., } u = c\frac{U^\alpha\eta_\alpha}{U^\gamma\xi_\gamma}, \text{ i.e., } u = 0, \qquad (23)$$

where the last expression holds when $\xi_\gamma = \frac{1}{c}U_\gamma$.

In fact, we firstly note that $\varphi_1 \neq 0$ under the hypothesis that the 2 time-like vectors ξ_γ and U_γ are oriented both towards the future or both towards the past. After that, if $\varphi_0 = 0$, Equation $(22)_1$, the components 0, 1, 2, 3 of Equation $(22)_2$ and the components 00, 01, 02, 03, 12, 13, 22, 33, 23 of Equation $(22)_3$ give a system whose solution is $\delta\lambda_1 = 0, \delta\sum_{01} = 0, \delta\sum_{12} = 0, \delta\sum_{13} = 0$ and the remaining unknowns are linked only by

$$6\vartheta_{1,1}\delta\lambda + 2\vartheta_{1,2}c\delta\lambda_0 + (\vartheta_{1,3} + 6\vartheta_{2,3})c^2\delta\Sigma_{00}$$

$$- 4\vartheta_{2,3}c^2(\delta\Sigma_{22} + \delta\Sigma_{33}) = 0,$$

$$10\vartheta_{1,2}\delta\lambda + 5\vartheta_{1,3}c\delta\lambda_0 + (3\vartheta_{1,4} + 6\vartheta_{2,4})c^2\delta\Sigma_{00}$$

$$- 4\vartheta_{2,4}c^2(\delta\Sigma_{22} + \delta\Sigma_{33}) = 0,$$

$$5\vartheta_{2,3}\delta\lambda_2 + 2\vartheta_{2,4}c\delta\lambda_{20} = 0, \quad 5\vartheta_{2,3}\delta\lambda_3 + 2\vartheta_{2,4}c\delta\lambda_{30} = 0, \tag{24}$$

where we have taken into account of $\delta\Sigma_{11} = \delta\Sigma_{00} - \delta\Sigma_{22} - \delta\Sigma_{33}$. We note that these equations are a little different from the corresponding ones in the 15 moments model, as reported in Equation (38) of [11]. Also the conclusions are a little different because in the 15 moments model this eigenvalue has multiplicity 7, while here we have 10 free unknowns linked by 4 equations and the eigenvalue (23) has multiplicity 6.

- For the other eigenvalues we have $\varphi_0 \neq 0$ and we define

$\delta\lambda = X_1$, $c\delta\lambda_0 = X_2$, $c\delta\lambda_1 = X_3$, $c^2\delta\Sigma_{00} = X_4$, $c^2\delta\Sigma_{01} = X_5$,

$c^2(\delta\Sigma_{22} + \delta\Sigma_{33}) = X_6$, $c\delta\lambda_2 = Y_1$, $c^2\delta\Sigma_{20} = Y_2$, $c^2\delta\Sigma_{12} = Y_3$,

$c\delta\lambda_3 = Z_1$, $c^2\delta\Sigma_{30} = Z_2$, $c^2\delta\Sigma_{13} = Z_3$, $c^2\delta\Sigma_{23} = Y_4$,

$c^2(\delta\Sigma_{22} - \delta\Sigma_{33}) = Z_4$.

It follows that $c^2\delta\sum_{11} = X_4 - X_6$. With this notation Equation (21) can be written as

$$-\frac{k_B}{m\rho c\varphi_0}\delta K_E = \sum_{h,k=1}^{6} a_{hk}X_hX_k + \sum_{h=1}^{3} b_{hk}Y_hY_k + \sum_{h=1}^{3} b_{hk}Z_hZ_k$$

$$+ \frac{4}{15}\vartheta_{2,4}(Y_4)^2 + \frac{1}{15}\vartheta_{2,4}(Z_4)^2,$$

with

$$a_{11}=\vartheta_{0,0},\quad a_{12}=\vartheta_{0,1},\quad a_{13}=\vartheta_{1,1}\frac{\varphi_1}{\varphi_0},\quad a_{14}=\vartheta_{0,2}+\frac{1}{3}\vartheta_{1,2},$$

$$a_{15}=\frac{2}{3}\vartheta_{1,2}\frac{\varphi_1}{\varphi_0},\quad a_{16}=0,\quad a_{22}=\vartheta_{0,2},\quad a_{23}=\frac{1}{3}\vartheta_{1,2}\frac{\varphi_1}{\varphi_0},$$

$$a_{24}=\vartheta_{0,3}+\frac{1}{6}\vartheta_{1,3},\quad a_{25}=\frac{1}{3}\vartheta_{1,3}\frac{\varphi_1}{\varphi_0},\quad a_{26}=0,\quad a_{33}=\frac{1}{3}\vartheta_{1,2},$$

$$a_{34}=\left(\frac{1}{6}\vartheta_{1,3}+\vartheta_{2,3}\right)\frac{\varphi_1}{\varphi_0},\quad a_{35}=\frac{1}{3}\vartheta_{1,3},\quad a_{36}=-\frac{2}{3}\vartheta_{2,3}\frac{\varphi_1}{\varphi_0},$$

$$a_{44}=\vartheta_{0,4}+\frac{2}{5}\vartheta_{2,4},\quad a_{45}=\left(\frac{1}{5}\vartheta_{1,4}+\frac{2}{5}\vartheta_{2,4}\right)\frac{\varphi_1}{\varphi_0},\quad a_{46}=-\frac{2}{15}\vartheta_{2,4},$$

$$a_{55}=\frac{2}{5}\vartheta_{1,4},\quad a_{56}=-\frac{4}{15}\vartheta_{2,4}\frac{\varphi_1}{\varphi_0},\quad a_{66}=\frac{1}{5}\vartheta_{2,4},$$

while $b_{h,k}$ are the same of the 15 moments model as reported on page 18 of [11]. From these results, it follows that the equations to determine eigenvalues and eigenvectors are

$$\sum_{k=1}^{6}b_{hk}X_k=0,\quad \sum_{k=1}^{3}b_{hk}Y_k=0,\quad \sum_{k=1}^{6}b_{hk}Z_k=0,\quad Y_4=0,\quad Z_4=0. \tag{25}$$

The last 4 of these equations are the same of the 15 moments model as reported Equation $(39)_{2-4}$ of [11]. So we have that the 2 eigenvalues with multiplicity 2 in Equation (40) of [11] are eigenvalues with multiplicity 2 also for the 14 moments model; they are the solutions of

$$\begin{vmatrix}\vartheta_{1,2} & \vartheta_{1,3} & 2\vartheta_{2,3}\\ \vartheta_{1,3} & \frac{6}{5}\vartheta_{1,4} & \frac{4}{5}\vartheta_{2,4}\\ 2\vartheta_{2,3} & \frac{4}{5}\vartheta_{2,4} & 0\end{vmatrix}\left(\frac{\varphi_1}{\varphi_0}\right)^2+\frac{4}{5}\vartheta_{2,4}\begin{vmatrix}\vartheta_{1,2} & \vartheta_{1,3}\\ \vartheta_{1,3} & \frac{6}{5}\vartheta_{1,4}\end{vmatrix}=0. \tag{26}$$

The remaining eigenvalues are the solutions of $\left|a_{h,k}\right| = 0$, i.e.,

$$\begin{vmatrix} \vartheta_{0,0} & \vartheta_{0,1} & \vartheta_{1,1}\frac{\varphi_1}{\varphi_0} & \vartheta_{0,2}+\frac{1}{3}\vartheta_{1,2} & \frac{2}{3}\vartheta_{1,2}\frac{\varphi_1}{\varphi_0} & 0 \\ \vartheta_{0,1} & \vartheta_{0,2} & \frac{1}{3}\vartheta_{1,2}\frac{\varphi_1}{\varphi_0} & \vartheta_{0,3}+\frac{1}{6}\vartheta_{1,3} & \frac{1}{3}\vartheta_{1,3}\frac{\varphi_1}{\varphi_0} & 0 \\ \vartheta_{1,1}\frac{\varphi_1}{\varphi_0} & \frac{1}{3}\vartheta_{1,2}\frac{\varphi_1}{\varphi_0} & \frac{1}{3}\vartheta_{1,2} & (\frac{1}{6}\vartheta_{1,3}+\vartheta_{2,3})\frac{\varphi_1}{\varphi_0} & \frac{1}{3}\vartheta_{1,3} & -\frac{2}{3}\vartheta_{2,3}\frac{\varphi_1}{\varphi_0} \\ \vartheta_{0,2}+\frac{1}{3}\vartheta_{1,2} & \vartheta_{0,3}+\frac{1}{6}\vartheta_{1,3} & (\frac{1}{6}\vartheta_{1,3}+\vartheta_{2,3})\frac{\varphi_1}{\varphi_0} & \vartheta_{0,4}+\frac{2}{5}\vartheta_{2,4} & (\frac{1}{5}\vartheta_{1,4}+\frac{2}{5}\vartheta_{2,4})\frac{\varphi_1}{\varphi_0} & -\frac{2}{15}\vartheta_{2,4} \\ \frac{2}{3}\vartheta_{1,2}\frac{\varphi_1}{\varphi_0} & \frac{1}{3}\vartheta_{1,3}\frac{\varphi_1}{\varphi_0} & \frac{1}{3}\vartheta_{1,3} & (\frac{1}{5}\vartheta_{1,4}+\frac{2}{5}\vartheta_{2,4})\frac{\varphi_1}{\varphi_0} & \frac{2}{5}\vartheta_{1,4} & -\frac{4}{15}\vartheta_{2,4}\frac{\varphi_1}{\varphi_0} \\ 0 & 0 & -\frac{2}{3}\vartheta_{2,3}\frac{\varphi_1}{\varphi_0} & -\frac{2}{15}\vartheta_{2,4} & -\frac{4}{15}\vartheta_{2,4}\frac{\varphi_1}{\varphi_0} & \frac{1}{5}\vartheta_{2,4} \end{vmatrix} = 0. \tag{27}$$

It is easy to prove that this equation depends on $\frac{\varphi_1}{\varphi_0}$ only through $\left(\frac{\varphi_1}{\varphi_0}\right)^2 = \frac{h^{\alpha\beta}\varphi_\alpha\varphi_\beta}{(U^\gamma\varphi_\gamma)^2}c^2$ (which is equal to $\left(\frac{c}{v}\right)^2$ if $U^\alpha = c\,\xi^\alpha$) and it is a second degree equation in $\left(\frac{\varphi_1}{\varphi_0}\right)^2$. In fact, by applying well known properties of the determinants, we can see that this equation can be written as

$$\begin{vmatrix} \vartheta_{0,0} & \vartheta_{0,1} & \vartheta_{1,1} & \vartheta_{0,2}+\frac{1}{3}\vartheta_{1,2} & \frac{2}{3}\vartheta_{1,2} & 0 \\ \vartheta_{0,1} & \vartheta_{0,2} & \frac{1}{3}\vartheta_{1,2} & \vartheta_{0,3}+\frac{1}{6}\vartheta_{1,3} & \frac{1}{3}\vartheta_{1,3} & 0 \\ \vartheta_{1,1}\left(\frac{\varphi_1}{\varphi_0}\right)^2 & \frac{1}{3}\vartheta_{1,2}\left(\frac{\varphi_1}{\varphi_0}\right)^2 & \frac{1}{3}\vartheta_{1,2} & \left(\frac{1}{6}\vartheta_{1,3}+\vartheta_{2,3}\right)\left(\frac{\varphi_1}{\varphi_0}\right)^2 & \frac{1}{3}\vartheta_{1,3} & -\frac{2}{3}\vartheta_{2,3}\left(\frac{\varphi_1}{\varphi_0}\right)^2 \\ \vartheta_{0,2}+\frac{1}{3}\vartheta_{1,2} & \vartheta_{0,3}+\frac{1}{6}\vartheta_{1,3} & \left(\frac{1}{6}\vartheta_{1,3}+\vartheta_{2,3}\right) & \vartheta_{0,4}+\frac{2}{5}\vartheta_{2,4} & \left(\frac{1}{5}\vartheta_{1,4}+\frac{2}{5}\vartheta_{2,4}\right) & -\frac{2}{15}\vartheta_{2,4} \\ \frac{2}{3}\vartheta_{1,2}\left(\frac{\varphi_1}{\varphi_0}\right)^2 & \frac{1}{3}\vartheta_{1,3}\left(\frac{\varphi_1}{\varphi_0}\right)^2 & \frac{1}{3}\vartheta_{1,3} & \left(\frac{1}{5}\vartheta_{1,4}+\frac{2}{5}\vartheta_{2,4}\right)\left(\frac{\varphi_1}{\varphi_0}\right)^2 & \frac{2}{5}\vartheta_{1,4} & -\frac{4}{15}\vartheta_{2,4}\left(\frac{\varphi_1}{\varphi_0}\right)^2 \\ 0 & 0 & -\frac{2}{3}\vartheta_{2,3} & -\frac{2}{15}\vartheta_{2,4} & -\frac{4}{15}\vartheta_{2,4} & \frac{1}{5}\vartheta_{2,4} \end{vmatrix} = 0. \tag{28}$$

So it gives 4 independent eigenvectors; other 6 come from (23), other 4 come from (26). The total is 14, as expected.

5.2. Wave speeds for the subsystem with 6 moments

This is a subsystem of that with 15 moments when $\lambda_{\mu\nu} = \frac{1}{4} g_{\mu\nu} \Sigma$; so Equation (21) becomes

$$\begin{aligned} \delta K_E = -\frac{m}{k_B}\varphi_\alpha \Big[& V_E^\alpha (\delta\lambda)^2 + 2T_E^{\alpha\mu}\delta\lambda\delta\lambda_\mu + \frac{1}{2}(A_E^{\alpha\mu\nu}g_{\mu\nu})\delta\lambda\delta\Sigma \\ & + A_E^{\alpha\beta\delta}\delta\lambda_\beta\lambda_\delta + \frac{1}{2}(A_E^{\alpha\beta\mu\nu}g_{\mu\nu})\delta\lambda_\beta\delta\Sigma + \frac{1}{16}(A_E^{\alpha\beta\gamma\mu\nu}g_{\beta\gamma}g_{\mu\nu})(\delta\Sigma)^2 \Big]. \end{aligned} \tag{29}$$

• We note that a first eigenvalue is

$$\varphi_0 = 0. \tag{30}$$

In fact, if $\varphi_0 = 0$, we find that the independent variables are linked only by

$$\delta\lambda_1 = 0, \quad 24\vartheta_{1,1}\delta\lambda + 8\vartheta_{1,2}c\delta\lambda_0 + \left(\vartheta_{1,3} - 10\vartheta_{2,3}\right)c^2\delta\Sigma = 0, \tag{31}$$

Consequently, the eigenvalue (30) has multiplicity 4.

• For the other eigenvalues we have $\varphi_0 \neq 0$ and we define $\delta\lambda = X_1$, $c\delta\lambda_0 = X_2$, $c\delta\lambda_1 = X_3$, $c^2\delta\Sigma = X_4$, $c\delta\lambda_2 = Y_1$, $c\delta\lambda_3 = Z_1$.

With this notation (29) can be written as

$$-\frac{k_B}{m\rho c\varphi_0}\delta K_E = \sum_{h,k=1}^{4} a_{hk}X_hX_k + \frac{1}{3}\vartheta_{1,2}\left[(Y_1)^2 + (Z_1)^2\right],$$

with

$$a_{11} = \vartheta_{0,0}, \quad a_{12} = \vartheta_{0,1}, \quad a_{13} = \vartheta_{1,1}\frac{\varphi_1}{\varphi_0}, \quad a_{14} = \frac{1}{4}(\vartheta_{0,2} - \vartheta_{1,2}),$$

$$a_{22} = \vartheta_{0,2}, \quad a_{23} = \frac{1}{3}\vartheta_{1,2}\frac{\varphi_1}{\varphi_0}, \quad a_{24} = \frac{1}{8}(2\vartheta_{0,3} - \vartheta_{1,3}), \quad a_{33} = \frac{1}{3}\vartheta_{1,2},$$

$$a_{34} = \left(\frac{1}{24}\vartheta_{1,3} - \frac{5}{12}\vartheta_{2,3}\right)\frac{\varphi_1}{\varphi_0}, \quad a_{44} = \frac{1}{16}\left(\vartheta_{0,4} - \frac{3}{5}\vartheta_{1,4} + \vartheta_{2,4}\right).$$

From these results, it follows that the equations to determine eigenvalues and eigenvectors are

$$\sum_{k=1}^{4} a_{hk}X_k = 0, \quad Y_1 = 0, \quad Z_1 = 0. \tag{32}$$

Then the other 2 eigenvalues are the solutions of the equation

$$\begin{vmatrix} \vartheta_{0,0} & \vartheta_{0,1} & \vartheta_{1,1}\frac{\varphi_1}{\varphi_0} & \frac{1}{4}(\vartheta_{0,2}-\vartheta_{1,2}) \\ \vartheta_{0,1} & \vartheta_{0,2} & \frac{1}{3}\vartheta_{1,2}\frac{\varphi_1}{\varphi_0} & \frac{1}{8}(2\vartheta_{0,3}-\vartheta_{1,3}) \\ \vartheta_{1,1}\frac{\varphi_1}{\varphi_0} & \frac{1}{3}\vartheta_{1,2}\frac{\varphi_1}{\varphi_0} & \frac{1}{3}\vartheta_{1,2} & (\frac{1}{24}\vartheta_{1,3}-\frac{5}{12}\vartheta_{2,3})\frac{\varphi_1}{\varphi_0} \\ \frac{1}{4}(\vartheta_{0,2}-\vartheta_{1,2}) & \frac{1}{8}(2\vartheta_{0,3}-\vartheta_{1,3}) & (\frac{1}{24}\vartheta_{1,3}-\frac{5}{12}\vartheta_{2,3})\frac{\varphi_1}{\varphi_0} & \frac{1}{16}(\vartheta_{0,4}-\frac{3}{5}\vartheta_{1,4}+\vartheta_{2,4}) \end{vmatrix} = 0,$$

that is,

$$\left(\frac{\varphi_1}{\varphi_0}\right)^2 \begin{vmatrix} \vartheta_{0,0} & \vartheta_{0,1} & \vartheta_{1,1} & \frac{1}{4}(\vartheta_{0,2}-\vartheta_{1,2}) \\ \vartheta_{0,1} & \vartheta_{0,2} & \frac{1}{3}\vartheta_{1,2} & \frac{1}{8}(2\vartheta_{0,3}-\vartheta_{1,3}) \\ \vartheta_{1,1} & \frac{1}{3}\vartheta_{1,2} & 0 & (\frac{1}{24}\vartheta_{1,3}-\frac{5}{12}\vartheta_{2,3}) \\ \frac{1}{4}(\vartheta_{0,2}-\vartheta_{1,2}) & \frac{1}{8}(2\vartheta_{0,3}-\vartheta_{1,3}) & (\frac{1}{24}\vartheta_{1,3}-\frac{5}{12}\vartheta_{2,3}) & \frac{1}{16}(\vartheta_{0,4}-\frac{3}{5}\vartheta_{1,4}+\vartheta_{2,4}) \end{vmatrix}$$

$$+\frac{1}{3}\vartheta_{1,2}\begin{vmatrix} \vartheta_{0,0} & \vartheta_{0,1} & \frac{1}{4}(\vartheta_{0,2}-\vartheta_{1,2}) \\ \vartheta_{0,1} & \vartheta_{0,2} & \frac{1}{8}(2\vartheta_{0,3}-\vartheta_{1,3}) \\ \frac{1}{4}(\vartheta_{0,2}-\vartheta_{1,2}) & \frac{1}{8}(2\vartheta_{0,3}-\vartheta_{1,3}) & \frac{1}{16}(\vartheta_{0,4}-\frac{3}{5}\vartheta_{1,4}+\vartheta_{2,4}) \end{vmatrix} = 0. \tag{33}$$

So it gives 2 independent eigenvectors, other 4 come from (30). The total is 6, as expected.

5.3. Wave speeds for the subsystem with 5 moments, i.e., Euler's equations

This is a subsystem of that with 6 moments when $\sum = 0$. Equation (29) now becomes

$$\delta K_E = -\frac{m}{k_B}\varphi_\alpha\left[V_E^\alpha(\delta\lambda)^2 + 2T_E^{\alpha\mu}\delta\lambda\delta\lambda_\mu + A_E^{\alpha\beta\delta}\delta\lambda_\beta\delta\lambda_\delta\right]. \tag{34}$$

• We note that a first eigenvalue is

$$\varphi_0 = 0 \text{ with multiplicity 3.} \tag{35}$$

In fact, if $\varphi_0 = 0$, we find that the independent variables are linked only by

$$\delta\lambda_1 = 0, \quad 3\vartheta_{1,1}\delta\lambda + \vartheta_{1,2}c\delta\lambda_0 = 0, \tag{36}$$

• For the other eigenvalues, we have $\varphi_0 \neq 0$ and we define $\delta\lambda = X_1$, $c\delta\lambda_0 = X_2$, $c\delta\lambda_1 = X_3$, $c\delta\lambda_2 = Y_1$, $c\delta\lambda_3 = Z_1$.

With this notation (29) can be written as

$$-\frac{k_B}{m\rho c\varphi_0}\delta K_E = \sum_{h,k=1}^{3} a_{hk}X_hX_k + \frac{1}{3}\vartheta_{1,2}\left[(Y_1)^2 + (Z_1)^2\right],$$

with

$$a_{11} = \vartheta_{0,0}, \quad a_{12} = \vartheta_{0,1}, \quad a_{13} = \vartheta_{1,1}\frac{\varphi_1}{\varphi_0}, \quad a_{22} = \vartheta_{0,2},$$

$$a_{23} = \frac{1}{3}\vartheta_{1,2}\frac{\varphi_1}{\varphi_0}, \quad a_{33} = \frac{1}{3}\vartheta_{1,2}.$$

From these results, it follows that the equations to determine eigenvalues and eigenvectors are

$$\sum_{k=1}^{3} a_{hk}X_k = 0, \quad Y_1 = 0, \quad Z_1 = 0. \tag{37}$$

Then the other 2 eigenvalues are the solutions of the equation

$$\begin{vmatrix} \vartheta_{0,0} & \vartheta_{0,1} & \vartheta_{1,1}\dfrac{\varphi_1}{\varphi_0} \\ \vartheta_{0,1} & \vartheta_{0,2} & \dfrac{1}{3}\vartheta_{1,2}\dfrac{\varphi_1}{\varphi_0} \\ \vartheta_{1,1}\dfrac{\varphi_1}{\varphi_0} & \dfrac{1}{3}\vartheta_{1,2}\dfrac{\varphi_1}{\varphi_0} & \dfrac{1}{3}\vartheta_{1,2} \end{vmatrix} = 0,$$

i.e.,

$$\left(\frac{\varphi_1}{\varphi_0}\right)^2 \begin{vmatrix} \vartheta_{0,0} & \vartheta_{0,1} & \vartheta_{1,1} \\ \vartheta_{0,1} & \vartheta_{0,2} & \dfrac{1}{3}\vartheta_{1,2} \\ \vartheta_{1,1} & \dfrac{1}{3}\vartheta_{1,2} & 0 \end{vmatrix} + \frac{1}{3}\vartheta_{1,2}\begin{vmatrix} \vartheta_{0,0} & \vartheta_{0,1} \\ \vartheta_{0,1} & \vartheta_{0,2} \end{vmatrix} = 0. \tag{38}$$

The corresponding 2 independent eigenvectors, plus the other 3 which come from (35) gives 5 independent eigenvectors, as expected.

Conclusion

The scalar functions, appearing in the closure of the new 15 moments model, have been expressed as polynomials of finite degree of the variable $\frac{1}{c^2}$; by using them it was easier to do the non relativistic limit of the final field equations expressed in terms of the physical variables. This proves that the manipulations, performed to the equations expressed in integral form, didn't change the result to recover the known equations for the classical case. The expressions of the above scalar functions have been found also for the easier case of polytropic gases. The wave equations for the subsystems with 14, 6, or 5 moments have been studied; this shows how they evolve from the 15 moments model to its subsystems.

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