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ON PERFECT HYPERGROUPS

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Abstract

This paper is devoted to show some results of harmonic analysis on perfect hypergroups. We give some ways to preserve the perfectness of hypergroups, under different kinds of transformations. We state and prove some properties of the set of Radon measures on the dual hypergroups. Also we explain some exact conditions on translation operators to translate perfect hypergroups, to another perfect hypergroups. Details computations with implemented formal examples are explicitly introduced.

1. Introduction

The main aim of this paper is to study some of the main topics of harmonic analysis on the product of perfect hypergroups. A hypergroup [1] is a locally compact Hausdorff space K with a certain convolution structure $*$ on the space of complex Radon measures on K, $M(K)$. The first to define hypergroups were Dunkl [2], Jewitt [3], and Spector [4].

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The main problem in harmonic analysis in various setting is the existence of a product, usually called convolution, for functions and measures [5, 6]. Hypergroups generalize locally compact groups. Hence, abundant results of harmonic analysis can be shown for hypergroups, in particular for commutative hypergroups. We will denote by $C(K)$, $C_b(K)$, $C_0(K)$, and $C_c(K)$ the spaces of continuous functions on K, that are bounded, vanish at infinity and that with compact support, respectively. The dual K^* of K is just the set of continuous characters with the compact-open topology in which case K^* must be locally compact. In this paper, we will be concerned with continuous characters on hypergroups. A locally bounded measurable function $\phi : K \to \mathbb{C}$ is said to be positive definite if

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \overline{c_j} \phi(z_i * z_j^{-}) \geqslant 0,$$

for all choice of $z_1, z_2, \ldots, z_n \in K$, $c_1, c_2, \ldots, c_n \in \mathbb{C}$ and $n \in \mathbb{N}$. We will denote by $P(K)$, $P^b(K)$ the class of positive definite functions on K and the class of bounded positive definite functions on K. The classical moment problem stated by Stieljes in the following form: For any sequence of real numbers $s_0, s_1, \ldots,$ find necessary and sufficient conditions for the existence of a measure μ on $[0, \infty[$ such that

$$s_n = \int_0^\infty x^n \, d\mu(x)$$

holds for $n = 0, 1, \ldots$. The F-moment problem

$$s_n = \int_F x^n \, d\mu(x)$$

carries the name of Stieltjes (for $F = \mathbb{R}_+$), Hamburger (for $F = \mathbb{R}$), Hausdorff (for $F = [0, 1]$), and Toeplitz (for $F = \mathbb{T}$). A function $\Phi : K \to \mathbb{C}$ is called a moment function on hypergroup K if there exist a measure $\mu \in E_+(K^*)$ such that Φ has the following representation:

$$\Phi(z) = \int_{K^*} \chi(z) d\mu(\chi), \text{ for } z \in K, \tag{1.1}$$

where $E_+(K^*)$ denote the set of Radon measure $\mu \in M(K^*)$ such that

$$\int_{K^*} |\chi(z)| d\mu(\chi) < \infty, \text{ for all } z \in K. \tag{1.2}$$

Clearly, we have $M_+^c(K^*) \subseteq E_+(K^*) \subseteq M_+^c(K^*)$, where $M_+^b(K^*)$, $M_+^c(K^*)$ are the spaces of bounded Radon measures and with compact support, respectively.

2. Dual Hypergroups

A hypergroup $(K, *)$ is called commutative if $(M(K), +, *)$ is a commutative algebra and Hermitian if the involution $^-$ is the identity map. Its easy to prove that every Hermitian hypergroup is commutative. A locally bounded measurable function $\chi : K \to \mathbb{C}$ is called a semicharacter if $\chi(e) = 1$ and $\chi(x * y^-) = \chi(x)\overline{\chi(y)}$ for all $x, y \in K$. Every bounded semicharacter is called a character, if the character is not locally null then (see [3, Proposition 1.4.33]) it must be continuous. We will denote by $H(K)$ the convex cone of moment functions on the hypergroup K. For $\Phi \in H(K)$ describe the set $E_+(K^*, \Phi)$ of representing measures for Φ, i.e., the set of $\mu \in E_+(K^*)$ satisfies (2.1).

Theorem 2.1. *If* $\{K_i\}_{i=1}^n$, $n \in \mathbb{N}$ *is a system of finitely generated hypergroup, then the dual* $\prod_{i=1}^n K_i^*$ *is homeomorphic and isomorphic with a closed subsemigroup of* $(\mathbb{C}^n, .)$ *for a suitable* $n \in \mathbb{N}$, *in particular* $\prod_{i=1}^n K_i^*$ *is locally compact with a countable base for the topology.*

Proof. As a direct application for Fubinis Theorem, the reader can easily prove that:

If Φ_1 and Φ_2 are two moment functions on K, represented via (2.1) by the two measures μ_1 and μ_2, respectively, then the product $\Phi_1 \cdot \Phi_2$ is a moment function represented by $\mu_1 * \mu_2$. Since for all $i = 0, 1, \ldots, n$ the hypergroup K_i is finitely generated, so there exist finitely many elements $e_1, e_2, \ldots, e_n$ belong to K_i such that for every element $z \in K_i$ there exist $\{k_j\}_{j=1}^n \subseteq \mathbb{N}_0$ such that

$$z = k_1 e_1 + \ldots + k_n e_n, \text{ with } k_j \in \mathbb{N}_0, \quad j = 1, 2, \ldots, n. \tag{2.1}$$

For $\chi \in \prod_{i=1}^n K_i^*$ we define a continuous homomorphism $I(\chi) \in \mathbb{C}^n$ by $I(\chi) = (\chi(e_1), \ldots, \chi(e_n))$. Clearly, $I : \prod_{i=1}^n K_i^* \to \mathbb{C}^n$ is one to one mapping. Suppose $(\chi_\alpha)_{\alpha \in \Lambda}$ is a net from the dual hypergroup $\prod_{i=1}^n K_i^*$ such that $\lim_\alpha I(\chi_\alpha)$ exists in $\mathbb{C}^n$. Since for every $z \in K_i$ and any $\chi \in \prod_{i=1}^n K_i^*$, we have

$$\chi(z) = \prod_{j=1}^n \chi(e_j)^{k_j},$$

so $\lim_\alpha \chi_\alpha(z)$ exists for each element z in the hypergroup $\prod_{i=1}^n K_i$. Putting $\chi(z) := \lim_\alpha \chi_\alpha(z)$, $z \in \prod_{i=1}^n K_i$ implies χ belongs to the dual hypergroup $\prod_{i=1}^n K_i^*$, $I(\chi) = \lim_\alpha I(\chi_\alpha)$ and χ_α converges to χ in $\prod_{i=1}^n K_i^*$.

3. The Product of Perfect Hypergroups

The set $E_+(K^*, \Phi)$ is a closed convex set in $M_+(K^*)$. The moment function $\Phi \in H(K)$ will be called determinate if the set $E_+(K^*, \Phi)$ is a one point set, and indeterminate if the set $E_+(K^*, \Phi)$ consists of more than one point. Similarly, the measure $\mu \in E_+(K^*)$ will be called determinate or indeterminate if the corresponding moment function Φ given by (2.1) is so. A commutative hypergroup $(K, *)$ with involution is called perfect if every Φ, belongs to the space of positive definite functions on K, is a determinate moment function. For any measure $\mu \in M^c(K^*)$ we will denote by $\widehat{\mu}$ the function

$$\widehat{\mu}(z) = \int_{K^*} \chi(z) d\mu(\chi), \quad z \in K,$$

and μ is called the generalized Laplace transform of μ. This transform has the following properties:

(1) $\widehat{\alpha\mu + \beta\nu} = \alpha\widehat{\mu} + \beta\widehat{\nu}$;

(2) $\widehat{\mu * \nu} = \widehat{\mu} \cdot \widehat{\nu}$;

(3) $\widehat{\mu} = 0 \Rightarrow \mu = 0$, where $\alpha, \beta \in \mathbb{C}$, $\mu, \nu \in M^c(K^*)$.

The following Lemma is an analogue of some results stated in [7, 8] for Hausdorff spaces.

Lemma 3.1. *Let P and Q be two perfect hypergroups and let $\Phi : B(P) \times B(Q) \to [0, \infty[$ denote a Radon bimeasure. Then there is a uniquely determined Radon measure γ on $P \times Q$ with the property*

$$\Phi(K, L) = \gamma(K \times L), \textit{ for all } K \in H(P),\ L \in H(Q).$$

Furthermore, the equality

$$\Phi(A, B) = \gamma(A \times B)$$

holds for all Borel sets $A \in B(P)$ *and* $B \in B(Q)$.

Lemma 3.2. *Let* $\{K_i\}_{i=1}^{n}$, $n \in \mathbb{N}$ *be a system of perfect hypergroup. The transformation* $\mu \to \hat{\mu}$ *is an injective algebra homomorphism of* $E(\prod_{i=1}^{n} K_i^*)$ *into the space* $\mathbb{C}^{\prod_{i=1}^{n} K_i}$.

Proof. Suppose that

$$\hat{\mu}_1(z) - \hat{\mu}_2(z) + i\,(\hat{\mu}_3(z) - \hat{\mu}_4(z)) = 0,$$

for all $z \in \prod_{i=1}^{n} K_i$, where $\mu_j \in E_+(\prod_{i=1}^{n} K_i^*)$ for $j = 1, \ldots, 4$. Since $\hat{\mu}_j(z^*) = \overline{\hat{\mu}_j(z)}$ for all $j = 1, \ldots, 4$, we also have

$$\hat{\mu}_1(z) - \hat{\mu}_2(z) - i\,(\hat{\mu}_3(z) - \hat{\mu}_4(z)) = 0,$$

for all $z \in \prod_{i=1}^{n} K_i$, hence

$$\hat{\mu}_1(z) = \hat{\mu}_2(z), \quad \hat{\mu}_3(z) = \hat{\mu}_4(z),$$

for all $z \in \prod_{i=1}^{n} K_i$. Since $\prod_{i=1}^{n} K_i$ is perfect hypergroup we conclude that $\mu_1 = \mu_2$ because they represent the same moment function. Similarly $\mu_3 = \mu_4$. This implies the injectivity of the generalized Laplace transform, the rest of the Lemma being obvious.

Theorem 3.3. *The product* $\prod_{i=1}^{n} K_i := K_1 \times K_2 \times \ldots \times K_n$ *of perfect hypergroups, is also perfect hypergroup.*

Proof. Firstly, we will prove that "if K_1 and K_2 are perfect hypergroups, then so is the product $K_1 \times K_2$" and we directly get the required by using mathematical induction. Suppose K_1 and K_2 are perfect hypergroups. For (χ, ζ) belong to the dual product $K_1^* \times K_2^*$ the mapping $(z, w) \to \chi(z)\zeta(w)$ is a character on $K_1 \times K_2$, and every character η on the product $K_1 \times K_2$ is of this form with $\chi(z) = \eta(z, 0)$ and $\zeta(w) = \eta(0, w)$. Let $\Phi \in P(K_1 \times K_2)$, for each $w \in K_2$ we have the following:

(1) $\Phi(\,.\,, w^* + w) \in P(K_1)$;

(2) $\Phi(\,.\,, w^* + w) + \Phi(., 0) - \Phi(\,.\,, w^*) - \Phi(\,.\,, w) \in P(K_1)$;

(3) $\Phi(\,.\,, w^* + w) + \Phi(., 0) - i\Phi(\,.\,, w^*) + i\Phi(\,.\,, w) \in P(K_1)$.

Since $(z_1^* + z_2, w^* + w) = (z_1, w)^* + (z_2, w)$, so (1) is obvious. Let $\{z_1, z_2, \ldots, z_n\} \subseteq K_1$ and $\{c_1, c_2, \ldots, c_n\} \subseteq \mathbb{C}$ be given, so $\{c_1, c_2, \ldots, c_n, -c_1, -c_2, \ldots, -c_n\} \subseteq \mathbb{C}$ (resp., $\{c_1, c_2, \ldots, c_n, ic_1, ic_2, \ldots, ic_n\} \subseteq \mathbb{C}$), hence

$$\sum_{k=1}^{n}\sum_{j=1}^{n} c_k\overline{c_j}[\Phi(z_j^* + z_k, w^* + w) + \Phi(z_j^* + z_k, 0)$$

$$- \Phi(z_j^* + z_k, w^*) - \Phi(z_j^* + z_k, w)] \geqslant 0$$

(resp.,

$$\sum_{k=1}^{n}\sum_{j=1}^{n} c_k\overline{c_j}[\Phi(z_j^* + z_k, w^* + w) + \Phi(z_j^* + z_k, 0)$$

$$- i\Phi(z_j^* + z_k, w^*) + i\Phi(z_j^* + z_k, w)] \geqslant 0),$$

which shows (2) and (3). For w belongs to the hypergroup K_2 we denote by σ_w, τ_w, and ν_w the uniquely determined representing measures in $E_+(K_1^*)$ for the functions stated in (1)-(3), observing that the measure

$$\mu_t := \frac{1}{2}((\sigma_w + \sigma_0 - \tau_w) + i(\sigma_w + \sigma_0 - \nu_w))$$

is the unique signed measure in $E_+(K_1^*)$ such that Φ have the integral representation

$$\Phi(z, w) = \int_{K_1^*} \chi(z) d\mu_w(\chi), \quad (z, w) \in K_1 \times K_2,$$

and the mapping $w \to \mu_w$ of the hypergroup K_1 into the convex set $E(K_1^*)$ is positive definite. In particular, for any set A belongs to the Borel set $B(K_1^*)$ the function $w \to \mu_w(A)$ belongs to the space of positive definite functions $P(K_1)$ and is therefore of the form

$$\mu_w(A) = \int_{K_2^*} \zeta(w) d\tau_A(\zeta), \quad w \in K_2,$$

for a uniquely determined measure τ_A belongs to the convex set $E_+(K_2^*)$. The mapping $A \hookrightarrow \tau_A$ of the Borel set $B(K_1^*)$ into $E_+(K_2^*)$ is a Radon vector measure [5], i.e.,

(i) $\tau_\Phi = 0$;

(ii) $\tau_{\bigcup_1^\infty A_n} = \sum_1^\infty \tau_{A_n}$, when (A_n) is a sequence of disjoint sets in $B(K_1^*)$;

(iii) $\tau_A = \sup\{\tau_F | F \in H(K_1^*), F \subseteq A\}$ for $A \in B(K_1^*)$.

The two conditions (i) and (ii) implies that the measure τ is increasing: If A_1, A_2 belong to the Borel set $B(K_1^*)$ and $A_1 \subseteq A_2$, then $\tau_{A_1} \leqslant \tau_{A_2}$.

For each $A \in B(K_1^*)$, the net $\{\tau_F | F \in H(K_1^*), F \subseteq A\}$ is increasing if the index set is ordered by inclusion, so by [5], Exercise 2.1.19,

$$\widetilde{\tau}_A = \sup\{\tau_F | F \in H(K_1^*), F \subseteq A\}$$

is a Radon measure on K_2^* and $\widetilde{\tau}_A \leqslant \tau_A$ so in particular $\widetilde{\tau}_A \in E_+(K_2^*)$. For w belongs to the dual hypergroup K_2^* we find

$$\int_{K_2^*} \zeta(w) d\widetilde{\tau}_A(\zeta) = \lim_F \int_{K_2^*} \zeta(w) d\tau_F(\zeta) = \lim_F \mu_w(F)$$

$$= \mu_w(A) = \int_{K_2^*} \zeta(w) d\tau_A(\zeta),$$

where the limit is a long the index set $\{F \in H(K_1^*), F \subseteq A\}$, from Lemma 3.1. We have $\widetilde{\tau}_A = \tau_A$ for every A belongs to the Borel set $B(K_1^*)$.

The function $\psi : B(K_1^*) \times B(K_2^*) \to [0, \infty[$ defined by

$$\psi(A, B) = \tau_A(B) \text{ for } A \in B(K_1^*), B \in B(K_2^*)$$

is a Radon bimeasure by (i)-(iii). Lemma 3.2 implies the existence of a Radon measure γ on $K_1^* \times K_2^*$ such that

$$\psi(A, B) = \int_{K_1^* \times K_2^*} 1_A \otimes 1_B d\gamma = \int_{K_2^*} 1_B \, d\tau_A,$$

for

$$A \in B(K_1^*), B \in B(K_2^*),$$

where the tensor product $f \otimes g$, for functions $f : K_1^* \to \mathbb{C}$ and $g : K_2^* \to \mathbb{C}$, on $K_1^* \times K_2^*$ is given by

$$f \otimes g(\chi, \zeta) = f(\chi)g(\zeta) \text{ for } (\chi, \zeta) \in K_1^* \times K_2^*.$$

By essential arguments from integration theory we then have

$$\int_{K_1^* \times K_2^*} 1_A \otimes h d\gamma = \int_{K_2^*} h d\tau_A,$$

for A belongs to the Borel set $B(K_1^*)$ and any τ_A-integrable function $h : K_2^* \to \mathbb{C}$, in particular

$$\mu_t(A) = \int_{K_2^*} \zeta(w) d\tau_A(\zeta) = \int_{K_1^* \times K_2^*} 1_A(\chi)\zeta(w) d\gamma(\chi, \zeta) \text{ for } w \in K_2^*.$$

Replacing w by $t^* + t$ and using the same technique we get

$$\int_{K_1^*} g d\mu_{t^*+t} = \int_{K_1^* \times K_2^*} g(\chi)|\zeta(w)|^2 d\gamma(\chi, \zeta),$$

for w belongs to the dual hypergroup K_2^* and any Borel measurable function $g : K_1^* \to [0, \infty[$, in particular

$$\int_{K_1^* \times K_2^*} |\chi(z)|^2 |\zeta(w)|^2 d\gamma(\chi, \zeta) = \int_{K_1^*} |\chi(z)|^2 d\mu_{t^*+t} < \infty,$$

showing that $\gamma \in E_+(K_1^* \times K_2^*)$. Therefore

$$\int_{K_1^* \times K_2^*} \chi(z)\zeta(w) d\gamma(\chi, \zeta) = \int_{K_1^*} \chi(z) d\mu_t = \Phi(z, w) \text{ for } (z, w) \in K_1^* \times K_2^*,$$

this implies Φ is a moment function with representing measure γ. Finally, we need to prove the determinacy of Φ. Suppose $\gamma_1, \gamma_2 \in E_+$ $(K_1^* \times K_2^*)$ satisfy

$$\Phi(z, w) = \int \chi_{(z, w)} d\gamma_1 \int \chi_{(z, w)} d\gamma_2 \text{ for } (z, w) \in K_1^* \times K_2^*,$$

where $\chi_{(z, w)} : K_1^* \times K_2^* \to \mathbb{C}$ is given by

$$\chi_{(z, w)}(\chi, \zeta) = \chi(z)\zeta(w) \text{ for } \chi \in K_1^*, \zeta \in K_2^*.$$

This implies

$$\int 1_A \bigotimes \chi_t d\gamma_1 = \int 1_A \bigotimes \chi_t d\gamma_2 \text{ for } w \in K_2 \quad A \in B(K_1^*).$$

Showing that, for the projections $\pi_1 : K_1^* \times K_2^* \to K_1^*$ and $\pi_2 : K_1^* \times K_2^* \to K_2^*$, the image measures $(1_{A \times K_2^*} \gamma_1)^{\pi_1}$ and $(1_{A \times K_2^*} \gamma_2)^{\pi_2}$ belong to the convex set $E_+(K_2^*)$ and represent the same function in the space of positive definite functions $P(K_2)$. This yields $\gamma_1(A, B) = \gamma_2(A, B)$ for all $A \in B(K_1^*)$, $B \in B(K_2^*)$ and hence $\gamma_1 = \gamma_2$.

4. Transformation of Hypergroups Between Topological Spaces

Theorem 1.1. *Let $K \subseteq X$ and $H \subseteq Y$ be $*$-hypergroups and let $h : K \to H$ be a surjective $*$-homomorphism. Then if K is perfect, so is H.*

Proof. The dual homomorphism h^* from the dual hypergroup H^* into K^*, defined by $h^*(\tau) := \tau \circ h$ is injective and its image $G = h^*(H^*)$, consisting of all character χ belongs to the dual hypergroup K^*, which is a closed subset of K^*. Furthermore, h^* is a homeomorphism from H^* to G. Let ϕ the class of positive definite functions $P(H)$ be given, then $\phi \circ h \in P(K)$ and has the unique integral representation

$$\phi(h(z)) = \int_{K^*} \chi(z) d\mu(\chi), \quad z \in K, \tag{4.1}$$

we will prove that μ is concentrated on G. For this aim let $a, b \in K$ and put

$$G_{a,\,b} := \{\chi \in K^* | \chi(a) = \chi(b)\}, \tag{4.2}$$

these family of sets $G_{a,b}$, $a, b \in K$ are open and satisfying

$$F^c = \bigcup_{h(a)=h(b)} G_{a,\,b},$$

hence it suffices to prove that $\mu(G_{a,b}) = 0$ if $h(a) = h(b)$. But under this assumption we get for $z \in K$

$$\begin{aligned}\int \chi(z)\chi(a)d\mu(\chi) &= \int \chi(z+a)d\mu(\chi)\\ &= \phi(h(z+a)) = \phi(h(z)) + \phi(h(a))\\ &= \phi(h(z)) + \phi(h(b)) = \phi(h(z+b))\\ &= \int \chi(z)\chi(b)d\mu(\chi),\end{aligned}$$

and therefore

$$\chi(a)\mu = \chi(b)\mu$$

by Lemma 3.1, i.e., $\operatorname{Re}\chi(a)\mu = \operatorname{Re}\chi(b)\mu$ and $\operatorname{Im}\chi(a)\mu = \operatorname{Im}\chi(b)\mu$. This implies

$$\int_{[\operatorname{Re}\chi(b) > \operatorname{Re}\chi(a)]} [\operatorname{Re}\chi(b) - \operatorname{Re}\chi(a)]d\mu = 0, \tag{4.3}$$

so that $\mu(\{\chi | \operatorname{Re}\chi(b) > \operatorname{Re}\chi(a)\}) = 0$. By the same technique we have $\mu(\{\chi | \operatorname{Im}\chi(b) > \operatorname{Im}\chi(a)\}) = 0$, i.e., we have $\mu(G_{a,b}) = 0$. Let ν belongs to the set of bounded Radon measure $M_+^b(H^*)$ be the image of μ under $(h^*)^{-1}$, then $\mu = \nu^{h^*}$ and for $w = h(z) \in H$ we get

$$\phi(w) = \phi(h(z)) = \int \chi(z) d\mu(\chi) = \int_{h^*(H^*)} \chi(z) d\nu^{h^*}(\chi)$$

$$= \int_{H^*} (h^*(\tau))(z) d\nu(\tau) = \int \tau(h(z)) d\nu(\tau) = \int \tau(t) d\nu(\tau),$$

which shows existence of a representing measure for ϕ. If ν' is another measure with this property then by assumption $\nu^{h^*} = (\nu'^{h^*})$ and therefore $\nu = \nu'$, h^* being a homeomorphism.

Some other generalization of our results obtained on generalized form of groups and rings can be found in [9, 10, 11].

5. Examples

The hypergroup $K := ([0, \infty[, *)$ is defined on the nonnegative real line under the convolution:

$$\delta_z * \delta_w = \frac{1}{2}[\delta_{z+w} + \delta_{z-w}], \quad 0 \leqslant w \leqslant z. \tag{5.1}$$

This hypergroup is a perfect hypergroup. The prove the last statement is a direct result from the following Theorem:

Theorem 5.1. *A necessary and sufficient condition that a function* $\Phi : \mathbb{R}_+^m \to \mathbb{R}$ *to have the integral representation*

$$\Phi(z) = \int_{\mathbb{R}^m} exp(< z, w >) d\mu(z), \text{ for } w \in \mathbb{R}_+^m, \tag{5.2}$$

with $\mu \in M_+(\mathbb{R}^m)$, *is that* Φ *is continuous and positive definite on the hypergroup* $(\mathbb{R}_+^m, +)$. *The measure* μ *is uniquely determined by* Φ.

6. Conclusion

The main aim of this paper is to give some ways to preserve the perfectness of hypergroups, under different kinds of transformations. We state and prove some properties of the set of Radon measures $M_+^b(K^*)$. Also we explain some exact conditions on translation operators to translate perfect hypergroups, to another perfect hypergroups.

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