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ON THE QUALITATIVE BEHAVIOURS OF A SECOND ORDER QUADRATIC RATIONAL DIFFERENCE EQUATION

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Abstract

The aim of this article is to give a clear analyze of local and global stability and periodicity of the solutions of the following quadratic rational difference equation:

$$U_{n+1} = aU_n^2 + \frac{bU_n^2 + cU_nU_{n-1} + dU_{n-1}^2}{\alpha U_n + \beta U_{n-1}}, \quad n = 0, 1, 2, \ldots,$$

where the initial conditions U_{-1} and U_0 are nonzero real numbers and the coefficients a, b, c, d, α and β are positive. Also, we confirm our theoretical results numerically by using MATLAB software.

1. Introduction

This research study objectives to focus on the stability, periodicity and other behaviours of the following rational difference equation:

$$U_{n+1} = aU_n^2 + \frac{bU_n^2 + cU_nU_{n-1} + dU_{n-1}^2}{\alpha U_n + \beta U_{n-1}}, \quad n = 0, 1, 2, \ldots, \tag{1}$$

where U_{-1} and U_0 are real numbers and the coefficients a, b, c, d, α and β are positive.

The qualitative study of nonlinear rational difference equations has been constantly growing for the latest years. In fact, no one can deny that these equations play a great role in modelling a huge numbers of real life phenomena such as in engineering, biology, physics, etc. Recently, there are many scholars and researchers have focused on describing these kind of equations such as on their stability, periodicity and boundedness character which need to be investigated in depth. Moreover, a big number of papers has been published in this area. For instance, in [1], Elabbasy et al. explored the periodicity and stability of the following fractional difference equation:

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1} + \gamma x_{n-2}}{Ax_n + Bx_{n-1} + Cx_{n-2}}.$$

Almatrafi et al. ([2]) investigated the behaviours of the following rational difference equation:

$$x_{n+1} = ax_n + \frac{bx_n^2 + cx_n x_{n-1} + dx_{n-1}^2}{\alpha x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}^2}.$$

In [3], authors studied the stability character and obtained the periodicity of the solution of

$$x_{n+1} = \frac{Ax_n^2 + Bx_n x_{n-1} + Cx_{n-1}^2}{\alpha x_n^2 + bx_n x_{n-1} + cx_{n-1}^2}.$$

The properties of the following recursive equation:

$$Y_{n+1} = AY_{n-1} + \frac{BY_{n-1}Y_{n-3}}{CY_{n-3} + DY_{n-5}},$$

were described by Alayachi et al. [4].

Bektesevic et al. ([5]) highlighted on explaining three cases of the following fractional difference equation:

$$x_{n+1} = \frac{\alpha x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}}{Ax_n^2 + Bx_n x_{n-1} + Cx_{n-1}}.$$

In [6], they analysed some behaviours of the following equation:

$$x_{n+1} = \frac{ax_{n-1} + bx_{n-2}}{c + dx_{n-1}x_{n-2}}.$$

The stability and periodicity of the following equation:

$$x_{n+1} = ax_{n-1} + \frac{bx_{n-1}}{cx_{n-1} - dx_{n-3}},$$

have been considered in [7].

Almatrafi et al. ([8]) explored some results of

$$x_{n+1} = ax_{n-1} - \frac{bx_{n-1}}{cx_{n-1} - dx_{n-3}}.$$

Amleh in [9] investigated some special cases of

$$x_{n+1} = \frac{(\alpha x_n + \beta x_n x_{n-1} + \gamma x_{n-1})x_n}{Ax_n + Bx_n x_{n-1} + Cx_{n-1}}.$$

The dynamical behaviours of the following difference equation:

$$U_{n+1} = \zeta U_{n-8} + \frac{\epsilon U_{n-8}^2}{\mu U_{n-8} + kU_{n-17}},$$

has considered by Alshareef et al. in [10].

Also, in [11], they studied the stability, periodicity and others of the following equation:

$$x_{n+1} = ax_n + bx_{n-1} + \frac{c + dx_{n-2}}{e + fx_{n-2}}.$$

For more related papers on this scope can be seen in [12-29].

2. Local Stability of Equation (1)

The local stability of Equation (1) is pointedly discussed in this part. Firstly, the equilibrium point of Equation (1) is given by

$$\overline{U} = a\overline{U}^2 + \frac{b\overline{U}^2 + c\overline{U}^2 + d\overline{U}^2}{\alpha\overline{\overline{U}} + \beta\overline{\overline{U}}},$$

or

$$\overline{U} = a\overline{U}^2 + \frac{(b + c + d)\overline{U}^2}{(\alpha + \beta)\overline{\overline{U}}},$$

thus

$$(\alpha + \beta)\overline{U}^2 = a(\alpha + \beta)\overline{U}^3 + (b + c + d)\overline{U}^2,$$

which gives that

$$a(\alpha + \beta)\overline{U} = (\alpha + \beta) - (b + c + d).$$

Then the unique equilibrium point is

$$\overline{U} = \frac{(\alpha + \beta) - (b + c + d)}{a(\alpha + \beta)}.$$

Let $f : (0, \infty)^2 \longrightarrow (0, \infty)$ be a function defined as

$$f(x, y) = ax^2 + \frac{bx^2 + cxy + dy^2}{\alpha x + \beta y}. \tag{2}$$

Now, we obtain partial derivatives of $f(x, y)$:

$$\frac{\partial f(x, y)}{\partial x} = 2ax + \frac{(\alpha x + \beta y)(2bx + cy) - \alpha(bx^2 + cxy + dy^2)}{(\alpha x + \beta y)^2},$$

$$\frac{\partial f(x, y)}{\partial y} = \frac{(\alpha x + \beta y)(2x + 2dy) - \beta(bx^2 + cxy + dy^2)}{(\alpha x + \beta y)^2}.$$

So, by evaluating these at $\overline{U}$ yields

$$\frac{\partial f\left(\overline{U}, \overline{U}\right)}{\partial x} = 2a\overline{U} + \frac{(\alpha + \beta)(2b + c)\overline{U}^2 - \alpha(b + c + d)\overline{U}^2}{(\alpha + \beta)^2\overline{U}^2}$$

$$= \frac{2(\alpha + \beta) - 2(b + c + d)}{\alpha + \beta} + \frac{(\alpha + \beta)(2b + c) - \alpha(b + c + d)}{(\alpha + \beta)^2}$$

$$= \frac{2(\alpha + \beta)^2 - 2(b + c + d)(\alpha + \beta) + (\alpha + \beta)(2b + c) - \alpha(b + c + d)}{(\alpha + \beta)^2}$$

$$= \frac{2(\alpha + \beta)^2 - (\alpha + \beta)(c + 2d) - \alpha(b + c + d)}{(\alpha + \beta)^2}$$

$$= 2 - \frac{(\alpha + \beta)(c + 2d) + \alpha(b + c + d)}{(\alpha + \beta)^2} = p_1,$$

$$\frac{\partial f(\overline{U}, \overline{U})}{\partial y} = \frac{(\alpha + \beta)(c + 2d)\overline{U}^2 - \beta(b + c + d)\overline{U}^2}{(\alpha + \beta)^2 \overline{U}^2}$$

$$= \frac{(\alpha + \beta)(c + 2d) - \beta(b + c + d)}{(\alpha + \beta)^2} = p_2.$$

The linearized Equation (1) about $\overline{U}$ we have

$$z_{n+1} - p_1 z_n - p_2 z_{n-2} = 0.$$

Theorem 2.1. *The equilibrium point of Equation* (1) *is locally asymptotically stable if*

$$(\alpha + \beta) < (b + c + d).$$

Proof. The sufficient condition for the asymptotic stability of the difference equation is

$$\sum_{i=1}^{k} |p_i| < 1.$$

Which means here, Equation (1) is asymptotically stable if

$$|p_1| + |p_2| < 1.$$

That is,

$$\left|2 - \frac{(\alpha + \beta)(c + 2d) + \alpha(b + c + d)}{(\alpha + \beta)^2}\right| + \left|\frac{(\alpha + \beta)(c + 2d) - \beta(b + c + d)}{(\alpha + \beta)^2}\right| < 1,$$

and so,

$$|2(\alpha + \beta)^2 - (\alpha + \beta)(c + 2d) - \alpha(b + c + d) + (\alpha + \beta)(c + 2d)$$

$$- \beta(b + c + d)| < (\alpha + \beta)^2.$$

Thus,

$$\left|(\alpha + \beta)^2\right| < (\alpha + \beta)(b + c + d).$$

Hence,

$$(\alpha + \beta) < (b + c + d).$$

This complete the required.

3. Periodicity of the Solution

The objective here is to examine the periodic solutions of Equation (1). The next theorem will confirm that Equation (1) has a periodic solutions of period two under needful condition.

Theorem 3.1. *Equation* (1) *has a periodic two solution if*

$$\frac{(\beta - d) + (\alpha - c)n - bn^2}{an(\alpha n + \beta)} = \frac{n((\beta - d)n^2 + (\alpha - c)n - b)}{a(\alpha + \beta n)},$$

$$n \neq 0, \pm 1, n \in \mathbb{R}.$$

Proof. We presume that there occurs a period two solution

$$\ldots, h, k, h, k, \ldots,$$

of Equation (1). It can be notice from Equation (1) that

$$h = ak^2 + \frac{bk^2 + chk + dh^2}{\alpha k + \beta h},$$

$$k = ah^2 + \frac{bh^2 + chk + dk^2}{\alpha h + \beta k}.$$

Therefore

$$\alpha hk + \beta h^2 = a\alpha k^2 + a\beta k^2 h + bk^2 + chk + dh^2, \tag{3}$$

$$\alpha hk + \beta k^2 = a\alpha h^3 + a\beta h^2 k + bh^2 + chk + dk^2. \tag{4}$$

Dividing Equation (3) by h^2 and Equation (4) by k^2 gives

$$\alpha\left(\frac{k}{h}\right) + \beta = a\alpha\left(\frac{k^3}{h^2}\right) + a\beta\left(\frac{k^2}{h}\right) + b\left(\frac{k}{h}\right)^2 + c\left(\frac{k}{h}\right) + d, \tag{5}$$

$$\alpha\left(\frac{h}{k}\right) + \beta = a\alpha\left(\frac{h^3}{k^2}\right) + a\beta\left(\frac{h^2}{k}\right) + b\left(\frac{h}{k}\right)^2 + c\left(\frac{h}{k}\right) + d. \tag{6}$$

Now, let $h = nk$, $n \neq 0, \pm 1$, $n \in \mathbb{R}$. Then from Equations (5) and (6), we have

$$\frac{\alpha}{n} + \beta = \frac{a\alpha}{n^2} k + \frac{a\beta}{n} k + \frac{b}{n^2} + \frac{c}{n} + d, \tag{7}$$

$$an + \beta = a\alpha n^2 h + a\beta nh + bn^2 + cn + d. \tag{8}$$

It follows from Equation (8) that

$$an\,(\alpha n + \beta)h = (\beta - d) + (\alpha - c)n - bn^2,$$

which yields

$$h = \frac{(\beta - d) + (\alpha - c)n - bn^2}{an(\alpha n + \beta)}. \tag{9}$$

Multiplying Equation (7) by n^2 leads to

$$\alpha n + \beta n^2 = a\alpha k + a\beta nk + b + cn + dn^2,$$

then

$$a(\alpha + \beta n)k = (\beta - d)n^2 + (\alpha - c)n - b.$$

Which gives

$$k = \frac{(\beta - d)n^2 + (\alpha - c)n - b}{a(\alpha + \beta n)}. \tag{10}$$

Since $h = nk$, easily from Equations (9) and (10) get that

$$\frac{(\beta - d) + (\alpha - c)n - bn^2}{an(\alpha n + \beta)} = \frac{n((\beta - d)n^2 + (\alpha - c)n - b)}{a(\alpha + \beta n)}.$$

This completes the proof.

4. Global Attractivity Results

Our task of this part is to explore and determine the global stability of Equation (1) under needful conditions. Here, the investigation of global stability will be presented in four cases in order to be increasing or decreasing function $f(x, y)$ in x and y.

Case (1):

Theorem 4.1. *Let* $f(x, y)$ *be increasing in* x *and* y. *Then the equilibrium point* $\overline{U}$ *of Equation* (1) *is a global attractor if* $a(\alpha + \beta) \neq 0$.

Proof. Assume that $f(x, y)$ is increasing in x and y and let (t, T) be a solution of the following system:

$$t = f(t, t) = at^2 + \frac{bt^2 + ct^2 + dt^2}{\alpha t + \beta t},$$

$$T = f(T, T) = aT^2 + \frac{bT^2 + cT^2 + dT^2}{\alpha T + \beta T}.$$

Or

$$(\alpha + \beta)t^2 = a(\alpha + \beta)t^3 + (b + c + d)t^2,$$

$$(\alpha + \beta)T^2 = a(\alpha + \beta)T^3 + (b + c + d)T^2.$$

Thus

$$(\alpha + \beta) = a(\alpha + \beta)t + (b + c + d), \tag{11}$$

$$(\alpha + \beta) = a(\alpha + \beta)T + (b + c + d). \tag{12}$$

Subtracting Equation (12) from Equation (11) leads to

$$a(\alpha + \beta)(t - T) = 0.$$

If $a(\alpha + \beta) \neq 0,$ then we have

$$T = t.$$

Then, the equilibrium point of Equation (1) is a global attractor.

Case (2):

Theorem 4.2. *Let* $f(x, y)$ *be decreasing in* x *and* y*. Then the equilibrium point* $\overline{U}$ *of Equation* (1) *is a global attractor if* $a(\alpha + \beta) \neq 0.$

Proof. Suppose that $f(x, y)$ is decreasing in x and y and let (t, T) be a solution of the following system:

$$t = f(t, t) = at^2 + \frac{bt^2 + ct^2 + dt^2}{\alpha t + \beta t},$$

$$T = f(T, T) = aT^2 + \frac{bT^2 + cT^2 + dT^2}{\alpha T + \beta T}.$$

Or

$$(\alpha + \beta)t^2 = a(\alpha + \beta)t^3 + (b + c + d)t^2,$$

$$(\alpha + \beta)T^2 = a(\alpha + \beta)T^3 + (b + c + d)T^2.$$

Thus

$$(\alpha + \beta) = a(\alpha + \beta)t + (b + c + d), \tag{13}$$

$$(\alpha + \beta) = a(\alpha + \beta)T + (b + c + d). \tag{14}$$

Subtracting Equation (14) from Equation (13) implies that

$$a(\alpha + \beta)(t - T) = 0, \quad a(\alpha + \beta) \neq 0.$$

Thus

$$T = t.$$

It gives that $\overline{U}$ is a global attractor of Equation (1).

Case (3):

Theorem 4.3. *Let $f(x, y)$ be increasing in x and decreasing in y. Then the equilibrium point $\overline{U}$ of Equation* (1) *is a global attractor if $b < \alpha + d$, $a\alpha < 0$ and $a(\alpha + \beta) < 0$.*

Proof. Suppose that $f(x, y)$ is increasing in x and decreasing in y and let (t, T) be a solution of the following system:

$$t = f(t, T) = at^2 + \frac{bt^2 + ctT + dT^2}{\alpha t + \beta T},$$

$$T = f(T, t) = aT^2 + \frac{bT^2 + cTt + dt^2}{\alpha T + \beta t}.$$

Or

$$\alpha t^2 + \beta tT = a\alpha t^3 + a\beta t^2 T + bt^2 + ctT + dT^2, \tag{15}$$

$$\alpha T^2 + \beta Tt = a\alpha T^3 + a\beta T^2 t + bT^2 + cTt + dt^2. \tag{16}$$

Subtracting Equation (16) from Equation (15) gives that

$$d(t^2 - T^2) = a\alpha(t^3 - T^3) + a\beta(t^2 T - T^2 t) + b(t^2 - T^2) + d(T^2 - t^2),$$

which implies that

$$(t - T)[(\alpha + d - b)(t + T) - a\alpha(t^2 + T^2) - a(\alpha + \beta)tT] = 0.$$

Hence, if $b < \alpha + d$, $a\alpha < 0$ and $a(\alpha + \beta) < 0$, then we have

$$T = t.$$

Therefore, the equilibrium point of Equation (1) is a global attractor.

Case (4):

Theorem 4.4. *Let* $f(x, y)$ *be decreasing in* x *and increasing in* y. *Then the equilibrium point* $\overline{U}$ *of Equation* (1) *is a global attractor if* $d < \beta + b$.

Proof. Suppose that $f(x, y)$ is decreasing in x and increasing in y and let (t, T) be a solution of the following system:

$$t = f(T, t) = aT^2 + \frac{bT^2 + cTt + dt^2}{\alpha T + \beta t},$$

$$T = f(t, T) = at^2 + \frac{bt^2 + ctT + dT^2}{\alpha t + \beta T}.$$

Or

$$\alpha tT + \beta t^2 = a\alpha T^3 + a\beta T^2 t + bT^2 + cTt + dt^2, \tag{17}$$

$$\alpha tT + \beta T^2 = a\alpha t^3 + a\beta t^2 T + bt^2 + ctT + dT^2. \tag{18}$$

Subtracting Equation (18) from Equation (17) gives that

$$\beta(t^2 - T^2) = a\alpha(T^3 - t^3) + a\beta(T^2 t - t^2 T) + b(T^2 - t^2) + d(t^2 - T^2),$$

which implies that

$$(t - T)[(\beta + b - d)(t + T) + a\alpha(t^2 + T^2) + a(\alpha + \beta)tT] = 0.$$

Hence, if $d < \beta + b$, gives

$$T = t.$$

Thus, $\overline{U}$ is a global attractor of Equation (1).

5. Numerical Solutions

This part will give some numerical examples to verify our results in this article which provide different types of solutions as well as periodic of solutions of Equation (1).

Example 5.1. Let $U_{-1} = -4$, $U_0 = 5$, $a = 0.05$, $b = 0.8$, $c = 0.5$, $d = 0.5$, $\alpha = 0.2$, and $\beta = 1.25$. Then this example demonstrates the local stability behaviour of Equation (1).

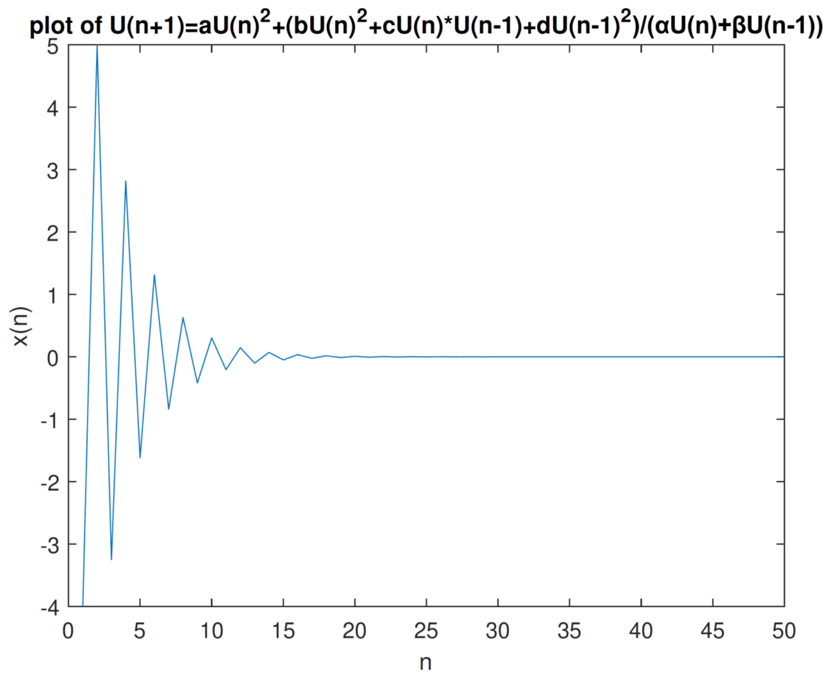

Figure 1. Local stability of Equation (1).

Example 5.2. Suppose that $U_{-1} = -3$, $U_0 = 6$, $a = 0.05$, $b = -0.3$, $c = -0.9$, $d = -0.5$, $\alpha = 10$, and $\beta = 1$. This figure gives the behaviour of the solution of Equation (1).

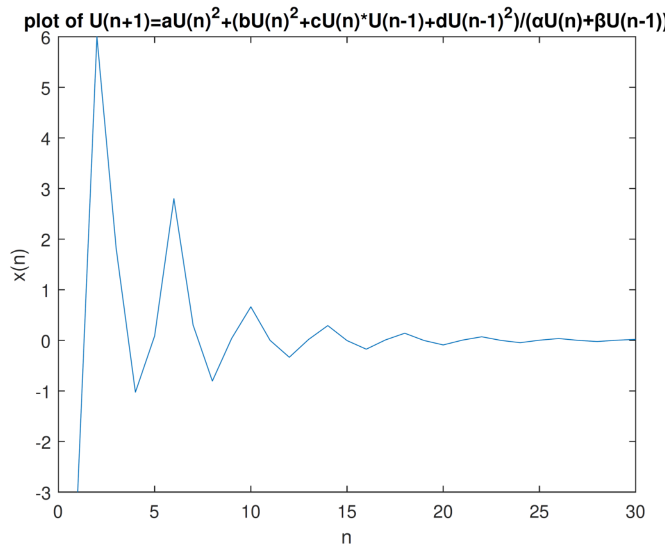

Figure 2. Solution behaviour of Equation (1).

Example 5.3. This example illustrates the behaviour of our problem under $U_{-1} = -10$, $U_0 = 10$, $a = 0.01$, $b = 0.6$, $c = 3$, $d = -0.1$, $\alpha = 5$, and $\beta = 0.01$.

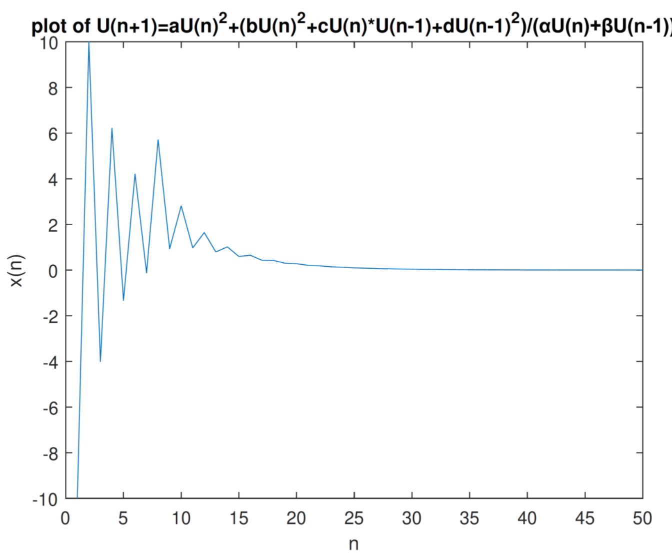

Figure 3. Dynamics of Equation (1).

Example 5.4. Figure 4. shows the stability of the solution of Equation (1) when we assume that $U_{-1} = -3$, $U_0 = 3$, $a = 0.01$, $b = 0.5$, $c = 4$, $d = -0.1$, $\alpha = 5$, and $\beta = 0.01$.

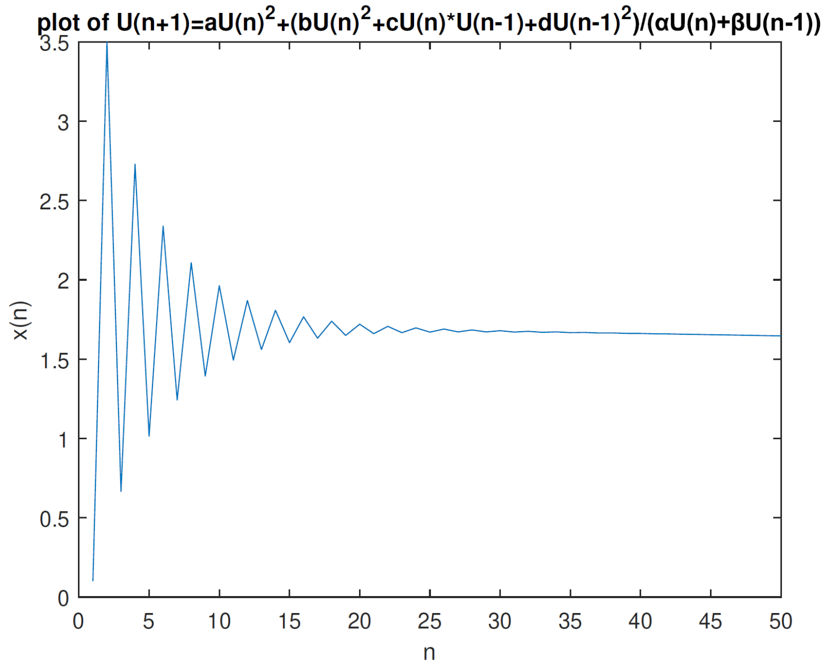

Figure 4. Stability of Equation (1) under $U_{-1} = -3$, $U_0 = 3$, $a = 0.01$, $b = 0.5$, $c = 4$, $d = -0.1$, $\alpha = 5$, and $\beta = 0.01$.

Example 5.5. Here, we present the plot of the solution when we have these values $U_{-1} = 0.1$, $U_0 = 3.5$, $a = 0.01$, $b = 1.3$, $c = 9$, $d = 0.5$, $\alpha = 10$, and $\beta = 1$.

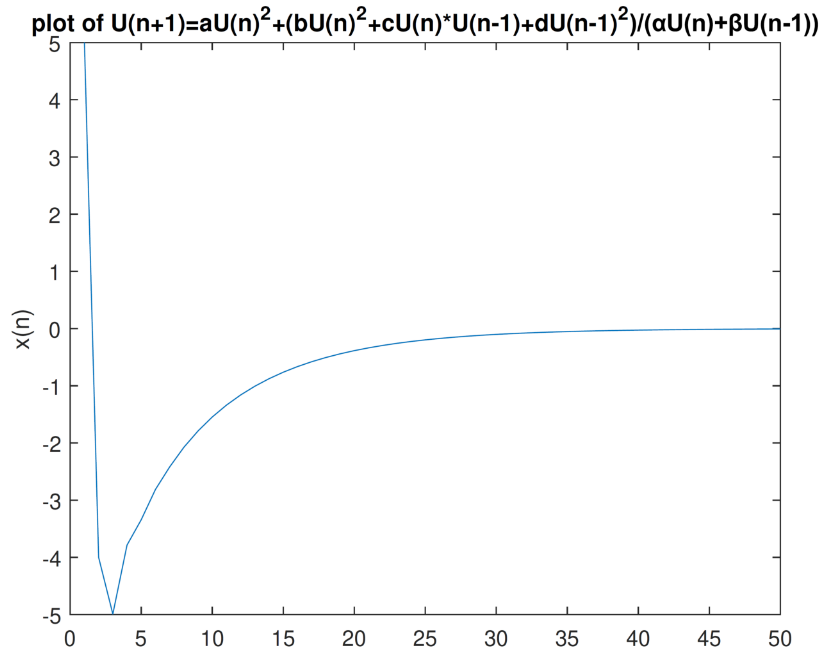

Figure 5. Plot of Equation (1).

Example 5.6. This example confirm the periodicity of the solution of Equation (1) when $a = 0.01 = b$, $c = 0.5$, $d = 1.5$, $\alpha = 15$, $\beta = 1$, $U_{-1} = q = 0.1$, and $U_0 = p = 3.2$.

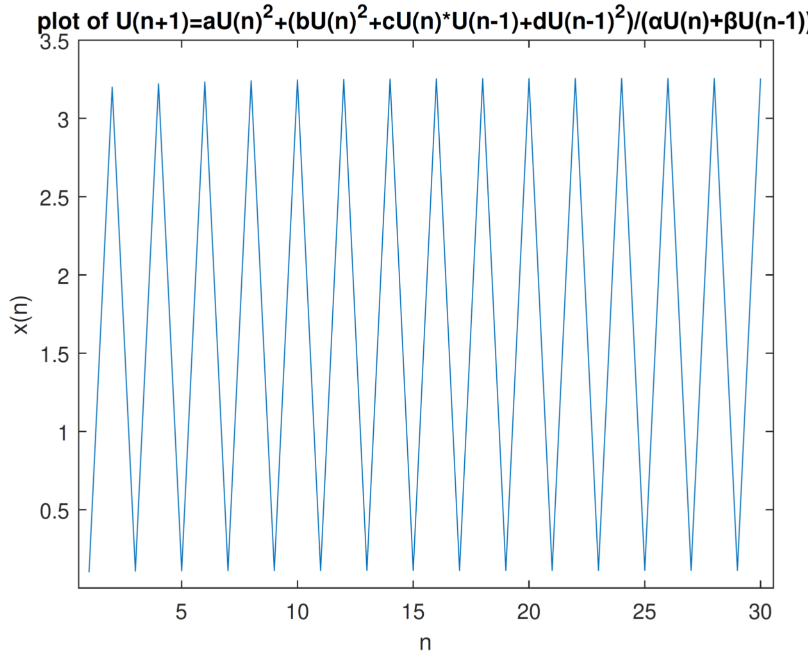

Figure 6. Periodicity of Equation (1).

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