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CRITICAL SETS IN SUDOKU

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Abstract

A common method for solving a Sudoku puzzle is the use of elimination techniques. At each step one tries to find a specific pattern and eliminates a value in the candidate list. We describe the solution of a Sudoku puzzle as a disparate selection of a set-valued mapping and define critical sets and elimination points for this model. Some of the known techniques can be understood as the search for critical sets and elimination points. Our approach unifies these solution techniques and provides a general elimination process.

1. Introduction

We consider a partially prepopulated Sudoku puzzle and examine solution strategies for completing this puzzle. A common strategy is the use of elimination techniques and the definition of specific patterns (e.g., naked pair, *X*-wing, ...) which results in a reduction of the candidate list.

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An introductory article to Sudoku puzzles had been published by Delahaye [3]. An overview on solution techniques for Sudoku puzzles can be found in Davis [2] and Maji et al. [8].

We describe the solution of a Sudoku puzzle as a disparate selection of a set-valued mapping and introduce formal definitions of some patterns. We use definitions of critical sets and elimination points in [5] and show, that some of the patterns describe critical sets and elimination points.

Let P be the set of cells of the Sudoku grid with the prepopulated values. Let p_x be the prepopulated value in cell $x \in P$. For each unpopulated cell x (not in P) there exists a "candidate list" L_x containing the possible values. A typical strategy to solve a Sudoku puzzle consists of a series of elimination steps reducing the size of the candidate list until L_x consists of a single value.

We set up a mathematical model by $\mathbf{N_9} = \{1, \ldots, 9\}$ and define the Sudoku grid by $X = \mathbf{N_9} \times \mathbf{N_9}$. We define a set of edges by

$$\begin{aligned} E = \{\{x, x'\} \subset X \mid x \neq x' \text{ and } x, x' &\text{ are in the same row or} \\ x, x' &\text{ are in the same column or} \\ x, x' &\text{ are in the same block}\}. \end{aligned}$$

The pair (X, E) denotes a simple graph with vertex set X and edge set E. We define $Y = \mathbf{N_9}$ and a set-valued mapping $F : X \longrightarrow 2^Y$ by

$$F(x) = \begin{cases} p_x, & \text{if } x \in P, \\ L_x, & \text{if } x \in X \setminus P. \end{cases}$$

Finally, in this section we collect some basic terms and notations. In a Sudoku puzzle the grid X is divided into rows, columns and blocks. We use the notation *racb* (where $a, b \in \mathbf{N_9}$) to denote the cell in row a and column b of a Sudoku grid. A unit of a Sudoku puzzle is a subset $U \subset X$ such that U is a row or a column or a block of the Sudoku grid. The 9 blocks of a Sudoku grid are depicted, e.g., in Figure 2.

In a graph (X, E), the points $x, x' \in X$ are called adjacent if $\{x, x'\} \in E$ and a subset of X is called independent if any two elements of this subset are not adjacent. In a Sudoku grid two cells x, x' are called adjacent if x and x' are in the same row, same column or same block.

The symbol $\sharp$ denotes the number of elements (cardinality) of a finite set. The expression $X \times Y$ denotes the cartesian product of X and Y. For a given set $W \subset X$ we define $F(W) = \bigcup_{x \in W} F(x)$. The restriction of F on a subset $W \subset X$ is denoted by $F_{|W}$. The graph of F is defined by $G(F) = \{(x, y) \in X \times Y | y \in F(x)\}$. A selection for F is a point-valued mapping $s : X \longrightarrow Y$ such that $s(x) \in F(x)$ for each $x \in X$. We identify a selection s of F with the set-valued mapping $F'(x) = \{s(x)\}, x \in X$. An introduction to set-valued mappings can be found in Berge [1].

2. Main Definitions

The main idea of this paper is based on the concept of disparate points, disparate sets and disparate selections introduced in [5].

Definition 2.1. The points $(x, y), (x', y') \in X \times Y$ are called *disparate* if $(x, y) \neq (x', y')$ and $y = y'$ implies $\{x, x'\} \notin E$.

Definition 2.2. A set $A \subset X \times Y$ is called *disparate* if any two points $a, a' \in A, a \neq a'$, are disparate.

Definition 2.3. A point-valued mapping $s : X \longrightarrow Y$ is called to be *disparate* if the graph $G(s)$ of s is a disparate set.

The completion of a Sudoku puzzle can be identified with a disparate selection of F. The term disparate reflects the dependencies within a Sudoku puzzle. All cells in a row, a column and a block are related by edges. Disparate means the corresponding values have to be different, i.e., each value appears only once.

We recall some definitions from [5] and define the complement set, the complement mapping, critical sets and elimination points.

Definition 2.4. The complement set of a point $a \in X \times Y$ is defined by

$$compl(\{a\}) = \{b \in X \times Y \mid a \text{ and } b \text{ are disparate}\}.$$

We introduce the complement mapping $F_{x,y} : X \setminus \{x\} \longrightarrow 2^Y$ of F depending on $(x, y) \in G(F)$ by $F_{x,y}(z) = F(z) \cap compl(\{(x, y)\})$ for each $z \in X \setminus \{x\}$. $F_{x,y}$ is a submapping of $F_{|(X \setminus \{x\})}$.

The complement set and complement mapping (and also critical sets) are defined in a simplified manner compared to [5], but this is sufficient for our purposes.

Definition 2.5. A nonempty set $W \subset X$ is called a *critical set* of F if there exists $(x, y) \in (X \setminus W) \times Y$ such that there does not exist a disparate set $A \subset G((F_{x,y})_{|W})$ with $\sharp A \geq \sharp W$.

In [5], generalized and t-critical sets had been introduced. The description of Definition 2.5 is equivalent to t-critical sets and each critical set is a generalized critical set (see [5, Lemma 12.3]). Also in [5], the relation of generalized critical sets to disparate selections had been shown and a method for the calculation of a disparate selection had been provided.

The consideration of critical sets is a useful tool in the proof of the marriage theorem, originally proved by Hall [6]. This theorem had been reproved by Halmos and Vaughan [7] using critical sets in the sense of [5, Lemma 14.2].

Closely related to critical sets are elimination points which are the basis for a reduction of the candidate list.

Definition 2.6. Let $W \subset X$ be a critical set of F. The elimination points of W are defined by $elim_F(W) = \{(x, y) \in G(F_{|(X \setminus W)}) \mid$ there does not exist a disparate set $A \subset G((F_{x,y})_{|W})$ with $\sharp A \geq \sharp W\}$.

The motivation for considering critical sets and elimination points is the following necessary condition of a disparate selection of F.

Theorem 2.7. *Let s be a disparate selection of F, let $W \subset X$ be a critical set of F and let $x \in X \setminus W$. $(x, s(x)) \notin elim_F(W)$.*

Proof. Define a disparate set $A = G(s) \cap (W \times Y)$ with $\sharp A = \sharp W$. By definition of the complement mapping and since $G(s)$ is disparate, $A \subset G((F_{x, s(x)})_{|W})$, i.e., $(x, s(x)) \notin elim_F(W)$. □

A possible elimination technique for a given value y in the candidate list of cell x, (i.e., $(x, y) \in G(F)$) is the search for a critical set $W \subset X$ of F such that $(x, y) \in elim_F(W)$. According to Theorem 2.7, the point (x, y) cannot be a solution point of the Sudoku puzzle, i.e., we can eliminate the value y in the candidate list of cell x.

In this paper, we show that several known elimination techniques for Sudoku puzzles are of the type described in Theorem 2.7. An elimination technique called "Unique Solution Constraints" described by Davis [2, Section 8] is not covered by our approach.

Combining the calculation methods of [5] and the results in this paper provide a general algorithm for solving Sudoku puzzles which considers the known patterns as part of the algorithm.

3. Naked and Hidden Tight Sets

Naked and hidden tight sets are known in the literature as naked and hidden singles, pairs, ... We adopt here an expression of Schrijver [9] and subsume these definitions under the term tight set.

Definition 3.1. Let $U \subset X$ be a unit. A subset $M \subset U$ is called a *naked tight set* if $M \neq \emptyset$, $M \neq U$ and $\sharp F(M) = \sharp M$.

This definition includes naked singles $(\sharp M = 1)$, pairs $(\sharp M = 2)$, triplets $(\sharp M = 3)$, quads $(\sharp M = 4)$, $\ldots$.

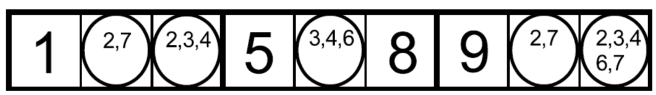

Figure 1. Naked pair, Davis [2, Figure 4].

Example 3.2. We depict row 1 of a Sudoku grid in Figure 1. The cells $r1c2$ and $r1c8$ form a naked tight set with $\sharp M = 2$ (naked pair) and the values $F(M) = \{2, 7\}$.

Naked tight sets describe a critical set of F.

Lemma 3.3. *Let $U \subset X$ be a unit and let $M \subset U$ be a naked tight set. $W = M$ is a critical set of F and $(x, y) \in elim_F(W)$ for each $x \in U \setminus M$ and $y \in F(M)$.*

Proof. Let $x \in U \setminus M$ and $y \in F(M)$. This implies $\sharp F(M) \setminus \{y\}) = \sharp M - 1$ and each disparate set $A \subset G((F_{x, y})_{|W})$ satisfies $\sharp A \leq \sharp W - 1$.

This shows $W = M$ is a critical set of F, since $U \setminus M$ and $F(M)$ are nonempty and $(x, y) \in elim_F(W)$ for each $x \in U \setminus M$ and $y \in F(M)$. □

Example 3.4. Using Lemma 3.3 the cells $\{r1c2, r1c8\}$ in Figure 1 form a critical set of F and $(r1c3, 2)$, $(r1c9, 2)$, $(r1c9, 7) \in elim_F(\{r1c2, r1c8\})$, i.e., we can eliminate the value 2 from the candidate list in cell $r1c3$ and we can eliminate the values 2, 7 from the candidate list in cell $r1c9$.

Hidden tight sets are the complement of naked tight sets and we can transfer our result on naked tight sets to hidden tight sets.

Definition 3.5. Let $U \subset X$ be a unit. A subset $M \subset U$ is called a *hidden tight set* if $M \neq \emptyset$, $M \neq U$ and $\sharp(\mathbf{N_9} \setminus F(U \setminus M)) = \sharp M$.

This definition includes hidden singles $(\sharp M = 1)$, pairs $(\sharp M = 2)$, triplets $(\sharp M = 3)$, quads $(\sharp M = 4), \ldots$.

We derive a critical set of F from a hidden tight set.

Lemma 3.6. *Let U be a unit and let $M \subset U$ be a hidden tight set. $W = U \setminus M$ is a critical set of F and $(x, y) \in elim_F(W)$ for each $x \in M$ and $y \in F(U \setminus M)$.*

Proof. We obtain $\sharp F(U \setminus M) = \sharp \mathbf{N_9} - \sharp(\mathbf{N_9} \setminus F(U \setminus M)) = \sharp U - \sharp M = \sharp(U \setminus M)$, i.e., $U \setminus M$ is a naked tight set and we can apply Lemma 3.3.

□

Please note, usually a Sudoku puzzle is (uniquely) solvable and in this case $\mathbf{N_9} = F(U)$.

4. Locked Candidates

The tight sets in Section 3 are all contained in a single unit. In the remaining sections we consider patterns which are contained in more than one unit.

Definition 4.1. Let $U_1, U_2 \subset X, U_1 \neq U_2$ be units. A value $y \in F(U_1 \cap U_2)$ is called a *locked candidate* of U_1 and U_2 if $y \notin F(U_1 \setminus U_2)$.

We number the 9 blocks of a Sudoku grid from 1 to 9 row-wise from left to right and the rows from top to bottom.

1		8	6	7	2		9	3
		9	8	1	4	6		2
	6		9	5	3	8		1
		6			7	1	8	5,9
	2	1		9	8	7	3	6
3,4,5 7,9	8	3,5,7	1	2,3	6	2,4,5	2,4,5	5,9
	1	4	3	8	9		6	7
6				4	1	9		8
8	9			6	5	3	1	4

Figure 2. Locked candidate, Davis [2, Figure 3].

Example 4.2. The value 2 in Figure 2 is a locked candidate of block $6 = U_1$ and row $6 = U_2$.

From a locked candidate we derive a critical set of F.

Lemma 4.3. *Let* $U_1, U_2 \subset X, U_1 \neq U_2$, *be units and let* y *be a locked candidate of* U_1 *and* U_2. $W = U_1$ *is a critical set of* F *and* $(x,\ y) \in elim_F(W)$ *for each* $x \in F^{-1}(y) \cap (U_2 \setminus U_1)$.

Proof. Let $x \in U_2 \setminus U_1$ and let $A \subset G((F_{x,y})_{|W})$ be a disparate set. There does not exist a point $z \in U_1 \setminus U_2$ such that $(z, y) \in A$, since y is a locked candidate. There does not exist a point $z \in U_1 \cap U_2$ such that $(z, y) \in A$ by definition of the complement mapping. This shows $\sharp A \leq 8 < \sharp U_1 = \sharp W$, i.e., $W = U_1$ is a critical set of F, since $U_2 \setminus U_1$ is nonempty and $(x, y) \in elim_F(W)$ for each $x \in F^{-1}(y) \cap (U_2 \setminus U_1)$. $\square$

Example 4.4. Using Lemma 4.3, block $6 = U_1$ in Figure 2 is a critical set of F and $(r6c1, 4), (r6c5, 2) \in elim_F(U_1)$, i.e., we can eliminate the value 4 from the candidate list in cell $r6c1$ and the value 2 from the candidate list in cell $r6c5$.

5. *K*-Wings

In the literature k-wings are known as X-wings $(k = 2)$, swordfish $(k = 3)$, jellyfish $(k = 4)$, and squirmbag $(k = 5)$. We subsume these definitions under the term k-wing.

Definition 5.1. A tuple $(R_1, \dots, R_k, C_1, \dots, C_k, y)$, where $2 \leq k \leq 8$, $R_1, \dots, R_k$ are rows, $C_1, \dots, C_k$ are columns and $y \in Y$ is called a row k-wing if $R_1, \dots, R_k$ are distinct, $C_1, \dots, C_k$ are distinct and $y \notin F(x)$ for each $x \in \bigcup_{i=1}^{k} R_i \setminus \bigcup_{i=1}^{k} C_i$.

This definition describes row k-wings. Analoguously it is possible to define column k-wings. In this case $y \notin F(x)$ for each $x \in \bigcup_{i=1}^{k} C_i \setminus \bigcup_{i=1}^{k} R_i$. In order to keep the description simple we deal here only with row k-wings. An analoguous result as in Lemma 5.3 holds for column k-wings.

	4	8	7	9				
	5		8	2		7		
2,9	2,3,9	7	5	4	1	2,3,9	6	8
3	8	5	2	1	9	4	7	6
7	6	2	3	5	4	8	9	1
4	1	9	6	7	8	2,3		5
8	7	6	4	3	5			
1,5,9	3,9	4	1,9	6	2	1,3, 5,9	8	7
				8	7	6		

Figure 3. 2-wing, Davis [2, Figure 7, left].

Example 5.2. In the rows 3 and 8 in Figure 3 the value 3 appears only in the columns 2 and 7. The rows 3 and 8, columns 2 and 7 with the value 3 form a row 2-wing in this Sudoku puzzle.

From a k-wing pattern we derive a critical set of F.

Lemma 5.3. *Let* $(R_1, \ldots, R_k, C_1, \ldots, C_k, y)$ *be a row k-wing,* $W = \bigcup_{i=1}^{k} R_i$ *is a critical set of* F *and* $(x, y) \in elim_F(W)$ *for each* $x \in F^{-1}(y) \cap (\bigcup_{i=1}^{k} C_i \setminus \bigcup_{i=1}^{k} R_i)$.

Proof. Let $x \in \bigcup_{i=1}^{k} C_i \setminus \bigcup_{i=1}^{k} R_i$. Suppose there exists a disparate set $A \subset G((F_{x,y})_{|W})$ such that $\sharp A \geq \sharp W$.

Let $1 \le i \le k$. Any disparate set in $G(F_{|R_i})$ contains at most 9 elements. We obtain

$$9 = \sharp W - (k-1) \cdot 9 \le \sharp A - \sharp\Big(A \cap \bigcup_{\substack{j=1,\\ j \ne i}}^{k} G(F_{|R_j})\Big) = \sharp(A \cap G(F_{|R_i})) \le 9,$$

i.e., $\sharp(A \cap G(F_{|R_i})) = 9$. All points in R_i are adjacent, i.e., there exists $x_i \in R_i$ such that $(x_i, y) \in A \cap G(F_{|R_i})$. In particular, $y \in F(x_i)$ and using Definition 5.1, $x_i \in \bigcup_{i=1}^{k} C_i$.

The set $\bigcup_{i=1}^{k} \{(x_i, y)\}$ is disparate, since all points are contained in A, i.e., x_i and x_j are not adjacent for $i, j = 1, \dots, k, i \ne j$. There exists $i_0 \in \{1, \dots, k\}$ such that $x \in C_{i_0}$ and there exists $i_1 \in \{1, \dots, k\}$ such that $x_{i_1} \in C_{i_0}$. Consequently, x_{i_1} and x are adjacent, i.e., (x_{i_1}, y) and (x, y) are not disparate. This contradicts with $(x_{i_1}, y) \in A \subset G((F_{x,y})_{|W})$ and the definition of the complement mapping.

$\bigcup_{i=1}^{k} C_i \setminus \bigcup_{i=1}^{k} R_i$ is nonempty, since $k \le 8$, and this shows that W is a critical set of F. It also shows that $(x, y) \in elim_F(W)$ for each $x \in F^{-1}(y) \cap (\bigcup_{i=1}^{k} C_i \setminus \bigcup_{i=1}^{k} R_i)$. □

Example 5.4. Let R_1 be row 3, R_2 be row 8, C_1 be column 2, C_2 be column 7 and $y = 3$ in Figure 3. Using Lemma 5.3, $R_1 \cup R_2$ forms a critical set of F and $(r6c7, 3) \in elim_F(R_1 \cup R_2)$, i.e., we can eliminate the value 3 from the candidate list in cell $r6c7$.

6. *XY*-Wings

Another strategy to eliminate values from the candidate list are *XY*-wings.

Definition 6.1. A tuple $(x_1, x_2, x_3, y_1, y_2, y_3) \in X^3 \times Y^3$ is called an *XY*-wing if x_1, x_2, x_3 are distinct, y_1, y_2, y_3 are distinct, x_1, x_2 are adjacent, x_1, x_3 are adjacent, $F(x_1) = \{y_1, y_2\}$, $F(x_2) = \{y_1, y_3\}$, and $F(x_3) = \{y_2, y_3\}$.

2	8	9	4		1		6	3
7	6		3	2	9	8		
1		3	8	6				2
3,8	1	2	7			3,4,9	8,9	6
9		6	2		3	1		5
3,5,8	3,5,7		1		6	3,9	2	
6			9			2		
	9	1	6		2		4	8
4	2		5	1		6	3	9

Figure 4. *XY*-wing, Davis [2, Figure 11].

Example 6.2. The tuple ($r4c8$, $r4c1$, $r6c7$, 8, 9, 3) in Figure 4 is an *XY*-wing. We observe the cells $r6c1$, $r6c2$, and $r4c7$ are adjacent to $r4c1$ and $r6c7$ and contain the value 3 in their candidate list.

An *XY*-wing describes a critical set of *F*.

Lemma 6.3. *Let $x_1, x_2, x_3 \in X$ be distinct points such that x_1, x_2 are adjacent and x_1, x_3 are adjacent. There exists $x \in X \setminus \{x_1, x_2, x_3\}$ such that x, x_2 are adjacent and x, x_3 are adjacent.*

Proof. We distinguish several cases.

Case 1: There exists a unit $U \subset X$ such that $x_1, x_2, x_3 \in U$.

We can choose any $x \in U \setminus \{x_1, x_2, x_3\}$ and obtain the desired properties.

Case 2: There does not exist a unit $U \subset X$ such that $x_1, x_2, x_3 \in U$.

There exist units $U_2 \subset X$ and $U_3 \subset X$ such that $x_1, x_2 \in U_2$ and $x_1, x_3 \in U_3$ and $U_2 \neq U_3$.

Case 2(a): U_2 or U_3 is a block.

This implies $\sharp(U_2 \cap U_3) = 3$, since $x_1 \in U_2 \cap U_3$. Using Case 2 $(U_2 \cap U_3) \setminus \{x_1, x_2, x_3\} \neq \emptyset$ and we can find $x \in (U_2 \cap U_3) \setminus \{x_1, x_2, x_3\}$ with the desired properties.

Case 2(b): U_2 and U_3 are no blocks.

One of U_2 and U_3 is a row and the other one is a column, since $x_1 \in U_2 \cap U_3$. There exists $x \in X \setminus (U_2 \cup U_3)$ such that x, x_2 are adjacent and x, x_3 are adjacent. □

Lemma 6.4. *Let* $(x_1, x_2, x_3, y_1, y_2, y_3) \in X^3 \times Y^3$ *be an XY-wing.* $W = \{x_1, x_2, x_3\}$ *is a critical set of* F *and* $(x, y_3) \in elim_F(W)$ *for each* $x \in X \setminus \{x_1, x_2, x_3\}$ *adjacent to* x_2 *and* x_3.

Proof. Let $x \in X \setminus \{x_1, x_2, x_3\}$ such that x, x_2 are adjacent and x, x_3 are adjacent. Suppose there exists a disparate set $A \subset G((F_{x, y_3})_{|W})$ such that $\sharp A \geq \sharp W$. This implies $A \subset \{(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_3, y_2)\}$ and $\sharp A \geq 3$. A simple consideration shows that such a disparate set A does not exist and this yields a contradiction.

By Lemma 6.3, there exists $x \in X \setminus \{x_1, x_2, x_3\}$ such that x, x_2 are adjacent and x, x_3 are adjacent, i.e., W is a critical set of F and $(x, y_3) \in elim_F(W)$ for each $x \in X \setminus \{x_1, x_2, x_3\}$ adjacent to x_2 and x_3.

□

Example 6.5. Using Lemma 6.4, the cells $W = \{r4c8,\ r4c1,\ r6c7\}$ in Figure 4 form a critical set of F and $(r6c1,\ 3)$, $(r6c2,\ 3)$, $(r4c7,\ 3)$ $\in elim_F(W)$, i.e., we can eliminate the value 3 from the candidate lists in the cells $r6c1$, $r6c2$, and $r4c7$.

7. Nishio Sets

Nishio sets had been introduced by Eppstein [4] on the basis of a Sudoku strategy proposed by the japanese author Tetsuya Nishio who published several books on Sudoku. We define Nishio sets in the sense of Eppstein [4] and show the relation between Nishio sets and critical sets.

The inverse image of F is defined by $F^{-1}(y) = \{x \in X | y \in F(x)\}$ for each $y \in Y$.

Definition 7.1. The Nishio set of a value $y \in Y$ is defined by

$$Nishio(y) = \bigcup \{V \subset F^{-1}(y) | V \text{ is an independent set and } \sharp V = 9\}.$$

	2	6			5			1
	9		3					
1	3		7					
	5	8			6			
				8				
			4			9	7	
					9		4	3
					4		8	9
9			5	3,6,7		2	1	

Figure 5. Nishio set, Eppstein [4, modification of Figure 1].

Example 7.2. The candidate list in cell $r9c5$ of Figure 5 contains the value 7. It is not possible to find an independent set V in the Sudoku grid with $\sharp V = 9$ such that V contains $r3c4$, $r6c8$ and $r9c5$ and 7 is in the candidate list of all cells in V. We recall the arguments of Eppstein [4]. If this was possible, this would force a 7 in these cells: $r5c6$, $r4c1$, $r2c3$, $r1c7$ and $r9c9$, but row 9 already contains a 7 in column 5. This shows $r9c5 \notin Nishio(7)$, i.e., $Nishio(7) \neq X$.

We define the projected level sets of $A \subset X \times Y$ and $y \in Y$ by $A_y = \{x \in X | (x, y) \in A\}$. This definition guarantees $\sharp A = \sum_{q \in Y} \sharp A_q$. If A is a disparate set, the projected level sets A_y are independent and $\sharp A_y \leq 9$ for each $y \in Y$.

From a Nishio set we derive a critical set of F.

Lemma 7.3. *Let* $y \in Y$. $W = X \setminus \{x\}$ *is a critical set of* F *and* $(x, y) \in elim_F(W)$ *for each* $x \in F^{-1}(y) \setminus Nishio(y)$.

Proof. Let $x \in F^{-1}(y) \setminus Nishio(y)$ and set $W = X \setminus \{x\}$. Suppose there exists a disparate set $A \subset G((F_{x, y})_{|W})$ such that $\sharp A \geq \sharp W$. By definition of the complement mapping $A \cup \{(x, y)\}$ is a disparate set, i.e., $A_y \cup \{x\}$ is an independent set.

Using $A_y \cup \{x\} \subset F^{-1}(y)$ and $x \notin Nishio(y)$, $\sharp(A_y \cup \{x\}) \leq 8$, i.e., $\sharp A_y \leq 7$. We obtain

$$80 = \sharp(X \setminus \{x\}) = \sharp W \leq \sharp A = \sharp A_y + \sum_{q \in Y, q \neq y} \sharp A_q \leq 7 + 8 \cdot 9 = 79,$$

and this is a contradiction. This shows $W = X \setminus \{x\}$ is a critical set of F and $(x, y) \in elim_F(W)$. $\square$

Example 7.4. Using Lemma 7.3, the cells $X \setminus \{r9c5\}$ in Figure 5 form a critical set of F and $(r9c5, 7) \in elim_F(X \setminus \{r9c5\})$, i.e., we can eliminate the value 7 from the candidate list in cell $r9c5$.

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