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STABILITY OF NON-MONOTONIC TRAVELLING WAVE SOLUTIONS OF DELAYED REACTION-DIFFUSION EQUATIONS

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Abstract

In this paper, we reconsider the stability of travelling wave solutions of reaction-diffusion equations with time delays, where the comparison principle does not hold. A new method will be developed to obtain the stability. Firstly, we define two auxiliary functions, and then obtain the stability of travelling wave fronts by using the known results and weight energy method. Our results complement the earlier results in this field.

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1. Introduction

The travelling wave describes the translation between two equilibrium states, and thus it is worth studying it. The stability of travelling wave solutions is an important issue in the theory of travelling wave solutions. Sattinger [15] used the method of spectral analysis to study the travelling wave fronts of a reaction-diffusion system and obtained the stability in weighted L^∞ spaces. The asymptotic stability of travelling wave solutions to nonlocal evolution equations and delayed reaction-diffusion equations are studied in [2] and [16, 18], respectively.

Mei et al. [10, 11, 12, 13] considered the following delayed equations:

$$u_t - du_{xx} + \delta u = \varepsilon b(u(x, t - r)), \tag{1.1}$$

and Mei et al. [10, 11] also considered the following nonlocal time-delayed reaction-diffusion equation:

$$u_t - du_{xx} + \delta u = \varepsilon \int_{\mathbb{R}} b(u(x - y, t - r)) f_\alpha(y) dy, \tag{1.2}$$

where constants $d, \delta, \varepsilon > 0$, $f_\alpha(y) = \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{y^2}{4\alpha}}$ and

$$b(u) = pue^{-au^q} \text{ or } b(u) = \frac{pu}{1 + au^q}.$$

They first established a comparison principle and then proved that travelling wave fronts of Equations (1.1) and (1.2) are asymptotic stable in some exponentially weighted L^∞ spaces. About the asymptotic stability of travelling wave fronts for Nicholson's blowflies equation, see Gourley-Kuang [5], Lin-Mei [7], Lv-Wang [9], Mei-Ou-Zhao [14], Zhang [23] and so on.

In this paper, we shall consider the Equation (1.1) with

$$u(x, s) = u_0(x, s), \quad x \in \mathbb{R}, s \in [-r, 0]. \tag{1.3}$$

Throughout this paper, we always assume that $\varepsilon = 1$,

$$b(u) = pue^{-au}, \text{ with } e < \frac{p}{\delta} < e^2.$$

Clearly, there are two constants 0 and $K = \frac{1}{a} \ln \frac{p}{\delta}$ such that $\delta u = b(u)$. Moreover, we have

$$b'(u) = pe^{-au}(1 - au)\begin{cases} \geq 0 & \text{for } 0 < u \leq \frac{1}{a}, \\ < 0 & \text{for } u > \frac{1}{a}. \end{cases}$$

Obviously, the comparison principle does not hold for Equation (1.1) with (1.3), and thus the earlier method fails. The existence, uniqueness of non-monotone travelling wave solution was obtained by Wu et al. [19, 20] and Faria-Trofimchuk [1, 3]. Wu et al. [21] obtained the stability of non-monotone travelling wave fronts by using the weight energy method and the method we used here is different from that in [21]. In order to obtain the stability of travelling wave solutions, we will define two auxiliary functions $b_{\pm}(w)$ which are non-decreasing functions. Then we obtain the stability of travelling wave solutions by using the known results and weight energy method. Although in paper [4, 6, 8] the authors considered the same problem as in this paper, the method we used here is different from those.

Denote $w_+^* = \frac{p}{a\delta e}$ and let

$$b_+(w) = \begin{cases} b(w), & w \in [0, \frac{1}{a} - \varrho], \\ \pi(w), & w \in [\frac{1}{a} - \varrho, \frac{1}{a} + \varrho], \\ \frac{p}{ae}, & w \in [\frac{1}{a} + \varrho, w_+^*]; \end{cases} \quad b_-(w) = \begin{cases} b(w), & w \in [0, \varsigma - \varrho], \\ \iota(w), & w \in [\varsigma - \varrho, \varsigma + \varrho], \\ b(w_+^*), & w \in [\varsigma + \varrho, w_+^*], \end{cases} \tag{1.4}$$

where $0 < \varsigma < \frac{1}{a}$, $b(\varsigma) = b(w_+^*)$, $0 < \varrho \ll 1$, and $\pi(w)$ and $\iota(w)$ are chosen to satisfy that $b_{\pm}(w) \in C^2([0, w_+^*])$, $b'_{\pm}(w) > 0$, $b''_{\pm}(w) \leq 0$, and $b_-(w) \leq b(w) \leq b_+(w)$ for $w \in [0, w_+^*]$. Moreover, $b_-(w) = \delta w$ has a minimal solution in $[0, w_+^*]$, and we denote it by w_-^*, see Figure 1. Actually, in this paper, we only use the information of $b_-(w)$ and we can assume that $b_+(w) := \max_{v \in [0, w]} b(v)$.

Note that $|b'(K)| = \delta(\ln(\frac{p}{\delta}) - 1) < \delta$ for $p/\delta \in (e, e^2)$ and $\lim_{\xi \to +\infty} \phi(\xi) = K$, we deduce that there exists a constant $\xi_* > 0$ such that when $\xi > \xi_*$, it holds that $|b'(\phi(\xi))| < \delta$. Besides that, it is easy to see that $K > w_-^*$. Let

$$w(\xi) = \begin{cases} e^{-2\lambda(\xi - \xi_0)}, & \xi \le \xi_0, \\ 1, & \xi > \xi_0, \end{cases} \tag{1.5}$$

where $\lambda_1(c) < \lambda < \min\{2\lambda_1(c), \lambda^*\}(\lambda_1(c)$ and λ^* are defined as in (1.6)), and ξ_0 satisfies $\phi(\xi - cr) \ge w_-^*$ and $|b'(\phi(\xi - cr))| < \delta$ for $\xi > \xi_0$.

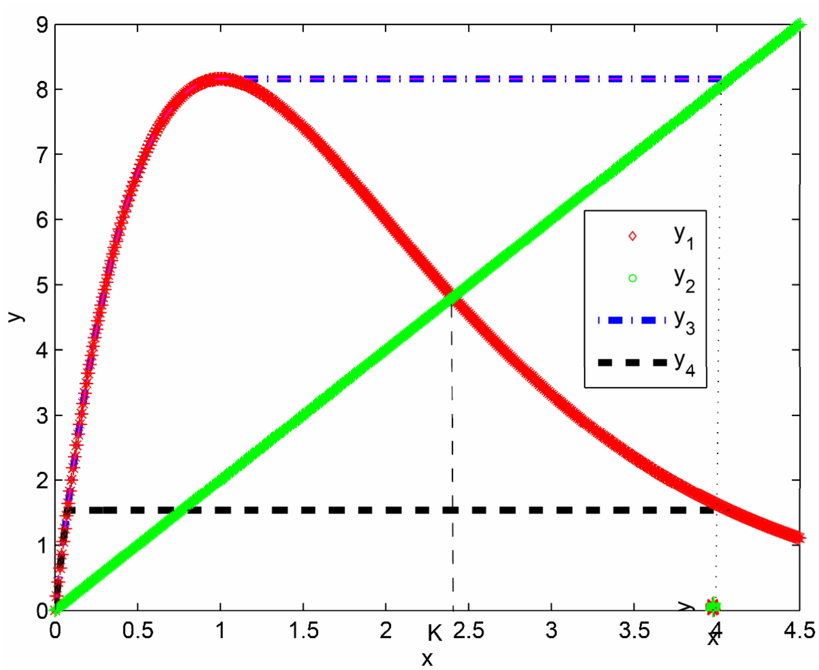

Figure 1. $y_1 = b(x)$, $y_2 = \delta x$, $y_3 = b_+(x)$ and $y_4 = b_-(x)$ with $p = 3e^2$, $\varrho = 0$, $\delta = 2$ and $a = 1$.

Now we state our main result.

Theorem 1.1. *Assume that* $e < p/\delta < e^2$ *and* $0 \le u_0(x, s) \le w_+^*$, *the initial perturbation* $u_0(x, s) - \phi(x + cs)$ *belongs to* $C([-\tau, 0], H_w^1(\mathbb{R}))$, *and the speed*

$$c \ge \frac{d}{\sigma}\lambda + \frac{p - \delta}{\lambda},$$

then the solution $u(x, t)$ *of* (1.1) *with* (1.3) *satisfies*

$$u(x, t) - \phi(x + ct) \in C([0, \infty); H_w^1(\mathbb{R})),$$

where the function $w(x)$ *and* λ *are defined by* (1.5). *In particular, the solution* $u(x, t)$ *converges to the travelling wave front* $\phi(x + ct)$ *exponentially in time*

$$\sup_{x \in \mathbb{R}} |u(x, t) - \phi(x + ct)| \le C\mathrm{e}^{-\mu t},$$

for some positive constants C *and* μ.

We give the following remarks:

(1) Following [10], when $b(u)$ is an increasing function, the author defined the weight function as

$$\omega(\xi) = \begin{cases} e^{-2\lambda(\xi - \xi_0)}, & \xi \le \xi_0, \\ 1, & \xi > \xi_0, \end{cases} \tag{1.6}$$

where $\lambda_1(c) < \lambda \le \lambda^*$, $\lambda_1(c)$ is the smaller root of $F(\lambda(c), c) = 0$, $\lambda^*(c^*)$ satisfies $F(\lambda^*(c^*), c^*) = 0$ and $F_{c^*}(\lambda^*(c^*), c^*) = 0$, and

$$F(\lambda(c), c) = c\lambda - d\lambda^2 + \delta - \varepsilon b'(0)e^{-\lambda cr}.$$

In this paper, we assume that $\varepsilon = 1$. It is clear that λ in this paper satisfies $\lambda \geq \lambda^*$ and thus the λ in this paper is smaller than that in [10]. On the other hand, the result of [10] also holds if we replace the weight function $\omega(x)$ by $w(x)$.

(2) Note that when $b(u)$ is not an increasing function, the comparison principle will not hold. Hence the result in this paper is non-trivial. Actually, whatever the restrict about the initial data and the travelling wave solution, we will not obtain that $u(z, t) - \phi(z) > 0$, where $z = x + ct$, and thus the method of [10, 11, 14] will fail.

(3) In the earlier results about the stability of travelling wave solutions, the assumption about the initial data says that $u_- \leq u_0 \leq u_+$, where u_- and u_+ are two constant equilibria. In this paper, we assume that $0 \leq u_0(x, s) \leq w_+^*$, which is different from the earlier results.

(4) We say that $e < p/\delta < \kappa$. For the value of the κ, we can give an estimation. It is easy to calculate

$$b'(w_+^*) = pe^{-\frac{p}{\delta e}}\left(1 - \frac{p}{\delta e}\right),$$

The condition $|b'(w_+^*)| < \delta$ shows that

$$x(x-1)e^{-x} < \frac{1}{e}, \tag{1.7}$$

where $x = \frac{p}{\delta e}$. Let $f(x) = x(x-1)e^{-x}$, then it follows from $f'(x) = (3x - x^2 - 1)e^{-x} = 0$ that

$$\max_{x>1} f(x) = f\left(\frac{3+\sqrt{5}}{2}\right) = \left(2+\sqrt{5}\right)e^{-\frac{3+\sqrt{5}}{2}}.$$

Since $e^{\frac{1+\sqrt{5}}{2}} > e^2 > 5 > 2 + \sqrt{5}$, we have $|b'(w_+^*)| < \delta$ for all $\frac{p}{\delta e} > 1$.

Let $T > 0$ be a number and B be a Banach space. We denote by $C^0([0, T]; B)$ the space of the B-valued continuous function on $[0, T]$, and by $L^2([0, T]; B)$ the space of the B-valued L^2-functions on $[0, T]$. The corresponding spaces of the B-valued L^2-functions on $[0, \infty)$ are defined similarly.

The rest of this paper is organized as follows. In Section 2, recall some known results and establish some property about the Cauchy problem of (1.1). Section 3 is concerned with the proof of the main results.

2. Preliminaries

In this section, we first recall some known results and then establish some properties about the Cauchy problem of (1.1). Now, we recall the existence of travelling wave solutions of (1.1) with $p/\delta > e.$ In paper [3], Faria-Trofimchuk considered the following delayed reaction-diffusion equation:

$$u_t(x, t) = d\Delta u(x, t) - u(x, t) + g(u(x, t - h)), \; u(x, t) \geq 0, \; x \in \mathbb{R}^m. \quad (2.1)$$

related to the Mackey-Glass type delay differential equations

$$u'(t) = -\, u(t) + g(u(t - h)), \quad u(t) \geq 0, \quad (2.2)$$

where $h \geq 0$ denotes the time delay.

(S) Equation (2.2) has exactly two steady states $u_1(t) \equiv 0$ and $u_2(t) \equiv K > 0,$ the second equilibrium being exponentially asymptotically stable and the first one being hyperbolic. Furthermore, $g \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ and is C^2-smooth in some vicinity of the equilibria, with $p := g'(0) > 1.$ The latter implies that the solution $u_1 = 0$ of (2.2) is unstable for all $h \geq 0.$

In the sequel, $\lambda_1(c)$ denotes the minimal positive root of the characteristic equation $(z/c)^2 - z - 1 + pe^{-zh} = 0$ for sufficiently large c, and λ the unique positive root of the equation $-z - 1 + pe^{-zh} = 0$, where $p > 1$. As shown in [3], $\lim_{c \to \infty} \lambda_1(c) = \lambda$. Under the above assumption (S), they obtained the following results.

Proposition 2.1. *Assume* (S) *holds. If the positive equilibrium* K *of Equation* (2.2) *is globally attracting, then there is* c^* *such that, for each* $\nu \in \mathbb{R}^m$, $\|\nu\| = 1$, *Equation* (2.1) *has a continuous family of positive travelling waves* $u(x, t) = \phi(x \cdot \nu + ct)$, $c > c^*$. *Furthermore, for some* $s_0 = s_0(c) \in \mathbb{R}$, *we have* $\phi(s - s_0, c) = \exp(\lambda_1(c)s) + O(\exp(2\lambda s))$ *as* $s \to -\infty$, *so that* $\phi'(s - s_0, c) = \lambda_1(c)\exp(\lambda_1(c)s) + O(\exp(2\lambda s)) > 0$ *on some semi-axis* $(-\infty, z]$. *Finally, if* $g'(K)he^{h+1} < -1$, *then the travelling profile* $\phi(t)$ *oscillates about* K *on every interval* $[z, +\infty)$.

Aguerrea et al. [1] established the uniqueness of travelling wave solutions obtained in Proposition 2.1. It is easy to see that the existence of travelling wave solutions of (1.1) can be obtained by Proposition 2.1. Moreover, the travelling wave solution oscillates about K on every interval $[z, +\infty)$, where $z \in \mathbb{R}$. Due to the continuous of the travelling wave solutions $\phi(z)$, we see that $\phi(z)$ is bounded, that is, $|\phi(z)| \le M_0$ for $z \in \mathbb{R}$, where $M_0 > 0$.

Note that Mei et al. [10] obtained the stability of travelling wave solutions of Equation (1.1) with increasing nonlinearity term. Now we recall the results of Mei et al. [14], see also [11]. In [14], they studied the following reaction-diffusion equations with nonlocal nonlinearity:

$$u_t - Du_{xx} + d(u) = \int_{\mathbb{R}} b(u(x - y, t - r)) f_\alpha(y) dy, \tag{2.3}$$

where $D > 0$ and the nonlinear functions $d(u)$ and $b(u)$ denote the death and birth rates of the mature population, respectively, and satisfy the following hypotheses:

(H1) There exist $u_- = 0$ and $u_+ > 0$ such that $d(0) = b(0) = 0$, $d(u_+) = b(u_+)$, and $d(u), b(u) \in C^2[0, u_+]$;

(H2) $b'(0) > d'(0)$ and $0 \le b'(u_+) < d'(u_+)$, and $d'(u_+)^2 > b'(0)b'(u_+)$;

(H3) For $0 \le u \le u_+$, $d'(u) \ge 0$, $b'(u) \ge 0$, $d''(u) \ge 0$, $b''(u) \le 0$, but either $d''(u) > 0$ or $|b''(u)| > 0$.

They defined a weight function as (1.6), where λ_1 and λ^* satisfy

$$c\lambda - D\lambda^2 + d'(0) - b'(0)e^{-\lambda cr} = 0,$$

and obtained the following result.

Proposition 2.2 ([14]). *Let* $d(u)$ *and* $b(u)$ *satisfy* (H1)-(H3). *For a given travelling wave front* $\phi(x + ct)$ *of* (2.3) *with* $c \ge c^*$ *and* $\phi(\pm\infty) = u_\pm$, *if the initial data satisfy*

$$u_- \le u_0(x, s) \le u_+, \quad (x, s) \in \mathbb{R} \times [-\tau, 0],$$

$$u_0(x, s) - \phi(x + cs) \in C([-\tau, 0]; L^1_\omega(\mathbb{R}) \cap H^1(\mathbb{R})),$$

then the solution $u(x, t)$ *of the Cauchy problem* (2.3) *with* (1.3) *satisfies*

$$u_- \le u(x, t) \le u_+, \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+,$$

$$u(x, t) - \phi(x + ct) \in C([0, \infty); L^1_\omega(\mathbb{R}) \cap H^1(\mathbb{R})),$$

where the weight function ω *is defined by* (1.6). *When* $c > c^*$, $u(x, t)$ *converges to* $\phi(x + ct)$ *exponentially in time*

$$\sup_{x \in \mathbb{R}} |u(x, t) - \phi(x + ct)| \le C\mathrm{e}^{-\nu t},$$

for some positive constants C *and* ν. *When* $c = c^*$, $u(x, t)$ *converges to* $\phi(x + ct)$ *algebraically in time*

$$\sup_{x \in \mathbb{R}} |u(x, t) - \phi(x + ct)| \leq Ct^{-\frac{1}{2}}.$$

Recently, Wang et al. [17] considered a general case by using the method of [14] and [11]. The authors found the assumption $d''(u) > 0$ or $b''(u) < 0$ in [14] can be removed. Hence the stability of travelling wave solutions of Equation (1.1) with $b(u)$ replacing by $b_{\pm}(u)$ can be established by using the method [14, 17].

Next, we consider the following Cauchy problem:

$$\begin{cases} u_t - du_{xx} + \delta u = b(u(x, t - r)), & x \in \mathbb{R}, t > 0, \\ u(x, s) = u_0(x, s), & x \in \mathbb{R}, s \in [-r, 0], \end{cases} \tag{2.4}$$

where $\delta > 0$ is a constant.

We denote $u^-(x, t; u_0^-)$ and $u^+(x, t; u_0^+)$ as the solution of Cauchy problem (2.4) with initial data u_0^- and u_0^+, respectively, where the nonlinearity term $b(u(x, t - r))$ is replaced by $b_-(u, (x, t - r))$ and $b_+(u(x, t - r))$.

Lemma 2.1. *Let* $u^-(x, t; u_0^-)$ *and* $u^+(x, t; u_0^+)$ *be the solution of Cauchy problem* (2.4) *with initial data* u_0^- *and* u_0^+, *respectively. Assume that* $0 \leq u_0^- \leq w_-^*$, $0 \leq u_0$, $u_0^+ \leq w_+^*$, *and further that* $u_0^-(x, s) \leq u_0(x, s) \leq u_0^+(x, s)$ *for* $x \in \mathbb{R}$ *and* $t \in [-r, 0]$, *then the solution* $u(x, t)$ *of* (2.4) *satisfies* $u^-(x, t; u_0^-) \leq u(x, t) \leq u^+(x, t; u_0^+)$.

Proof. We only prove that $u^-(x, t; u_0^-) \leq u(x, t)$, because the proof of $u(x, t) \leq u^+(x, t; u_0^+)$ is similar. Set $v(x, t) = u(x, t) - u^-(x, t; u_0^-)$. It follows from $u_0^-(x, s) \leq u_0(x, s)$ for $x \in \mathbb{R}, t \in [-r, 0]$ that $v(x, t-r) \geq 0$ for $t \in [0, r]$. Note that $b_-(u) \leq b(u)$ and the non-decreasing of $b_-(u)$, we have for $t \in [0, r]$

$$v_t - dv_{xx} + \delta v = b(u(x, t-r)) - b_-(u^-(x, t-r))$$

$$\geq b_-(u(x, t-r)) - b_-(u^-(x, t-r)) \geq 0.$$

Noting that $v_0(x, s) = u_0(x, s) - u_0^-(x, s) \geq 0$, it follows from the parabolic principle that $v(x, t) \geq 0$ for $t \in [0, r]$. Repeating this procedure to each of the intervals $[nr, (n+1)r], n = 1, 2, \cdots$, it follows that $v \geq 0$ in $\mathbb{R} \times \mathbb{R}_+$, that is, $u(x, t) \geq u^-(x, t; u_0^-)$ in $\mathbb{R} \times \mathbb{R}_+$. This completes the proof. □

Lemma 2.2. *Assume that* $b(u)$ *is replaced by* $b_-(u)$ (*or* $b_+(u)$), *and the initial data satisfy*

$$0 \leq u_0^-(x, s) \leq w_-^* \ (or\ 0 \leq u_0^+(x, s) \leq w_+^*), \quad (x, s) \in \mathbb{R} \times [-\tau, 0].$$

Then the Cauchy problem (2.4) *has a unique solution* $u^-(x, t; u_0^-)$ (*or* $u^+(x, t; u_0^+)$) *with respect to* $b_-(u)$ (*or* $b_+(u)$), *which satisfies*

$$0 \leq u^-(x, t; u_0^-) \leq w_-^* \ (or\ 0 \leq u^+(x, t; u_0^+) \leq w_+^*), \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+.$$

The proof of this lemma is standard and we omit it here. Combining the above two lemmas, one sees that the solution $u(x, t)$ of (2.4) satisfies that $u^-(x, t, u_0^-) \leq u(x, t) \leq w_+^*$, where $0 \leq u_0^- \leq w_-^*, 0 \leq u_0, u_0^+ \leq w_+^*$, and $u_0^-(x, s) \leq u_0(x, s) \leq u_0^+(x, s)$.

Note that Equation (1.1) with the nonlinearity terms $b_-(u)$ and $b_+(u)$ has the same linear equation at $u = 0,$ thus travelling wave solutions to Equation (1.1) with the nonlinearity terms $b_-(u)$ and $b_+(u)$ have the same minimum speed c^*. Now, in order to use Proposition 2.2, we look for $u_0^-(x, s)$ such that all the assumptions in Proposition 2.2 hold. By the assumption "$u_0(x, s) - \phi(x + cs)$ belongs to $C([-t, 0], H_w^1(\mathbb{R}))$" in Theorem 1.1 and the definition of $w(x)$, we claim that there exists a constant $\sigma_0 \in \mathbb{R}$ such that $0 \le u_0(x, s) - \phi(x + cs) + \phi_-(x + cs) \le w_-^*$ for $x \le \sigma_0$, where $\phi_-(\xi)$ is the travelling wave solution to Equation (1.1) with $b(u)$ replaced by $b_-(u)$. Indeed, by "$u_0(x, s) - \phi(x + cs)$ belongs to $C([-\tau, 0], H_w^1(\mathbb{R}))$" and $w(\xi) = e^{-2\lambda(\xi-\xi_0)}$ for $\xi \le \xi_0$, we know that $|u_0(x, s) - \phi(x + cs)|e^{-2\lambda(x-\xi_0)} \to 0$ as $x \to -\infty$, i.e., $u_0(x, s)$ is very close to $\phi(x + cs)$ near $x = -\infty$. From Proposition 2.1, we know that $\phi_-(x + cs)$ $\exp(-\lambda_1(c)(x + cs)) \to C$ as $x + cs \to -\infty$. Thus, we have $u_0(x, s) - \phi(x + cs)$ $+ \phi_-(x + cs) \ge 0$ for $x \le \sigma_0$ provided that $|\sigma_0|$ is sufficiently large. And $u_0(x, s) - \phi(x + cs) + \phi_-(x + cs) \le w_-^*$ for $x \le \sigma_0$ obviously holds since $u_0(x, s)$ is very close to $\phi(x + cs)$ near $x = -\infty$. By the way, we can deduce that $u_0(x, s) > 0$ for $x \le -x^*$, where x^* is large enough. Let

$$u_0^-(x, s) = \begin{cases} u_0(x, s) - \phi(x + cs) + \phi_-(x + cs), \; x \le \sigma_0, \; s \in [-r, 0], \\ \min\{\kappa_1 e^{x-\sigma_0}, \; u_0(x, s), \; w_-^*\}, \; x > \sigma_0, \; s \in [-r, 0], \end{cases}$$

where

$$\kappa_1 = u_0(\sigma_0, s) - \phi(\sigma_0 + cs) + \phi_-(\sigma_0 + cs).$$

We remark that we can choose $\phi_-(x)$ such that $\phi(x) \ge \phi_-(x)$ for $x \le \sigma_0$ because of the translational invariance. Here we only consider the $u_0^-(x, s)$ and assume that $u_0^+(x, s)$ satisfies the assumptions of Lemmas

2.1 and 2.2. It is easy to see that $u_0^-(x, s) \le u_0(x, s) \le u_0^+(x, s)$ for $(x, s) \in \mathbb{R} \times [-r, 0]$. By using the assumption "$u_0(x, s) - \phi(x + cs)$ belongs to $C([-\tau, 0], H_w^1(\mathbb{R}))$" and the definition of $u_0^-(x, s)$, we have that $u_0^-(x, s)$ is a continuous function and

$$u_0^-(x, s) - \phi_-(x + cs) \in C([-r, 0], L_\omega^1(\mathbb{R}) \cap H^1(\mathbb{R})).$$

Indeed, it is easy to see that $u_0^-(x, s) - \phi_-(x + cs) \in C([-r, 0], L_\omega^1((-\infty, \sigma_0]) \cap H^1((-\infty, \sigma_0]))$. On the other hand, from the definition of $u_0^-(x, s)$ and $\lim_{x \to \infty} u_0(x, s) = u_+ > w_-^*$, we have, for $x \gg 1$, $u_0^-(x, s) = w_-^*$, and by using the known facts that ϕ_- convergent w_-^* exponentially, we have the above result. It follows from Proposition 2.2 that there exists two constants $s_0 > 0$ and $\xi_1 \in \mathbb{R}$ such that

$$u^-(x, t; u_0^-) \ge \epsilon_1 w_-^* \text{ for } t > s_0, \quad x + ct > \xi_1, \tag{2.5}$$

where $0 < \epsilon_1 < 1$ satisfies $-\delta < b'(\epsilon_1 w_-^*) < 0$. We remark that s_0 depends on the choice of ϵ_1.

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by using weight energy method. Let $\xi = x + ct$ and

$$v(\xi, t) = u(x, t) - \phi(x + ct), \quad v_0(\xi, s) = u_0(x, s) - \phi(x + cs),$$

then v satisfies

$$\begin{cases} v_t + cv_\xi - dv_{\xi\xi} + \delta v = b(v(\xi - cr, t - r) + \phi(\xi - cr)) - b(\phi(\xi - cr)), \\ v(\xi, s) = v_0(\xi, s), \qquad s \in [-r, 0]. \end{cases} \tag{3.1}$$

In order to obtain the element estimate, we need the following lemma.

Lemma 3.1. *Assume that all the assumptions in Theorem* 1.1 *hold. Then for any* $t > 0$, *there exists a constant* $C := C(t) > 0$ *such that*

$$\|v(t)\|_{L_w^2} \leq C.$$

Proof. Note that v satisfies that

$$\begin{cases} v_t + cv_\xi - dv_{\xi\xi} + \delta v = b'(\phi(\xi - cr))v(\xi - cr, t - r) + \dfrac{b''(\eta)}{2} v^2(\xi - cr, t - r), \\ v(\xi, s) = v_0(\xi, s), \qquad\qquad s \in [-r, 0], \end{cases} \tag{3.2}$$

where $\eta = \phi(\xi - cr) + \theta v(\xi - cr, t - r)$ and $0 \leq \theta \leq 1$.

Let $w(\xi) > 0$ be the weight function defined in (1.5). Multiplying the first equation of (3.2) by $e^{2\mu t} w(\xi) v(\xi, t)$, where $\mu > 0$ will be specified later in Lemma 3.3, we have

$$\begin{aligned} &\left(\frac{1}{2} e^{2\mu t} wv^2\right)_t + \left(\frac{c}{2} e^{2\mu t} wv^2 - de^{2\mu t} wvv_\xi\right)_\xi + de^{2\mu t} w' vv_\xi \\ &\qquad + de^{2\mu t} w' v_\xi^2 + \left(-\frac{c}{2}\frac{w'}{w} - \mu + \delta\right) wv^2 e^{2\mu t} \\ &= b'(\phi(\xi - cr)) wvv(\xi - cr, t - r) e^{2\mu t} + \frac{b''(\eta)}{2} v^2(\xi - cr, t - r) vwe^{2\mu t}. \end{aligned} \tag{3.3}$$

For $\sigma \in (1/2, 1)$ to be chosen later, one gets from the Cauchy-Schwarz inequality that

$$|de^{2\mu t} w' vv_\xi| \leq d\sigma e^{2\mu t} w(v_\xi)^2 + \frac{d}{4\sigma} e^{2\mu t} \left(\frac{w'}{w}\right)^2 wv^2. \tag{3.4}$$

We claim that

$$(w(\xi)v^2(\xi, t))\Big|_{\xi=-\infty}^{\xi=\infty} \geq 0. \tag{3.5}$$

We only prove that $w(-\infty)v^2(-\infty) = 0$ because $w(\infty)v^2(\infty) \geq 0$. Note that

$$(u - \phi)^2 \leq (u^+ - \phi)^2, \quad \text{if } u - \phi \geq 0,$$

$$(u - \phi)^2 \leq (u^- - \phi)^2, \quad \text{if } u - \phi \leq 0,$$

we have

$$\begin{aligned} v^2 = (u - \phi)^2 &\leq (u^+ - \phi)^2 + (u^- - \phi)^2 \\ &= (u^+ - \phi_+ + \phi_+ - \phi)^2 + (u^- - \phi_- + \phi_- - \phi)^2 \\ &\leq 2[(u^+ - \phi_+)^2 + (u^- - \phi_-)^2 + (\phi_+ - \phi)^2 + (\phi_- - \phi)^2]. \end{aligned}$$

Following [9], we have $w(-\infty)(u^{\pm}(-\infty, t) - \phi_{\pm}(-\infty)) = 0$ for all $t \geq 0$. Note that $\lambda < 2\lambda_1(c)$ and $\phi_+(\xi) - \phi(\xi) = o(e^{2\lambda_1(c)\xi})$, we have $w(-\infty)(\phi_+(-\infty) - \phi(-\infty)) = 0$. Combining above discussion, we obtain $w(-\infty)v^2(-\infty) = 0$. Hence we prove (3.5). On the other hand, by using heat kernel, we can obtain that

$$\begin{aligned} v(\xi, t) = {} & e^{-(\frac{(1-d)c^2}{4d^2}+\delta)t} \int_{\mathbb{R}} (u_0(y) - \phi(y)) \frac{1}{\sqrt{4\pi dt}} e^{-\frac{(\xi-ct-y)^2}{4dt}} dy \\ & + \int_0^t \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d(t-s)}} e^{-(\frac{(1-d)c^2}{4d^2}+\delta)(t-s)} e^{-\frac{(\xi-c(t-s)-y)^2}{4d(t-s)}} \\ & \times [b(v(y - cr, s - r) + \phi(y - cr)) - b(\phi(y - cr))]dyds, \end{aligned}$$

which implies that

$$v_\xi(\xi, t) = e^{-(\frac{(1-d)c^2}{4d^2}+\delta)t}\int_{\mathbb{R}}(u_0(\xi - ct - y) - \phi(\xi - ct - y))\frac{1}{\sqrt{4\pi dt}}\frac{y}{2dt}e^{-\frac{y^2}{4dt}}dy$$

$$+\int_0^t\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi d(t-s)}}\frac{y}{2d(t-s)}e^{-(\frac{(1-d)c^2}{4d^2}+\delta)(t-s)}e^{-\frac{y^2}{4d(t-s)}}$$

$$\times[b(v(\xi - c(t-s) - y - cr, s - r) + \phi(y - cr))$$

$$-b(\phi(\xi - c(t-s) - y - cr))]dyds.$$

Let $\xi \to \infty$, noting that $|v_\xi| < \infty$ and using Fubini's theorem, and then we have $v_\xi(\infty, t) = 0$. Therefore, we have

$$-(w(\xi)v(\xi, t)v_\xi(\xi, t))\Big|_{\xi=-\infty}^{\xi=\infty} \geq 0. \tag{3.6}$$

We first consider the case $t \in [0, r]$. Substituting (3.4) and (3.5)-(3.6) into (3.3), and integrating the results with respect to (ξ, t) over $\mathbb{R}\times[0, t]$, we obtain

$$e^{2\mu t}\|v(t)\|_{L_w^2}^2 + 2d(1-\sigma)\int_0^t e^{2\mu s}\|v_\xi(s)\|_{L_w^2}^2 ds$$

$$+\int_0^t\int_{\mathbb{R}} e^{2\mu s}\left(-c\frac{w'}{w} - 2\mu - \frac{d}{2\sigma}\left(\frac{w'}{w}\right)^2 + 2\delta\right)w(\xi)v^2(\xi, s)d\xi ds$$

$$= \|v_0(0)\|_{L_w^2}^2 + 2\int_0^t\int_{\mathbb{R}} e^{2\mu s}b'(\phi(\xi - cr))w(\xi)v(\xi, s)v(\xi - cr, s - r)d\xi ds$$

$$+2\int_0^t\int_{\mathbb{R}} e^{2\mu s}\frac{b''(\eta)}{2}v^2(\xi - cr, s - r)w(\xi)v(\xi, s)d\xi ds. \tag{3.7}$$

Here and after, L_w^2 indicates $L_w^2(\mathbb{R})$.

For the second term of the right side in (3.7), by using the Cauchy-Schwarz inequality $2xy \le x^2 + y^2$, we can estimate that

$$\left| 2\int_0^t \int_{\mathbb{R}} e^{2\mu s} b'(\phi(\xi - cr)) w(\xi) v(\xi, s) v(\xi - cr, s - r) d\xi ds \right|$$

$$\le \int_0^t \int_{\mathbb{R}} e^{2\mu s} |b'(\phi(\xi - cr))| w(\xi) v^2(\xi, s) d\xi ds$$

$$+ \int_0^t \int_{\mathbb{R}} e^{2\mu s} |b'(\phi(\xi - cr))| w(\xi) v^2(\xi - cr, s - r) d\xi ds$$

$$= \int_0^t \int_{\mathbb{R}} e^{2\mu s} |b'(\phi(\xi - cr))| w(\xi) v^2(\xi, s) d\xi ds$$

$$+ e^{2\mu r} \int_{-r}^{t-r} \int_{\mathbb{R}} e^{2\mu s} |b'(\phi(\xi))| w(\xi + cr) v^2(\xi, s) d\xi ds$$

$$\le \int_0^t \int_{\mathbb{R}} e^{2\mu s} |b'(\phi(\xi - cr))| w(\xi) v^2(\xi, s) d\xi ds$$

$$+ e^{2\mu r} \int_0^t \int_{\mathbb{R}} e^{2\mu s} |b'(\phi(\xi))| w(\xi + c\tau) v^2(\xi, s) d\xi ds$$

$$+ e^{2\mu r} \int_{-r}^0 \int_{\mathbb{R}} e^{2\mu s} |b'(\phi(\xi))| w(\xi + cr) v^2(\xi, s) d\xi ds. \qquad (3.8)$$

For the last term of the right side in (3.7), by using the facts that $b''(u) = pae^{-au}(-2 + au)$, $\lim_{u \to +\infty} b''(u) = 0$, and $|v| \le |u| + |\phi|$, we have

$$2 \left| \int_0^t \int_{\mathbb{R}} e^{2\mu s} \frac{b''(\eta)}{2} v^2(\xi - cr, s - r) w(\xi) v(\xi, s) d\xi ds \right|$$

$$\le \max_{u > 0} |b''(u)| (w_+^* + M_0) \int_0^t \int_{\mathbb{R}} e^{2\mu s} v^2(\xi - cr, s - r) w(\xi) d\xi ds$$

$$\le C^* \int_{-r}^0 e^{2\mu s} \| v_0(s) \|_{L_w^2}^2 ds,$$

where $C^* = \max_{u>0}|b''(u)|(w_+^* + M_0)e^{2\mu r}$ and $|\phi| \le M_0$. Substituting the above inequality and (3.8) into (3.7), we have that

$$e^{2\mu t}\|v(t)\|_{L_w^2}^2 + 2d(1-\sigma)\int_0^t e^{2\mu s}\|v_\xi(s)\|_{L_w^2}^2 ds$$

$$+ \int_0^t \int_{\mathbb{R}} e^{2\mu s} B_{\mu, w}(\xi) w(\xi) v^2(\xi, s) d\xi ds$$

$$\le \|v_0(0)\|_{L_w^2}^2 + e^{2\mu r}\int_{-r}^0 \int_{\mathbb{R}} e^{2\mu s}|b'(\phi(\xi))| w(\xi + cr) v^2(\xi, s) d\xi ds$$

$$+ C^* \int_{-r}^0 e^{2\mu s}\|v_0(s)\|_{L_w^2}^2 ds$$

$$\le C_1\left(\|v_0(0)\|_{L_w^2}^2 + \int_{-r}^0 \|v_0(s)\|_{L_w^2}^2 ds\right), \tag{3.9}$$

where the constant $C_1 > 0$, $B_{\mu, w}(\xi) > 0$ (see the following Lemma 3.3)

$$B_{\mu, w} = A_w(\xi) - 2\mu - (e^{2\mu r} - 1)|b'(\phi(\xi))| \frac{w(\xi + cr)}{w(\xi)},$$

$$A_w(\xi) = -c\frac{w'(\xi)}{w(\xi)} - \frac{d}{2\sigma}\left(\frac{w'(\xi)}{w(\xi)}\right)^2 + 2\delta$$

$$- |b'(\phi(\xi - cr))| - |b'(\phi(\xi))| \frac{w(\xi + cr)}{w(\xi)}.$$

In order to prove Lemma 3.1, we must prove $B_{\mu, w}(\xi) \ge C$ for some positive constant C. To do this, we need the following key lemmas.

Lemma 3.2. *There exists positive constant* C_2 *such that*

$$A_w(\xi) \ge C_2, \quad \xi \in \mathbb{R}.$$

Proof. Case 1: $\xi \leq \xi_0$. In this region of ξ, $w(\xi) = e^{-2\lambda(\xi - \xi_0)}$ and $w(\xi + cr) = e^{-2\lambda(\xi-\xi_0)} e^{-2\lambda cr}$. Noting that $\phi(\xi)$ is a nonnegative bounded function, we have

$$
\begin{aligned}
A_w(\xi) &= -c\frac{w'(\xi)}{w(\xi)} - \frac{d}{2\sigma}\left(\frac{w'(\xi)}{w(\xi)}\right)^2 + 2\delta \\
&\quad - |b'(\phi(\xi - cr))| - |b'(\phi(\xi))| \frac{w(\xi + cr)}{w(\xi)} \\
&\geq 2c\lambda - \frac{2d}{\sigma}\lambda^2 + 2\delta - 2\max_{z \in \mathbb{R}} b'(z) \\
&= 2\left(c\lambda - \frac{d}{\sigma}\lambda^2 + \delta - p\right) \\
&:= C_3 > 0,
\end{aligned}
$$

where we have used the properties that $b''(\frac{2}{a}) = 0$ and $\lim_{u \to \infty} b'(u) = 0$, and fact that $c > \frac{d}{\sigma}\lambda + \frac{p - \delta}{\lambda}$.

Case 2: $\xi > \xi_0$. In this case, $w(\xi) = w(\xi + cr) = 1$ and we have

$$
\begin{aligned}
A_w(\xi) &= 2\delta - |b'(\phi(\xi - cr))| - |b'(\phi(\xi))| \\
&:= C_4 > 0,
\end{aligned}
$$

where we have used the facts that $|b'(\phi(\xi - cr))| < \delta$ for $\xi > \xi_0$.

Finally, let

$$C_2 := \min\{C_3, C_4\}.$$

Then we have $A_w(\xi) \geq C_2 > 0$. The proof is completed. □

Lemma 3.3. *Let* $\mu_1 > 0$ *be the unique root of the following equation*:

$$C_2/2 - 2\mu - p(e^{2\mu r} - 1) = 0.$$

Then $B_{\mu, w}(\xi) \geq C_2/2 := C_5 > 0$ *on* $\mathbb{R}$ *for all* $0 < \mu \leq \mu_1$.

Proof. Note that $|b'(u)| \leq p$ for $u \in \mathbb{R}$ and

$$\frac{w(\xi + cr)}{w(\xi)} = \begin{cases} e^{-2\lambda cr} < 1, & \xi \leq \xi_0 - cr, \\ e^{2\lambda(\xi - \xi_0)} < 1, & \xi_0 - cr < \xi \leq \xi_0, \\ 1, & \xi > \xi_0 + c\tau. \end{cases}$$

It is easy to see that, for $0 < \mu \leq \mu_1$,

$$\begin{aligned} B_{\mu, w}(\xi) &= A_w(\xi) - \mu - (e^{2\mu r} - 1)|b'(\phi(\xi))| \frac{w(\xi + cr)}{w(\xi)} \\ &\geq C_2 - 2\mu - p(e^{2\mu r} - 1) \\ &\geq C_2/2 := C_5 > 0. \end{aligned}$$

The proof is completed. □

Using the above Lemma 3.3, it follows from (3.9) that

$$\begin{aligned} &e^{2\mu t}\|v(t)\|_{L_w^2}^2 + 2d(1 - \sigma)\int_0^t e^{2\mu s}\|v_\xi(s)\|_{L_w^2}^2 ds + C_6\int_0^t e^{2\mu s}\|v(s)\|_{L_w^2}^2 ds \\ &\quad \leq C_1\left(\|v_0(0)\|_{L_w^2}^2 + \int_{-r}^0 \|v_0(s)\|_{L_w^2}^2 ds\right). \end{aligned}$$

Dropping the positive terms $\int_0^t e^{2\mu s}\|v_\xi(s)\|_{L_w^2}^2 ds$ and $\int_0^t e^{2\mu s}\|v(s)\|_{L_w^2}^2 ds$, we obtain the basic estimate, for $t \in [0, r]$,

$$e^{2\mu t}\|v(t)\|_{L_w^2}^2 \leq C_1\left(\|v_0(0)\|_{L_w^2}^2 + \int_{-\tau}^0 \|v_0(s)\|_{L_w^2}^2 ds\right). \tag{3.10}$$

We remark that the condition $t \in [0, r]$ is only used in estimating the last term of (3.7), thus when we consider the case that $t \in [0, 2r]$, we only estimate the last term of (3.7). Similar to the above discussion, we have

$$2\left|\int_0^t \int_{\mathbb{R}} e^{2\mu s} \frac{b''(\eta)}{2} v^2(\xi - cr, s - r) w(\xi) v(\xi, s) d\xi ds\right|$$

$$\leq \max_{u>0} |b''(u)| (w_+^* + M_0) \int_0^t e^{2\mu s} \|v(s-r)\|_{L_w^2}^2 ds$$

$$\leq \max_{u>0} |b''(u)| (w_+^* + M_0) e^{2\mu r} \left(\int_{-r}^0 e^{2\mu s} \|v_0(s)\|_{L_w^2}^2 ds + \int_0^r e^{2\mu s} \|v(s)\|_{L_w^2}^2 ds \right)$$

$$\leq \max_{u>0} |b''(u)| (w_+^* + M_0) e^{2\mu r} \left(\int_{-r}^0 e^{2\mu s} \|v_0(s)\|_{L_w^2}^2 ds \right.$$

$$\left. + C_1 (\|v_0(0)\|_{L_w^2}^2 + \int_{-r}^0 \|v_0(s)\|_{L_w^2}^2 ds) \right),$$

where we have used (3.10). And then using Lemma 3.3, one can obtain, for $t \in [0, 2r]$,

$$e^{2\mu t} \|v(t)\|_{L_w^2}^2 \leq C_1^* \left(\|v_0(0)\|_{L_w^2}^2 + \int_{-\tau}^0 \|v_0(s)\|_{L_w^2}^2 ds \right).$$

For any $t \in \mathbb{R}_+$, there exists a positive integer N such that $t \in [Nr, (N+1)r]$, then repeat the above discussion and last we can prove that $\|v(t)\|_{L_w^2} \leq C$, where $C = C(t)$. This completes the proof of Lemma 3.1. □

Now, we rewrite (3.1) as

$$\begin{cases} v_t + cv_\xi - dv_{\xi\xi} + \delta v = b'(\gamma) v(\xi - cr, t - r), \\ v(\xi, s) = v_0(\xi, s), \qquad s \in [-r, 0], \end{cases} \tag{3.11}$$

where $\gamma = \phi(\xi - cr) + \theta v(\xi - cr, t - r)$ and $0 \leq \theta \leq 1$.

Let $w(\xi) > 0$ be the weight function defined in (1.5). Multiplying the first equation of (3.11) by $e^{2\mu t} w(\xi) v(\xi, t)$, where $\mu > 0$ will be defined in Lemma 3.3, we have

$$\left(\frac{1}{2} e^{2\mu t} w v^2\right)_t + \left(\frac{c}{2} e^{2\mu t} w v^2 - d e^{2\mu t} w v v_\xi\right)_\xi + d e^{2\mu t} w' v v_\xi + d e^{2\mu t} w v_\xi^2$$

$$+ \left(-\frac{c}{2}\frac{w'}{w} - \mu + \delta\right) w v^2 e^{2\mu t}$$

$$= b'(\gamma) w v v(\xi - cr, t - r) e^{2\mu t}. \tag{3.12}$$

Substituting (3.4) into (3.12), and integrating the results with respect to (ξ, t) over $\mathbb{R} \times [s_0, t]$ (s_0 is defined as (2.5)), we obtain

$$e^{2\mu t} \|v(t)\|_{L_w^2}^2 + 2d(1 - \sigma) \int_{s_0}^{t} e^{2\mu s} \|v_\xi(s)\|_{L_w^2}^2 ds$$

$$+ \int_{s_0}^{t} \int_{\mathbb{R}} e^{2\mu s} \left(- c\frac{w'}{w} - 2\mu - \frac{d}{2\sigma}\left(\frac{w'}{w}\right)^2 + 2\delta\right) w(\xi) v^2(\xi, s) d\xi ds$$

$$= e^{2\mu s_0} \|v(s_0)\|_{L_w^2}^2 + 2\int_{s_0}^{t} \int_{\mathbb{R}} e^{2\mu s} b'(\gamma) w(\xi) v(\xi, s) v(\xi - cr, s - r) d\xi ds. \tag{3.13}$$

Taking the similar steps to obtain (3.9) and using Lemma 3.1, we get

$$e^{2\mu t} \|v(t)\|_{L_w^2}^2 + 2d(1 - \sigma) \int_{s_0}^{t} e^{2\mu s} \|v_\xi(s)\|_{L_w^2}^2 ds$$

$$+ \int_{s_0}^{t} \int_{\mathbb{R}} e^{2\mu s} D_{\mu, w}(\xi) w(\xi) v^2(\xi, s) d\xi ds$$

$$\le e^{2\mu s_0} \|v(s_0)\|_{L_w^2}^2 + e^{2\mu r} \int_{s_0 - r}^{s_0} \int_{\mathbb{R}} e^{2\mu s} |b'(\gamma(\xi + cr, s + r))| w(\xi + cr) v^2(\xi, s) d\xi ds$$

$$\le e^{2\mu s_0} \|v(s_0)\|_{L_w^2}^2 + p e^{2\mu r} \int_{s_0 - r}^{s_0} e^{2\mu s} \|v(s)\|_{L_w^2} ds$$

$$\le C, \tag{3.14}$$

where

$$D_{\mu, w} = C_w(\xi) - 2\mu - (e^{2\mu r} - 1)|b'(\gamma(\xi + cr,\ s + r))|\frac{w(\xi + cr)}{w(\xi)},$$

$$C_w(\xi) = -c\frac{w'(\xi)}{w(\xi)} - \frac{d}{2\sigma}\left(\frac{w'(\xi)}{w(\xi)}\right)^2 + 2\delta$$

$$- |b'(\gamma(\xi,\ s))| - |b'(\gamma(\xi + cr,\ s + r))|\frac{w(\xi + cr)}{w(\xi)}.$$

We remark that in the above inequality the constant C depends on s_0, which is defined in (2.5). The proof of $D_{\mu, w}(\xi) \geq C$ for some positive constant C is similar to that $B_{\mu, w}(\xi) \geq C$. In fact, we only prove when $\xi \geq \xi_0$, $|b'(\gamma)| < \delta$. It follows from the Remarks 4 in the Introduction that $|b'(w_+^*)| < \delta$ for $p/\delta \in (e, e^2)$. We divide the proof into two cases.

Case 1: $\boldsymbol{e < \frac{p}{\delta} \leq 2e}$**.** In this case, we note that $0 \leq \phi(\xi) \leq K \leq w_+^*$ and $u(\xi, t) \leq w_+^*$ (see Lemmas 2.1 and 2.2), and thus we have

$$\gamma(\xi,\ s) = (1 - \theta)\phi(\xi) + \theta u(\xi,\ s) \leq w_+^*.$$

It is easy to see that $b''(u) = ape^{-au}(au - 2)$. Consequently, $b'(u)$ is decreasing in $\left(0, \frac{2}{a}\right)$. Hence we get

$$b'(\gamma(\xi,\ s)) = b'((1 - \theta)\phi(\xi) + \theta u(\xi,\ s)) \geq b'(w_+^*).$$

On the other hand, we can prove

$$w_-^* = \frac{p^2}{a\delta^2 e} e^{-\frac{p}{\delta e}} > \frac{1}{a}.$$

Indeed, let

$$f(x) = \frac{x^2}{e} e^{-\frac{x}{e}},$$

then $f(e) = 1$ and

$$f'(x) = \frac{x}{e} e^{-\frac{x}{e}} (2 - \frac{x}{e}).$$

It is clear that $f(x)$ increases firstly and then decreases. Meanwhile, we remark that $f(e^2) = \frac{1}{a} e^{3-e} > \frac{1}{a}$. So we get a conclusion that $f(x) > \frac{1}{a}$ for $x \in (e, e^2)$. By using (2.5) and the definition of ξ_0 (see (1.5)), we have

$$(1 - \theta)\phi(\xi) + \theta u(\xi, t) \geq \epsilon_1 w_-^*,$$

where we choose $\epsilon_1 \in (0, 1)$ such that $|b'(\epsilon_1 w_-^*)| < \delta$. Noting that $b'(\frac{1}{a}) = 0$ and $w_-^* > \frac{1}{a}$, it is easy to show that there exists $\epsilon_1 \in (0, 1)$ such that $-\delta < b'(\epsilon_1 w_-^*) < 0$. Combining above discussion, we have

$$-\delta < b'(w_+^*) \leq b'((1 - \theta)\phi(\xi) + \theta u(\xi, s)) \leq b'(\epsilon_1 w_-^*) < 0.$$

Case 2: $2e < \frac{p}{\delta} < e^2$. When $u > 1/a$, $b'(u)$ firstly decreases and then increases, and obtains the local minimum at $u = \frac{2}{a}$. More precisely, $b'\left(\frac{2}{a}\right) = -pe^{-2} < \delta$, where we used the assumption that $p/\delta \in (e, e^2)$. We note that

$$|b'(\epsilon_1 w_-^*)|, |b'(w_-^*)|, |b'(\frac{2}{a})| < \delta,$$

and thus we get a conclusion that $|b'(\gamma(\xi, s))| < \delta$ by using the property of $b'(u)$, Similarly, we can prove $|b'(\gamma(\xi + cr, s + r))| < \delta$. Therefore, one can prove that $C_w(\xi) \geq C_6$ and $D_{\mu, w} \geq C_7$, where C_6 and C_7 are some positive constants. Consequently, one can obtain that

$$\|v(t)\|_{L^2_w} \leq C\mathrm{e}^{-\mu t}, \quad t > s_0. \tag{3.15}$$

Next, differentiating (3.1) with respect to ξ, similar to Lemma 3.1, we first obtain $\|v_\xi(t)\|^2_{L^2_w} \leq C$ for $t > 0$, and then taking similar steps to get (3.15), we have

$$\|v_\xi(t)\|_{L^2_w} \leq C\mathrm{e}^{-\mu t}, \quad t > s_0. \tag{3.16}$$

Since $b'''(u)$ and $\phi'(\xi)$ are bounded functions and $v_0(s) \in H^1_w$ for $s \in [-r, 0]$, the proof of (3.16) is very similar to that of (3.15) and thus we omit it here.

Combining (3.15) with (3.16) and noting that $w(\xi) \geq 1$ on $\mathbb{R}$, we obtain the following decay rate.

Lemma 3.4. *It holds that*

$$\|v(t)\|^2_{H^1} \leq \|v(t)\|^2_{H^1_w} \leq C\mathrm{e}^{-\mu t}, \quad t > s_0,$$

where $C > 0$.

Using Sobolev embedding theorem $H^1(\mathbb{R}) \hookrightarrow C^0(\mathbb{R})$ and Lemma 3.4, we get the desire result. This completes the proof of Theorem 1.1. □

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