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A NEW TWO-DIMENSIONAL RELATION COPULA INSPIRING A GENERALIZED VERSION OF THE FARLIE-GUMBEL-MORGENSTERN COPULA

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Abstract

Copulas are useful mathematical tools in probabilistic modelling and simulation; they allow the construction or understanding of various dependence structures in random processes. As a result, constructing various types of copulas is quite important. In this paper, we first present and investigate a new copula based on a simple but original ratio function. Among the properties, we show that it does not belong to the Archimedean family, it is symmetric, it is not radially symmetric, it has interesting quadrant dependence properties, it has the tail independence property, and it satisfies comprehensive concordance properties with the famous Farlie-Gumbel-Morgenstern copula. Secondly, in order to make it more flexible, we propose and develop some parametric generalizations of this copula. One generalization is of particular interest; it can be considered as a new generalized version of the Farlie-Gumbel-Morgenstern copula involving a weighted ratio function. Some graphical illustrations illustrate the findings.

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1. Introduction

In recent years, the literature has paid a lot of attention to random variable dependence features. In particular, an elegant way of determining or identifying the dependence structure of several random variables is proposed by the concept of copula. On the mathematical aspect, an n-dimensional copula is defined as a multivariate distribution function supported into $[0, 1]^n$, with uniform univariate marginals on $[0, 1]$. There are a few different methods for building copulas, including inversion, geometric, and algebraic methods. Also, the elliptical, Archimedean, and extreme value families of copulas can be used to classify the majority of copulas. The elliptical family includes Gaussian and Student copulas. The Archimedean family includes Ali-Mikhail-Haq, Clayton, Gumbel, and Frank copulas, whereas the extreme value family includes Galambos, Hüsler-Reiss, and Student-EV. Certain copulas, such as the famous Farlie-Gumbel-Morgenstern (FGM) copula, are unrelated to these families. On this subject, we may redirect the reader to [16], [24], [9], [5], [3], [1], [2], [23], and [4]. Concrete applications of copulas may be found in [11], [10], [13], [25], [22], and [19].

According to recent literature, it is still of interest to construct copulas that are basic in nature so that we may determine the underlying functions of interest and various correlation measures. As a matter of fact, a wide range of applications involving various sorts of data need the creation of new copulas. This demand motivated this study. We look into two-dimensional copulas of the ratio-type that do not appear to have garnered much attention in the literature. They are built on the sum, product, and ratio of simple functions that are coupled to provide a unique dependence structure. To be more specific, the following basic two-dimensional function is at the basis of our current research:

$$S(u, v) = uv\frac{(2-u)(2-v)}{2-uv}, \quad (u, v) \in [0, 1]^2. \tag{1}$$

We motivate the interest in this special function by the following arguments that will be proved later: (i) $S(u, v)$ is a new valid two-dimentional copula, called the new ratio (NR) copula; (ii) it has an original structural dependency, and it is not a member of the Archimedean family in particular; (iii) it is symmetric; (iv) it is not radically symmetric; (v) it possesses the positive quadrant dependence (PQD) property; (vi) it has manageable dependence measures; (vii) it has the tail independence property; and (viii) it has interesting concordance relationships with the FGM copula. Motivated by the simple structure of $S(u, v)$, we derive three of its extensions or generalizations depending on one or two tuning parameters. These parametric generalizations are of interest because they offer more flexibility in the dependence structure of the NR copula. One of the proposed extensions has the feature of being a generalized version of the FGM copula. The determination of original extensions of the FGM copula is also an active branch of research in probability and statistics. See [9], [3], [1], and [2], among others. Thus, the proposed FGM extension can be viewed as a contribution to this branch. As a last remark, graphics are used to demonstrate various crucial characteristics of the introduced copulas.

We organized the rest of the paper as follows. Section 2 recalls some basic concepts in copula theory with a brief retrospective on the FGM copula, then studies the NR copula and some of its features. Generalizations of this copula are the object of Section 3. In Section 4, there is a conclusion as well as some future work perspectives.

2. A New Two-Dimensional Copula

2.1. Mathematical foundation

Before studying the copula candidate in Equation (1), the necessary mathematical foundation on the concept of copula is now recalled. A two-dimensional copula is any function $C(u, v)$, $(u, v) \in [0, 1]^2$, that has the following properties: for any $(u, v) \in [0, 1]^2$, $C(u, 0) = C(0, v) = 0$, $C(u, 1) = u$, $C(1, v) = v$, and for any $(u_1, u_2, v_1, v_2) \in \{(u_1, u_2, v_1, v_2) \in [0, 1]^4;$ $u_1 \leq u_2, v_1 \leq v_2\}$, we have

$$C(u_2, v_2) - C(u_2, v_1) + C(u_1, v_1) - C(u_1, v_2) \geq 0.$$

This inequality is referred to as two-increasing property. If $C(u, v)$ is differentiable with respect to u and v, the two-increasing property is equivalent to $\partial^2 C(u, v)/(\partial u \partial v) \geq 0$. According to the two-dimensional version of the main result in [21], any joint cumulative distribution function $F_{(X, Y)}(x, y)$, $(x, y) \in \mathbb{R}^2$, of a continuous random vector (X, Y) may be expressed as

$$F_{(X, Y)}(x, y) = C(F_X(x), F_Y(y)),$$

where $F_X(x)$ and $F_Y(y)$ denote the cumulative distribution functions of X and Y, respectively, and $C(u, v)$ denotes the related copula. The following concepts and properties are related to copulas.

- The copula $C(u, v)$ is said to be symmetric if and only if $C(u, v) = C(v, u)$ for any $(u, v) \in [0, 1]^2$.

- The reflected (or survival) copula associated with a copula $C(u, v)$ is given by $\widehat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v)$. The copula $C(u, v)$ is said to be radially symmetric if and only if $C(u, v) = \widehat{C}(u, v)$ for any $(u, v) \in [0, 1]^2$.

• The copula $C(u, v)$ satisfies the positive quadrant dependence (PQD) property if and only if $C(u, v) \geq uv$ for any $(u, v) \in [0, 1]^2$, and the negative quadrant dependence (NQD) property if and only if $C(u, v) \leq uv$ for any $(u, v) \in [0, 1]^2$.

• The copula $C(u, v)$ has the tail independence property if and only if the following limit results are satisfied:

$$\lim_{u \to 0} \frac{C(u, u)}{u} = 0, \quad \lim_{u \to 1} \frac{1 - 2u + C(u, u)}{1 - u} = 0.$$

• The dependence, in the correlation sense, associated with a copula $C(u, v)$ can be evaluated via the medial correlation coefficient or Spearman measure defined by

$$M = 4C\left(\frac{1}{2}, \frac{1}{2}\right) - 1, \quad \rho = 12 \int_0^1 \int_0^1 C(u, v) du dv - 3,$$

respectively.

We may refer to [16] for all the details behind these concepts and properties. Let us consider the FGM copula as an inevitable example of a copula. To begin, it is defined as the following two-dimensional function:

$$C_*(u, v) = uv[1 + \beta(1 - u)(1 - v)], \quad (u, v) \in [0, 1]^2, \tag{2}$$

with $\beta \in [-1, 1]$. It was first studied by [8], [12] and [14]. Its interest remains in the following properties: (i) it is a very simple one-parameter copula; (ii) it is singular among the main families of copulas, and it is not a member of the Archimedean family in particular; (iii) it is symmetric; (iv) it is not radially symmetric except for $\beta = 0$; (v) it has various quadrant dependence properties depending on the sign of β; it has the PQD property for $\beta \in (0, 1]$, the independence property for $\beta = 0$, and the NQD property for $\beta \in [-1, 0)$; (vi) it allows the modelling of any random vector (X, Y) with moderate correlation, since its Spearman measure is equal to $\beta/3$; and (vii) it has the tail independence property. The use of the FGM copula in modern applied fields is still active. It is involved in the dependence of components to assess the reliability of a random system (see [7] and [15]), it is crucial in statistical modelling in a variety of domains, including economics (see [18]), educational engineering (see [20]), and finance ([6]), the estimation of its parameters is now well established (see [17]), and it has inspired some useful extensions (see [9], [3], [1], and [2]). As mentioned in the introductive section, in this study, attempts will be made to connect the NR and FGM copulas. In particular, a new generalized FGM copula is proposed.

2.2. Primary research on the NR copula

The NR copula as presented in Equation (1) is a valid copula, as formulated in the next result.

Proposition 2.1. *The two-dimensional function* $S(u, v)$ *defined in Equation* (1) *is a valid copula.*

Proof. As a first condition, since uv is a factor term, we have $S(u, 0) = S(0, v) = 0.$ Furthermore, we have

$$S(u, 1) = u \times 1 \times \frac{(2-u)(2-1)}{2-u \times 1} = u\,\frac{2-u}{2-u} = u.$$

With a similar approach, we get $S(1, v) = v.$ Since $(u, v) \in [0, 1]^2,$ it is clear that $(2 - uv)^3 \geq 1 > 0.$ Therefore, the function $S(u, v)$ is differentiable with respect to u and v. By the use of standard derivative formulas and simplifications, we obtain

$$\frac{\partial^2 S(u, v)}{\partial u \partial v} = \frac{u^3v^3 - 6u^2v^2 + 24uv - 16(u + v - 1)}{(2 - uv)^3}.$$

Moreover, since $u + v - uv = u(1 - v) + v \leq 1 - v + v = 1$, we have $u + v - 1 \leq uv$. Therefore,

$$u^3v^3 - 6u^2v^2 + 24uv - 16(u + v - 1) \geq u^3v^3 - 6u^2v^2 + 24uv - 16uv$$

$$= u^3v^3 - 6u^2v^2 + 8uv = uv(4 - uv)(2 - uv) \geq 0.$$

Hence $\partial^2 S(u, v)/(\partial u \partial v) \geq 0$, proving the two-increasing property. This achieves the proof. □

As a direct remark, the NR copula has the following alternative expression:

$$S(u, v) = uv\left[1 + \frac{1}{1 - uv/2}(1 - u)(1 - v)\right]. \tag{3}$$

We can note a similiture in form with the FGM copula. This point will be discussed later. The NR copula density is given by

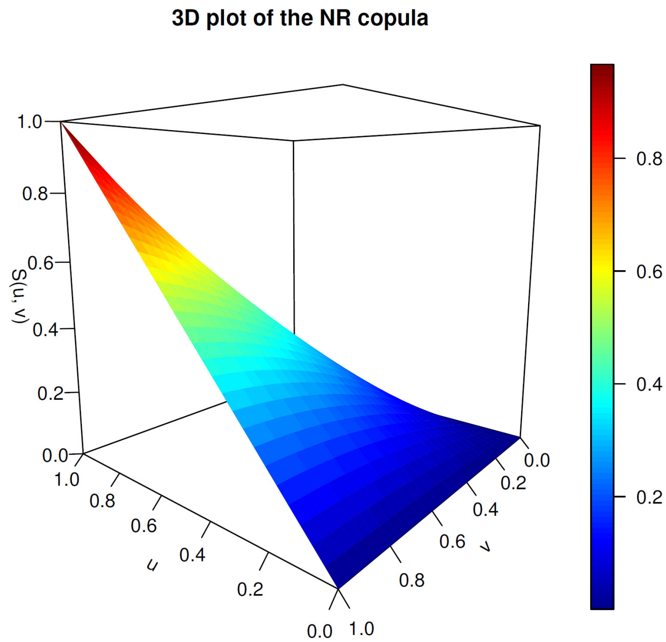
$$s(u, v) = \frac{\partial^2 S(u, v)}{\partial u \partial v} = \frac{u^3v^3 - 6u^2v^2 + 24uv - 16(u + v - 1)}{(2 - uv)^3},$$

$$(u, v) \in [0, 1]^2.$$

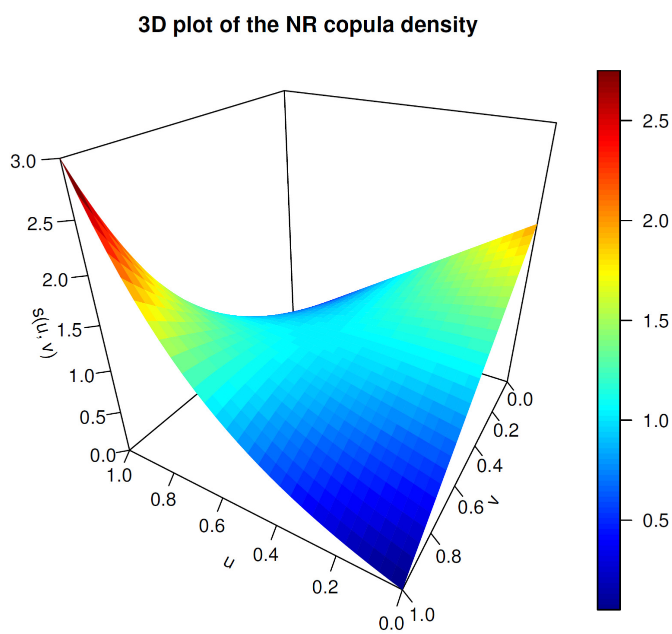
As a last remark, the reflected NR copula is specified by

$$\hat{S}(u, v) = u + v - 1 + S(1 - u, 1 - v) = uv\frac{uv - u - v + 3}{u + v + 1 - uv}, \quad (u, v) \in [0, 1]^2. \tag{4}$$

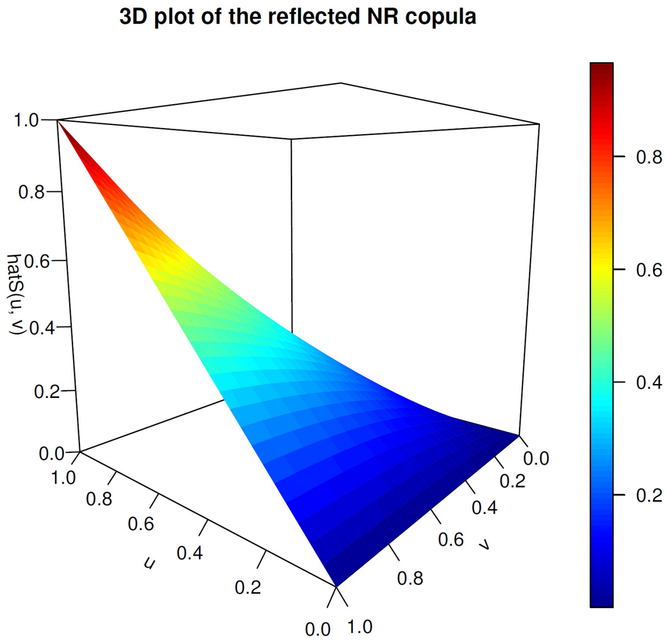
We use a visual method to finish this presentation. Figure 1 shows the NR copula, NR copula density, and reflected NR copula as three-dimensional plots. Figure 2 depicts the corresponding contour plots.


3D plot of the NR copula
S(u, v)
u
v

(a)

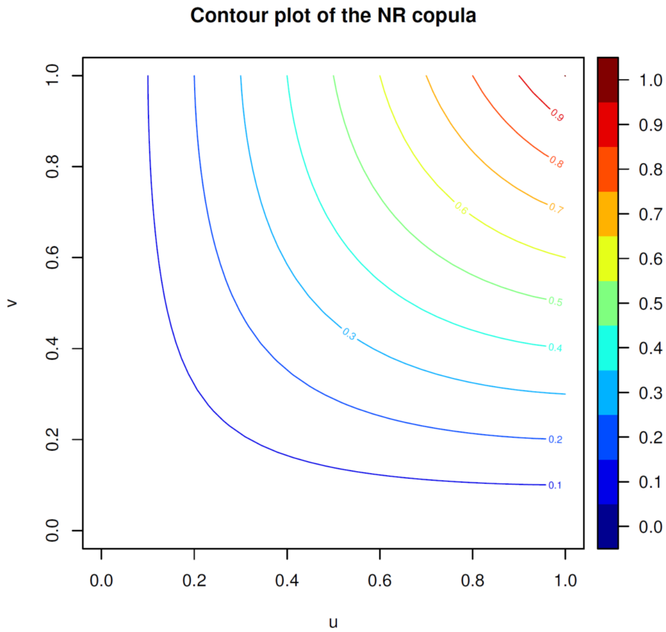

3D plot of the NR copula density
s(u, v)
u
v

(b)

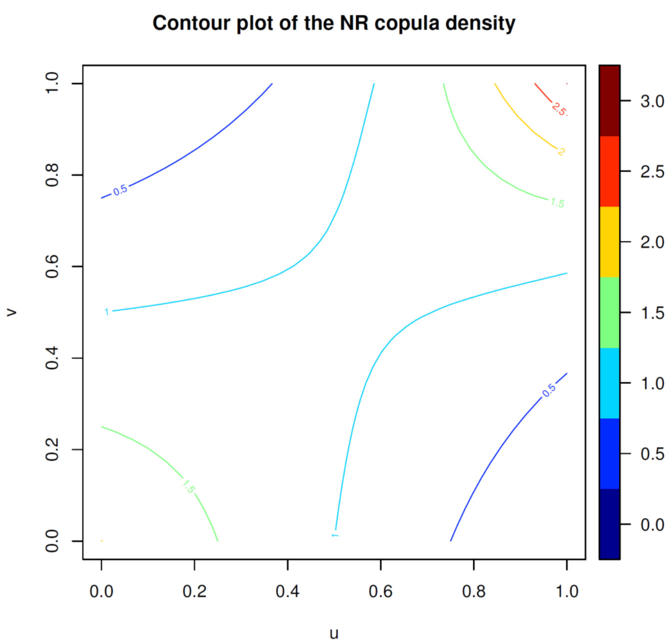


(c)

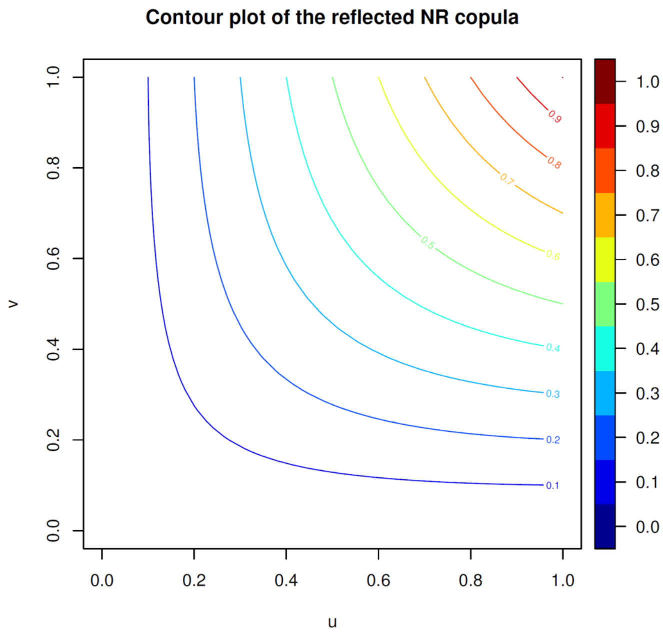
Figure 1. Three-dimensional plots of the (a) NR copula, (b) NR copula density, and (c) reflected NR copula.


Contour plot of the NR copula
v
u
0.0
0.2
0.4
0.6
0.8
1.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

(a)


Contour plot of the NR copula density
v
u
0.0
0.2
0.4
0.6
0.8
1.0
0.5
1.0
1.5
2.0
2.5
3.0

(b)



(c)

Figure 2. Contour plots of the (a) NR copula, (b) NR copula density, and (c) reflected NR copula.

It can be seen in Figures 1 and 2 the expected triangular form and contour plot for the NR copula. We notice that the NR copula density is a bit singular; it has relatively low in values, exepted at the neighbourhood of the extremal points (0, 1) and (1, 0).

2.3. Properties

The remainder of this section is devoted to a detailed examination of several elements of the NR copula.

Proposition 2.2. *The following list of properties holds*:

(1) *The NR copula does not belong to the Archimedean family*.

(2) *The NR copula is symmetric, but not radially symmetric*.

(3) *The NR copula has the PQD property*.

(4) *The NR copula has the tail independence property*.

(5) *Based on the FGM copula defined in Equation* (2), *the following concordance property holds*: $S(u, v) \geq C_*(u, v)$.

(6) *The NR copula has the following medial correlation coefficient and Spearman measure*: $M \approx 0.2857143$ *and* $\rho \approx 0.38476$, *respectively*.

Proof. Let us go over each item one by one.

(1) We proceed with an example of a non-associative case, we have

$$S\left(\frac{1}{5}, S\left(\frac{1}{2}, \frac{1}{3}\right)\right) \approx 0.07420719, \quad S\left(S\left(\frac{1}{5}, \frac{1}{2}\right), \frac{1}{3}\right) \approx 0.07511704.$$

Since the two above values are not equal, the NR copula is not associative; it can not belong to the Archimedean family (see [16]).

(2) For any $(u, v) \in [0, 1]^2$, it is clear that $S(u, v) = S(v, u)$ implying that the NR copula is symmetric. Based on Equation (4), it exists (u, v) such that $S(u, v) \neq \hat{S}(u, v)$, implying that the NR copula is not radially symmetric.

(3) For any $(u, v) \in [0, 1]^2$, one can notice that $(1-u)(1-v)/(1-uv/2) \geq 0$. By virtue of Equation (3), we have

$$S(u, v) = uv\left[1 + \frac{1}{1 - uv/2}(1-u)(1-v)\right] \geq uv.$$

Therefore, the NR copula has the PQD property.

(4) We have

$$\lim_{u \to 0} \frac{S(u, u)}{u} = \lim_{u \to 0} u\frac{(2-u)^2}{2-u^2} = 0,$$

and

$$\lim_{u \to 1} \frac{1 - 2u + S(u, u)}{1 - u} = \lim_{u \to 1}(1 - u)\frac{2 + u^2}{2 - u^2} = 0.$$

These convergences to 0 imply the tail independence property.

(5) For any $(u, v) \in [0, 1]^2$, $uv \geq 0$, $(1 - u)(1 - v) \geq 0$ and $1/(1 - uv/2) \geq 1 \geq \beta$. It follows from Equations (3) and (2) that

$$S(u, v) = uv\left[1 + \frac{1}{1 - uv/2}(1 - u)(1 - v)\right] \geq uv[1 + (1 - u)(1 - v)]$$

$$\geq uv[1 + \beta(1 - u)(1 - v)] = C_*(u, v).$$

The stated concordance property is proved.

(6) We have immediately

$$M = 4S\left(\frac{1}{2}, \frac{1}{2}\right) - 1 = \frac{2}{7} \approx 0.2857143.$$

For the Spearman measure, we recall that it is given by

$$\rho = 12\int_0^1 \int_0^1 S(u, v)dudv - 3.$$

Therefore

$$\int_0^1 \int_0^1 S(u, v)dudv$$

$$= \int_0^1 \left[\int_0^1 uv\frac{(2 - u)(2 - v)}{2 - uv}du\right]dv$$

$$= \frac{1}{2}\int_0^1 (2 - v)\frac{v(4 - 3v + \log(256)) + 8(1 - v)\log(2 - v) - 8\log(2)}{v^2}dv$$

$$= -\frac{49}{4} + \pi^2 - \frac{1}{2}\log(8)\log(16) + \log(256) \approx 0.282064.$$

Thus $\rho \approx 12 \times 0.282064 - 3 = 0.38476$.

The results of Proposition 2.2 are proved. □

The qualities discussed in Proposition 2.2 demonstrate the importance of the NR copula in both the mathematical and practical realms. This encourages the creation of additional copulas which generalize it in some way, which is the aim of the rest of the study.

3. Attempts of Generalization

Thus, the NR copula has some qualities, but it remains free of parameters, making it not very flexible in the structural dependency sense. In this section, we attempt to generalize it by introducing one or two parameters. The key to our generalization is the representation in Equation (3), aiming to provide a hybrid copula between the NR and FGM copulas. This hybrid copula may be expressed in the following general form:

$$S_H(u, v) = uv\left[1 + \frac{\beta}{1 - \alpha uv}(1 - u)(1 - v)\right], \quad (u, v) \in [0, 1]^2, \tag{5}$$

where the values of α and β need to be determined. In some senses, the validity of $S_H(u, v)$ in a copula sense is a “parameter game”, which is less easy to win than it seems to be at first glance.

3.1. First attempt

Based on Equations (3) and (5), our first attempt aims to generalize the NR copula first by considering the case of $\beta = 1$.

Proposition 3.1. *Let us consider the following two-dimensional function*:

$$\breve{S}(u, v) = uv\left[1 + \frac{1}{1 - \alpha uv}(1 - u)(1 - v)\right], \quad (u, v) \in [0, 1]^2. \tag{6}$$

Then, *for* $\alpha < 1$, $\breve{S}(u, v)$ *defines a valid copula*; *the NR copula corresponds to the case where* $\alpha = 1/2$.

Proof. It is clear that $\breve{S}(u, 0) = \breve{S}(0, v) = 0.$ Furthermore, we have

$$\breve{S}(u, 1) = u \times 1 \times \left[1 + \frac{1}{1 - \alpha u \times 1}(1 - u)(1 - 1)\right] = u,$$

and, similarly $\breve{S}(1, v) = v.$ Since $\alpha < 1,$ we have $1 - \alpha uv > 0,$ the function $\breve{S}(u, v)$ is differentiable with respect to u and v. Standard derivative formulas and simplifications yield

$$\frac{\partial^2 \breve{S}(u, v)}{\partial u \partial v} = \frac{(1 - \alpha)\alpha^2 u^3 v^3 - 3(1 - \alpha)\alpha u^2 v^2 + 2(2 - \alpha)uv - 2(u + v - 1)}{(1 - \alpha uv)^3}.$$

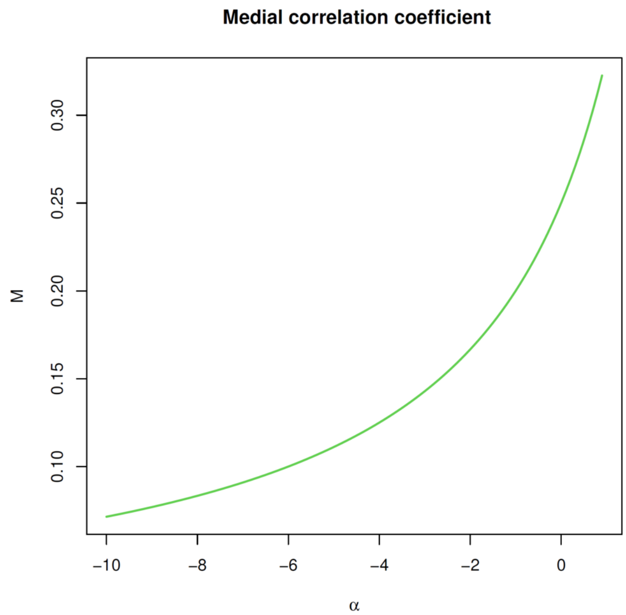
Since $(u, v) \in [0, 1]^2$ and $\alpha < 1,$ owing to $1 - \alpha uv > 0$ and $u + v - uv = u(1 - v) + v \leq 1 - v + v = 1,$ implying that $u + v - 1 \leq uv,$ we get

$$\begin{aligned}
&(1 - \alpha)\alpha^2 u^3 v^3 - 3(1 - \alpha)\alpha u^2 v^2 + 2(2 - \alpha)uv - 2(u + v - 1) \\
&\quad \geq (1 - \alpha)\alpha^2 u^3 v^3 - 3(1 - \alpha)\alpha u^2 v^2 + 2(2 - \alpha)uv - 2uv \\
&\quad = (1 - \alpha)\alpha^2 u^3 v^3 - 3(1 - \alpha)\alpha u^2 v^2 + 2(1 - \alpha)uv \\
&\quad = (1 - \alpha)uv(\alpha^2 u^2 v^2 - 3\alpha uv + 2) = (1 - \alpha)uv(1 - \alpha uv)(2 - \alpha uv) \geq 0.
\end{aligned}$$

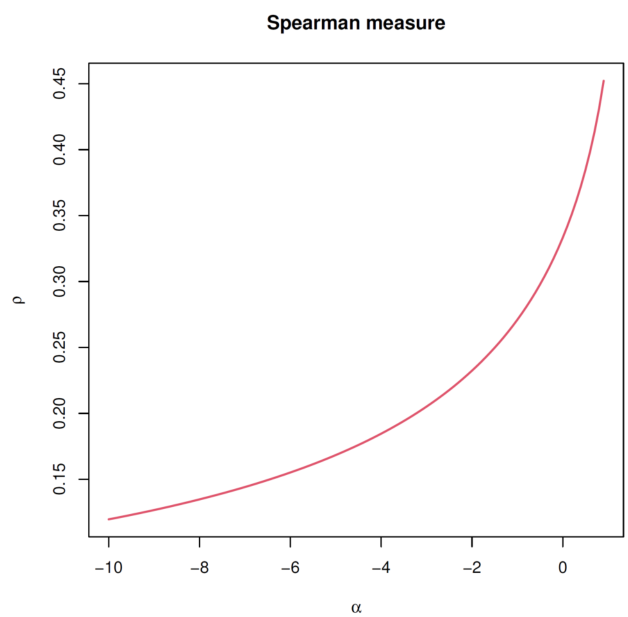
Hence $\partial^2 \breve{S}(u, v)/(\partial u \partial v) \geq 0,$ demonstrating the two-increasing property. By combining all the above properties, the fact that $\breve{S}(u, v)$ is a copula is proved. This ends the proof of Proposition 3.1. □

The copula presented in Equation (6) is naturally called the extended NR (ENR) copula. A list of its basic properties is briefly presented below. The ENR copula does not belong to the Archimedean family, it is symmetric, it is not radially symmetric, it has the PQD property and tends to the independence property by applying $\alpha \to -\infty,$ it has the tail independence property, and, for $\alpha \in [0, 1),$ it satisfies the following concordance property: $S(u, v) \geq C_*(u, v).$ For $\alpha < 0,$ if we define $C_*(u, v)$ with $\beta = 1,$ the reversed inequality holds, i.e., $S(u, v) \leq C_*(u, v).$

Some values of its medial correlation and coefficient and Spearman measure are depicted in Figure 3 for several values of $\alpha.$



(a)



(b)

Figure 3. Two-dimensional plots of the (a) medial correlation coefficient and (b) Spearman measure of the ENR copula with and $\alpha \in [-10, 0.9]$.

In particular, from Figure 3, we can note that the Spearman measure can be adjusted to 0 when α tends to $-\infty$ and to approximately 0.45 when α is in the neighbourhood of 1. This improves a bit the correlation capabilities of the FGM copula when $\beta \in [0, 1]$.

3.2. Second attempt

An attempt for another generalized version based on Equation (5) is now provided.

Proposition 3.2. *Let us consider the following two-dimensional function*:

$$\bar{S}(u, v) = uv\left[1 + \frac{\beta}{1 - \alpha uv}(1 - u)(1 - v)\right], \quad (u, v) \in [0, 1]^2. \tag{7}$$

First, *let us impose that* $\beta \in [-1, 1]$, *as the corresponding parameter* β *of the FGM copula. Then*, *for* $\alpha \leq 0$, $\bar{S}(u, v)$ *defines a valid copula. The FGM copula is obtained by taling* $\alpha = 0$.

Proof. The first steps are immediate, we have $\bar{S}(u, 0) = \bar{S}(0, v) = 0$, and

$$\bar{S}(u, 1) = u \times 1 \times \left[1 + \frac{\beta}{1 - \alpha u \times 1}(1 - u)(1 - 1)\right] = u.$$

Similarly, we have $\bar{S}(1, v) = v$. Since $\alpha \leq 0$, we have $1 - \alpha uv \geq 1 > 0$, the function $\bar{S}(u, v)$ is differentiable with respect to u and v. After using standard derivative operations, we obtain

$$\frac{\partial^2 \bar{S}(u, v)}{\partial u \partial v} = 1 + \beta \frac{\alpha^2 u^3 v^3 - \alpha uv(3uv - 1) + (2u - 1)(2v - 1)}{(1 - \alpha uv)^3}.$$

Since $\beta \in [-1, 1]$, the following lower bound holds:

$$\frac{\partial^2 \bar{S}(u, v)}{\partial u \partial v} \geq 1 - |\beta| \frac{|\alpha^2 u^3 v^3 - \alpha uv(3uv - 1) + (2u - 1)(2v - 1)|}{(1 - \alpha uv)^3}$$

$$\geq 1 - \frac{|\alpha^2 u^3 v^3 - \alpha uv(3uv - 1) + (2u - 1)(2v - 1)|}{(1 - \alpha uv)^3}.$$

Therefore, we have $\partial^2 \overline{S}(u, v)/(\partial u \partial v) \geq 0$ if

$$\frac{\alpha^2 u^3 v^3 - \alpha uv(3uv - 1) + (2u - 1)(2v - 1)}{(1 - \alpha uv)^3} \in [-1, 1]. \tag{8}$$

Let us examine the validity of this condition. We begin by investigating the following inequality:

$$\alpha^2 u^3 v^3 - \alpha uv(3uv - 1) + (2u - 1)(2v - 1) \leq (1 - \alpha uv)^3. \tag{9}$$

We have

$$\begin{aligned}
&\alpha^2 u^3 v^3 - \alpha uv(3uv - 1) + (2u - 1)(2v - 1) - (1 - \alpha uv)^3 \\
&\quad = \alpha^3 u^3 v^3 + \alpha^2 u^3 v^3 - 3\alpha^2 u^2 v^2 - 3\alpha u^2 v^2 + 4\alpha uv + 4uv - 2u - 2v \\
&\quad = \alpha^2(\alpha + 1)u^3 v^3 - 3\alpha(\alpha + 1)u^2 v^2 + 4(\alpha + 1)uv - 2(u + v) \\
&\quad = (\alpha + 1)uv(\alpha^2 u^2 v^2 + 4 - 3\alpha uv) - 2(u + v).
\end{aligned}$$

Let us distinguising the case $\alpha \leq -1$ and the case $\alpha \in (-1, 0]$.

- Suppose that $\alpha \leq -1$. First, it is clear that $-2(u + v) \leq 0$. Since $\alpha \leq -1$, we have $\alpha + 1 \leq 0$ and $\alpha^2 u^2 v^2 + 4 - 3\alpha uv \geq 4 > 0$. Hence

$$\begin{aligned}
&\alpha^2 u^3 v^3 - \alpha uv(3uv - 1) + (2u - 1)(2v - 1) - (1 - \alpha uv)^3 \\
&\qquad = (\alpha + 1)uv(\alpha^2 u^2 v^2 + 4 - 3\alpha uv) - 2(u + v) \leq 0,
\end{aligned}$$

implying that

$$\alpha^2 u^3 v^3 - \alpha uv(3uv - 1) + (2u - 1)(2v - 1) \leq (1 - \alpha uv)^3.$$

• Suppose that $\alpha \in (-1, 0]$. Then $\alpha + 1 \geq 0$, and, since $(u, v) \in [0, 1]^2$, $\alpha^2 u^2 v^2 \leq \alpha^2$ and $-3\alpha uv \leq -3\alpha$, we obtain

$$(\alpha + 1)(\alpha^2 u^2 v^2 + 4 - 3\alpha uv) \leq (\alpha + 1)(\alpha^2 + 4 - 3\alpha) = \alpha(1 - \alpha)^2 + 4 \leq 4.$$

Therefore

$$\alpha^2 u^3 v^3 - \alpha uv(3uv - 1) + (2u - 1)(2v - 1) - (1 - \alpha uv)^3$$
$$\leq 4uv - 2(u + v) = -2v(1 - u) - 2u(1 - v) \leq 0,$$

which implies that

$$\alpha^2 u^3 v^3 - \alpha uv(3uv - 1) + (2u - 1)(2v - 1) \leq (1 - \alpha uv)^3.$$

Hence, for $\alpha \leq 0$, Equation (9) is proved.

Let us now examine the complementary inequality:

$$-(1 - \alpha uv)^3 \leq \alpha^2 u^3 v^3 - \alpha uv(3uv - 1) + (2u - 1)(2v - 1). \tag{10}$$

Suppose that $\alpha < 1$, including $\alpha \leq 0$. We have

$$\alpha^2 u^3 v^3 - \alpha uv(3uv - 1) + (2u - 1)(2v - 1) + (1 - \alpha uv)^3$$
$$= -\alpha^3 u^3 v^3 + \alpha^2 u^3 v^3 + 3\alpha^2 u^2 v^2 - 3\alpha u^2 v^2 - 2\alpha uv + 4uv - 2u - 2v + 2$$
$$= \alpha^2 (1 - \alpha) u^3 v^3 - 3\alpha(1 - \alpha) u^2 v^2 + 2(2 - \alpha) uv - 2(u + v - 1).$$

By using the inequality $u + v - 1 \leq uv$, we obtain

$$\alpha^2 (1 - \alpha) u^3 v^3 - 3\alpha(1 - \alpha) u^2 v^2 + 2(2 - \alpha) uv - 2(u + v - 1)$$
$$\geq \alpha^2 (1 - \alpha) u^3 v^3 - 3\alpha(1 - \alpha) u^2 v^2 + 2(2 - \alpha) uv - 2uv$$
$$= \alpha^2 (1 - \alpha) u^3 v^3 - 3\alpha(1 - \alpha) u^2 v^2 + 2(1 - \alpha) uv$$
$$= (1 - \alpha) uv(1 - \alpha uv)(2 - \alpha uv) \geq 0.$$

This implies that

$$-(1-\alpha uv)^3 \leq \alpha^2 u^3 v^3 - \alpha uv(3uv-1)+(2u-1)(2v-1).$$

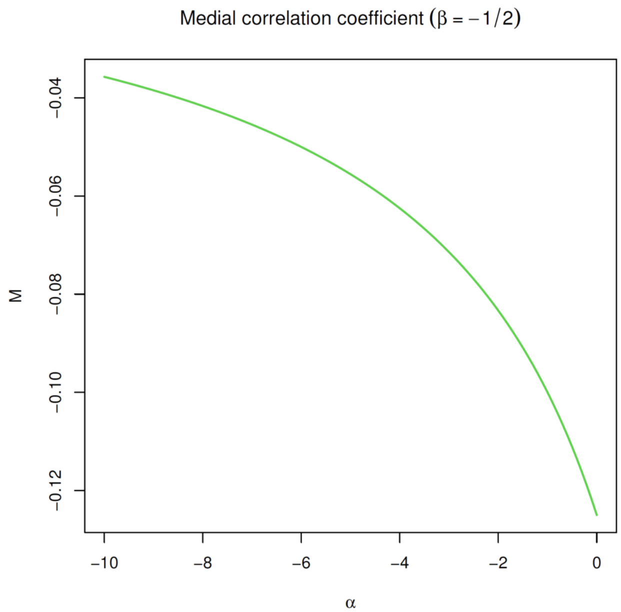
Hence, for $\alpha < 1$, Equation (10) is proved.

Finally, for $\alpha \leq 0$, the appartenance in Equation (8) is satisfied, implying that $\partial^2 \overline{S}(u, v)/(\partial u \partial v) \geq 0$; the two-increasing condition is satisfied. As a result, $\overline{S}(u, v)$ is a valid copula. Proposition 3.2 is proved.

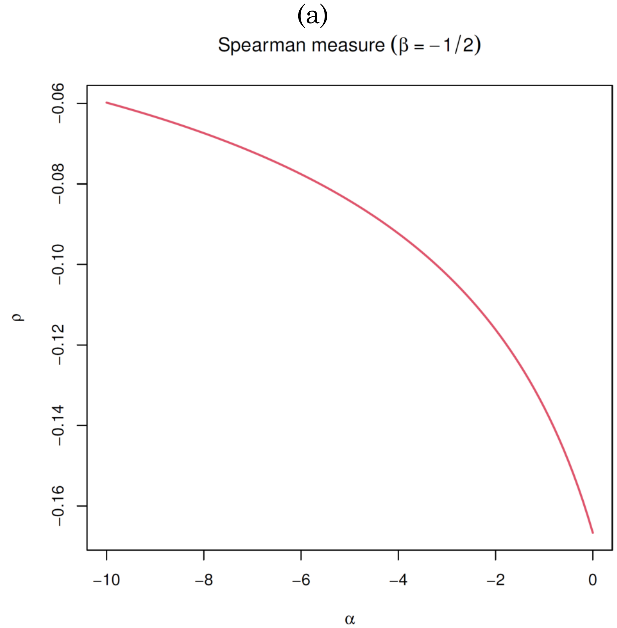
□

The copula presented in Equation (7) is called the ratio extended FGM (REFGM) copula. As primary properties, we can say that it does not belong to the Archimedean family, it is symmetric, it is not radically symmetric, it has the PQD property for $\beta \in (0, 1)$, the independence property for $\beta = 0$ and the NQD property for $\beta \in (-1, 0)$, it has the tail independence property, and, since $\alpha \leq 0$, it satisfies the following concordance properties: $\overline{S}(u, v) \leq C_*(u, v)$ for $\beta \in [0, 1]$, and $\overline{S}(u, v) \geq C_*(u, v)$ for $\beta \in [-1, 0)$.

Figures 4 and 5 plot the related medial correlation and coefficient and Spearman measure for several values of α for $\beta = -1/2$ and $\beta = 1/2$, respectively.

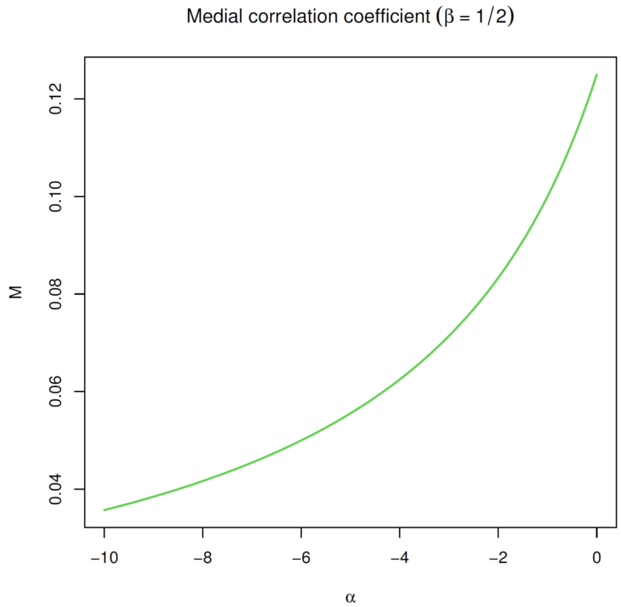


(a)

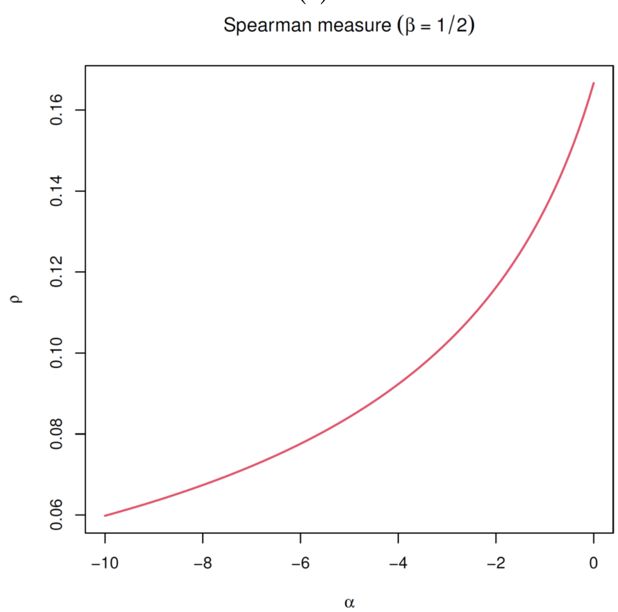


(b)

Figure 4. Two-dimensional plots of the (a) medial correlation coefficient and (b) Spearman measure of the REFGM copula for $\beta = -1/2$ and $\alpha \in [-10, 0]$.



(a)



(b)

Figure 5. Two-dimensional plots of the (a) medial correlation coefficient and (b) Spearman measure of the REFGM copula for $\beta = 1/2$ and $\alpha \in [-10, 0]$.

In Figures 4 and 5, we see the monotonic versatility of M and ρ, depending on the sign of β, and also the possible range of values for these measures.

By its definition, the REFGM copula thus contains the FGM copula, and constitutes a generalized FGM version. However, it does not contain the NR copula since the case $\alpha = 1/2$ is excluded. Also, the condition $\beta \in [-1, 1]$ is imposed from the beginning, with model to the possible values of the parameter β into the definition of the FGM copula. The REFGM copula is new to our knowledge, and can attract further research in the areas of new extended FGM copulas, following the spirit of the studies of [9], [3], [1], and [2].

In view of Proposition 3.2, it is natural to ask the following question: "Can we define a general copula defined as in Equation (5) allowing the case $\alpha \in (0, 1)$?" An attempt at an answer is given in the next part.

3.3. Third attempt

An alternative result to the one in Proposition 3.2 considering the case $\alpha \in (0, 1)$ is proposed in the next result, but under the constraint of $\beta \in [0, 1]$.

Proposition 3.3. *Let us consider the following two-dimensional function*:

$$\ddot{S}(u, v) = uv\left[1 + \frac{\beta}{1 - \alpha uv}(1 - u)(1 - v)\right], \quad (u, v) \in [0, 1]^2,$$

which corresponds to the one in Equation (7). *Suppose that* $\beta \in [0, 1]$ *and* $\alpha < 1$. *For mathematical convenience, we directly exclude the case* $\alpha \neq 0$, *corresponding to the FGM copula. If either*

C1: $\alpha < \beta$, $4\alpha\beta - 3\alpha - \beta \geq 0$ *and* $\alpha\beta - 3\alpha + 2\beta \geq 0$,

C2: $\alpha > \beta$, $4\alpha\beta - 3\alpha - \beta \leq 0$ *and* $\alpha\beta - 3\alpha + 2\beta \geq 0$, C3: $\alpha < \beta,\ 4\alpha\beta - 3\alpha - \beta < 0,$ *by setting*

$$x_{\pm} = \frac{3}{2\alpha} \pm \frac{\sqrt{(\alpha - \beta)(4\alpha\beta - 3\alpha - \beta)}}{2\alpha(\beta - \alpha)}, \tag{11}$$

$\min(x_-, x_+) \geq 1$ *or* $\max(x_-, x_+) \leq 0,$

C4: $\alpha > \beta,\ 4\alpha\beta - 3\alpha - \beta > 0,$ *and, with the notations of* C3, $\max(x_-, x_+) \geq 1$ *and* $\min(x_-, x_+) \leq 0,$

then $\ddot{S}(u, v)$ *defines a valid copula. The NR copula is obtained by taking* $\beta = 1$ *and* $\alpha = 1/2$, *satisfying the conditions in* C3 *with* $x_- = 2$ *and* $x_+ = 4.$

Proof. We proceed a bit differently to the proof of Proposition 3.2. Obviously, we have $\ddot{S}(u, 0) = \ddot{S}(0, v) = 0,$ and

$$\ddot{S}(u, 1) = u \times 1 \times \left[1 + \frac{\beta}{1 - \alpha u \times 1}(1 - u)(1 - 1)\right] = u.$$

Similarly, we have $\ddot{S}(1, v) = v.$ Since $\alpha < 1,$ we have $1 - \alpha uv > 0,$ the function $\ddot{S}(u, v)$ is differentiable with respect to u and $v.$ Several calculus steps and re-arrangements give

$$\frac{\partial^2 \ddot{S}(u, v)}{\partial u \partial v} = 1 + \beta \frac{\alpha^2 u^3 v^3 - \alpha uv(3uv - 1) + (2u - 1)(2v - 1)}{(1 - \alpha uv)^3}$$

$$= \frac{(1 - \alpha uv)^3 + \beta[\alpha^2 u^3 v^3 - \alpha uv(3uv - 1) + (2u - 1)(2v - 1)]}{(1 - \alpha uv)^3}$$

$$= \frac{\alpha^2(\beta - \alpha)u^3 v^3 + 3\alpha(\alpha - \beta)u^2 v^2 + (\alpha\beta - 3\alpha + 4\beta)uv - 2\beta u - 2\beta v + \beta + 1}{(1 - \alpha uv)^3}.$$

Since $(u, v) \in [0, 1]^2$ and $\alpha < 1$, we have $1 - \alpha uv > 0$, and since $\beta \leq 1$, we have $\beta + 1 \geq 2\beta$. Therefore

$$\frac{\partial^2 \ddot{S}(u, v)}{\partial u \partial v}$$

$$\geq \frac{\alpha^2(\beta - \alpha)u^3v^3 + 3\alpha(\alpha - \beta)u^2v^2 + (\alpha\beta - 3\alpha + 4\beta)uv - 2\beta(u + v - 1)}{(1 - \alpha uv)^3}.$$

By using $u + v - 1 \leq uv$ and $\beta \geq 0$, we get

$$\frac{\partial^2 \ddot{S}(u, v)}{\partial u \partial v} \geq uv \frac{P(uv)}{(1 - \alpha uv)^3},$$

where

$$P(x) = \alpha^2(\beta - \alpha)x^2 + 3\alpha(\alpha - \beta)x + \alpha\beta - 3\alpha + 2\beta, \quad x \in [0, 1].$$

Thus, we have $\partial^2 \ddot{S}(u, v)/(\partial u \partial v) \geq 0$ if $P(x) \geq 0$ for any $x \in [0, 1]$, and we will focus on this last condition in the next.

For $\alpha = \beta$ with $\beta \in (0, 1]$, we have $P(x) = \alpha(\alpha - 1) \leq 0$; this case is thus excluded in our strategy of proof.

For $\alpha \neq 0$ and $\alpha \neq \beta$, let us factorize $P(x)$. The discriminant of $P(x)$ is equal to

$$\Delta = \alpha^2[9(\alpha - \beta)^2 - 4(\beta - \alpha)(\alpha\beta - 3\alpha + 2\beta)] = \alpha^2(\alpha - \beta)(4\alpha\beta - 3\alpha - \beta).$$

If $\Delta \leq 0$ and $\alpha\beta - 3\alpha + 2\beta \geq 0$, we have $P(x) \geq 0$. Based on the definition of Δ, the related conditions correspond to C1 and C2.

On the other hand, if $\Delta > 0$, $P(x)$ has two different roots given as $x_{\pm}$ as described in Equation (11), and we can write $P(x)$ as

$$P(x) = \alpha^2(\beta - \alpha)(x_- - x)(x_+ - x).$$

Let us distinguish the case $\alpha < \beta$ and the case $\alpha > \beta$.

• If $\alpha < \beta$, since $x \in [0, 1]$, we have $P(x) = \alpha^2(\beta - \alpha)(x_- - x)(x_+ - x) \geq 0$ if $\min(x_-, x_+) \geq 1$ or $\max(x_-, x_+) \leq 0$. This corresponds to the condition C3.

• If $\alpha > \beta$, since $x \in [0, 1]$, we have $P(x) = \alpha^2(\beta - \alpha)(x_- - x)(x_+ - x) \geq 0$ if $\max(x_-, x_+) \geq 1$ and $\min(x_-, x_+) \leq 0$. This corresponds to the condition C4.

Hence, under either C1, C2, C3, or C4, we have $\partial^2 \ddot{S}(u, v)/(\partial u \partial v) \geq 0$, demonstrating the two-increasing property. By combining all the above properties, the fact that $\ddot{S}(u, v)$ is a copula is proved, concluding the proof of Proposition 3.3. □

It is worth noting that Proposition 3.3 includes the case $\alpha \in (0, 1]$, which was excluded from Proposition 3.2. In this sense, Propositions 3.2 and 3.3 are complementary. In addition, Proposition 2.1 can be viewed as a special case of Proposition 3.3 by taking $\beta = 1$ and $\alpha = 1/2$.

4. Conclusion

In this study, we first developed a novel two-dimensional dependence strategy. It is based on a new ratio copula. We have highlighted its main characteristics, showing that it does not belong to the Archimedean family, it is symmetric, it is not radically symmetric, it has the positive quadrant dependence property, it has the tail independence property, and it satisfies comprehensive concordance properties involving the Farlie-Gumbel-Morgenstern copula. These nice properties motivate the creation of generalizations of this copula. In this regard, three attempts have been made, including a new generalized version of the Farlie-Gumbel-

Morgenstern copula. The usage of a special weighted ratio function positioned at a specific place in the former definition of the former FGM copula is the basis for this generalization.

Future work based on our findings includes.

• The development of a multivariate generalized NR copula of the following form:

$$S_{(d)}(u_1, \ldots, u_d) = \prod_{i=1}^{d} u_i \left[1 + \frac{\beta}{1 - \alpha \prod_{i=1}^{d} u_i} \prod_{i=1}^{d} (1 - u_i) \right],$$

$$(u, \ldots, u_d) \in [0, 1]^d,$$

where the possible values for α and β need to be determined.

• The in-depth study of the reflected versions of the proposed copulas is also an interesting perspective.

• The use of the proposed copulas in applied areas, such as economics, educational engineering, and finance.

• Various two-dimensional statistical models can be constructed, such as data fitting models and regression models.

These practical features, which we will leave as viewpoints, require additional examination.

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