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GLOBAL SOLUTION FOR THE COUPLED YANG-MILLS-BOLTZMANN SYSTEM IN A BIANCHI TYPE 1 SPACE-TIME

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Abstract

Global existence of a solution for the coupled Yang-Mills-Boltzmann system is proved, in a Bianchi type I space-time background. This system rules the collisional evolution of a kind of fast moving, massive particles with a non-Abelian charge.

1. Introduction

A plasma is a train of charged particles evolving at very high speed and under the effect of forces they create collectively. For electrons, in the Abelian case, these created forces are electromagnetic forces which in turn re-influence the movement of these particles, and when we consider that collisions between particles cannot be neglected, the self-sustaining phenomenon is governed by the Maxwell-Boltzmann system, which has been widely studied in the literature; see [3, 10, 11, 15, 16, 17, 19, 26, 27] and references therein, for the relativistic case. In this paper, we consider a more general plasma with particles having non-Abelian charges called Yang-Mills charges. In the collisionless case where the Boltzmann equation is replaced by the Vlasov one, some authors have already studied this kind of phenomenon; see [4, 5, 13, 14, 20, 21, 34] for the coupled system and [1, 2] for the studying of Vlasov equation in the presence of a Yang-Mills field.

At the contrary of the collisionless case, which is widely studied in the literature, there are few results in the collision case. More recently, Dongo et al. have proven in [6, 7, 9] the first results for the local in time existence and uniqueness, for the relativistic Boltzmann equation in the presence of an external force that is of Yang-Mills type. They have also proved in [8] the first result for the coupled Yang-Mills-Boltzmann system, taking as background the spatially homogeneous Bianchi type I space-time with locally rotationally symmetry and obtained local in time classical solutions. The Bianchi space-times models are natural physically motivated cosmological models which have widely been used in literature, see [15, 16, 17, 24, 30] and references therein. In this paper, we extend the local in time result obtained in [8] to the global one. We follow the method used in [10, 11].

We then study the relativistic Yang-Mills-Boltzmann system in a curved Bianchi type I space-time, which has as unknowns: The Yang-Mills field subject to the Yang-Mills equations which derives from the Yang-Mills potential, and the Yang-Mills's particles distribution function. We take particles in temporal gauge, we couple them with the Yang-Mills field as if they were a priori independent, and we assume that the Yang-Mills current, source of the Yang-Mills field, is generated by the distribution function of particles, which is subject to the Boltzmann equation.

The paper is organised as follows:

In Section 2, we introduce the background space-time and unknown functions.

In Section 3, we present the Yang-Mills-Boltzmann system in the corresponding space-time.

In Section 4, we define some function spaces and recall the local existence result obtained in [8].

In Section 5, we establish a global existence theorem for the coupled system.

2. The Background Space-Time and Unknown Functions

In all what follows, unless otherwise specified, Greek indices range from 0 to 3 and Latin ones from 1 to 3. We use the Einstein summation convention.

We consider the collision evolution of a kind of fast moving massive and charged particles in the time-oriented Bianchi type 1 space-time with locally rotationally symmetric in the form

$$g = -dt^2 + h^2(t)dx_1^2 + r^2(t)(dx_2^2 + dx_3^2), \tag{1}$$

where $h > 0$ and $r > 0$ are two continuously differentiable functions of time t. We assume that there exists a constant $C > 0$ such that

$$\left|\frac{\dot{h}}{h}\right| \leq C, \qquad \left|\frac{\dot{r}}{r}\right| \leq C. \tag{2}$$

$(\mathfrak{G}, [,])$ is a Lie algebra of a Lie group G, endowed with an Ad-invariant scalar product denoted by a dot "$\cdot$", which enjoys the following property:

$$u \cdot [v, w] = [u, v] \cdot w, \quad \forall u, v, w \in \mathfrak{G}, \tag{3}$$

where $[,]$ stands for the Lie brackets of the Lie algebra $\mathfrak{G}$. We consider that $\mathfrak{G}$ is a vector space on $\mathbf{R}$ with dimension $N \geq 2$ and (ε_I), $I = 1, \ldots, N$ an orthonormal basis of $\mathfrak{G}$.

The massive particles have the same rest mass $m > 0$, normalized to the unity, i.e., $m = 1$. We denote by $T(\mathbb{R}^4)$, the tangent bundle of $\mathbb{R}^4$ with coordinates (x^α, p^β), where $p = (p^\beta) = (p^0, \overline{p})$ stands for the momentum of each particle and $\overline{p} = (p^i)$, $i = 1, 2, 3$. The charged particles move on the mass hyperboloid $\mathbb{P}(\mathbb{R}^4) \subset T(\mathbb{R}^4)$, whose equation is $\mathbf{P}_{t,x}(p) = g_{\alpha\beta}p^\alpha p^\beta = -1$ or equivalently, using the expression (1) of g, we have

$$p^0 = [1 + h^2(t)(p^1)^2 + r^2(t)((p^2)^2 + (p^3)^2)]^{\frac{1}{2}}, \tag{4}$$

the choice of $p^0 > 0$ symbolizes the fact that, naturally, particles eject towards the future.

Denote by A a Yang-Mills potential represented by a 1-form on $\mathbb{R}^4$ which takes its values in $\mathfrak{G}$. Then the Yang-Mills potential is locally defined as follows: $A = (A_\mu)$, where $A_\mu = A_\mu^I \varepsilon_I$, $I = 1, 2, \ldots, N$. We require that A satisfies the temporal gauge condition, which means that

$$A_0 = 0. \tag{5}$$

Particles evolve, under the action of their own gravitational field represented by the metric tensor $g = (g_{\alpha\beta})$ given by (1) that informs about gravitational effects, and in addition, under the action of the non-Abelian forces generated by the Yang-Mills field $F = (F_{\alpha\beta})$, where $F_{\alpha\beta}$ is a function from $\mathbb{R}^4$ to $\mathfrak{G}$.

The Yang-Mills field is the curvature of the Yang-Mills potential. Its components can be written in the basic (ε_I) as $F_{\alpha\mu} = F^I_{\alpha\mu}\varepsilon_I$, where $F^I_{\alpha\mu}$ are linked to potential components by:

$$F^I_{\lambda\mu} = \nabla_\lambda A^I_\mu - \nabla_\mu A^I_\lambda + [A_\lambda, A_\mu]^I, \text{ with } [A_\lambda, A_\mu]^I = C^I_{bc}A^b_\lambda A^c_\mu, \tag{6}$$

where C^I_{bc} are structure constants of $\mathfrak{G}$ and ∇_α, the covariant derivative.

The 2-form F verifies Bianchi identities

$$\hat{\nabla}_\alpha F_{\lambda\mu} + \hat{\nabla}_\lambda F_{\mu\alpha} + \hat{\nabla}_\mu F_{\alpha\lambda} = 0, \tag{7}$$

and the relation

$$\hat{\nabla}_\alpha \hat{\nabla}_\beta F^{\alpha\beta} = 0, \tag{8}$$

where $\hat{\nabla}_\alpha$ is the gauge covariant derivative defined by

$$\hat{\nabla}_\alpha = \nabla_\alpha + [A_\alpha, .]. \tag{9}$$

The Yang-Mills field $F = (F^{0i}, F_{ij})$ is subject to the Yang-Mills system.

Taking into account the temporal gauge (5) and relation (6), we have

$$\partial_0 A_i = F_{0i}, \qquad F^I_{ij} = C^I_{bc}A^b_i A^c_j, \quad i \neq j. \tag{10}$$

We also suppose that, the non-Abelian charge q of Yang-Mills particles takes its values in an orbit of $\mathfrak{G}$, which is a sphere ϑ whose equation is:

$$(\vartheta): \quad q \cdot q = \|q\|^2 = e^2, \tag{11}$$

where $\|\cdot\|$ stands for the norm deduced from the scalar product of $\mathfrak{G}$. The relation (11) allows to express the q^N component of q as a function of $\tilde{q} = (q^I)$, $I = 1, 2, \ldots, N-1$.

We denote by f the unknown distribution function which measures the probability density of the presence of particles in the plasma. f is a function defined on $T(\mathbb{R}^4) \times \mathfrak{G}$ and will be subject to the Boltzmann equation. Using relations (4), (11) and the fact that we are studying an homogeneous phenomenon, we obtain that the distribution function of Yang-Mills particles is definitely a function of independent variables $(t, p^i, q^I) = (t, \bar{p}, \tilde{q})$. Then, $f = f(t, \bar{p}, \tilde{q})$, $t \in \mathbb{R}$, $\bar{p} \in \mathbb{R}^3$, $\tilde{q} \in \mathbb{R}^{N-1}$.

According to relations (4) and (11), the phase space of such Yang-Mills particles is in fact the subset $\mathbb{P} = \mathbb{P}_{t,x} \times \vartheta$ of $T(\mathbb{R}^4) \times \mathfrak{G}$. Trajectories of particles satisfy the following differential system:

$$\frac{dx^\alpha}{ds} = p^\alpha, \quad \frac{dp^\alpha}{ds} = -\Gamma^\alpha_{\lambda\mu} p^\lambda p^\mu + p^\beta q \cdot F^\alpha_\beta, \quad \frac{dq^I}{ds} = -C^I_{bc} p^\alpha A^b_\alpha q^c, \tag{12}$$

where $\Gamma^\alpha_{\lambda\mu}$ are the Christoffel symbols of the Levi-Civita connection associated to g, which are computed using (1) and give:

$$\begin{cases} \Gamma^1_{10} = \dfrac{\dot{h}}{h}, \Gamma^2_{20} = \Gamma^3_{30} = \dfrac{\dot{r}}{r}, \quad \Gamma^0_{11} = h\dot{h}, \Gamma^0_{33} = \Gamma^0_{11} = r\dot{r}, \\ \Gamma^\alpha_{\lambda\mu} = 0 \quad \text{otherwise.} \end{cases} \tag{13}$$

The last equation in (12), called Wong's equation, expresses the fact that the covariant derivative of gauge of q along a trajectory is null.

The charged particles also create a current $J = (J^{\beta})$ (locally) called the Yang-Mills current. Recall that J^{β} in the Bianchi type 1 space-time that we consider is a function from $\mathbb{R}$ to $\mathfrak{G}$, such that $J^{\beta}(t) = J^{\beta, I}\varepsilon_I$, where

$$J^{\beta, I}(t) = \int_{\mathbb{R}^3 \times \vartheta} p^{\beta} q^I f(t, p, q) \omega_p \omega_q, \; I = 1, \dots, N, \tag{14}$$

in which $\omega_p = hr^2 \dfrac{dp^1 dp^2 dp^3}{p^0}$ and ω_q stands for the canonical volume element of ϑ.

3. The Yang-Mills-Boltzmann System

On its general form, the Yang-Mills-Boltzmann (YMB) system is written as:

$$\hat{\nabla}_{\alpha} F^{\alpha\beta} = J^{\beta}, \tag{15}$$

$$L_Y f = \mathcal{L}(f, f), \tag{16}$$

where (15) represent the Yang-Mills (YM) system, and the expression (14) shows that, the Yang-Mills current is generated by the distribution function of particles. System (16) is the Boltzmann equation, where L_Y is the Lie derivative of f with respect to the vector field $Y = \left(p^{\alpha}, -\Gamma^{\alpha}_{\lambda\mu} p^{\lambda} p^{\mu} + p^{\beta} q \cdot F^{\alpha}_{\beta}, -C^{I}_{bc} p^{\alpha} A^{b}_{\alpha} q^{c}\right)$ given by the differential system (12) and $\mathcal{L}(f, f)$ is the collision operator, presented in Subsection 3.2.

3.1. Initial data for the Yang-Mills system

The Cauchy's data for the YMB system are given as follows:

For the Yang-Mills potential A, we consider a given element a of $\mathfrak{G}$ such that $A|_{t=0} = a$. Locally, $a = (a_i)$. For the magnetic part of the Yang-Mills field F denoted F_{ij}, we consider a given element Φ of $\mathfrak{G}$ such that $F_{ij}|_{t=0} = (\Phi_{ij})$, where $\Phi = (\Phi_{ij})$. Φ_{ij} and a_i will be linked by

$$\Phi_{ij} = \nabla_i a_j - \nabla_j a_i + [a_i, a_j] = [a_i, a_j]. \tag{17}$$

For the electrical part (F^{0i}) of the Yang-Mills field F, we consider a given element E of $\mathfrak{G}$ such that $F^{0i}|_{t=0} = E^i$, where $E = (E^i)$.

For the distribution function of Yang-Mills particles f, we consider $f_0 : \mathbb{P} \longrightarrow \mathbb{R}^+$ such that $f|_{t=0} = f_0$.

3.2. The Boltzmann equation in *f*

After some computations, the Boltzmann equation in f for the Yang-Mills charged particles in the Bianchi type 1 space-time can be written as follows:

$$\frac{\partial f}{\partial t} + P^i \frac{\partial f}{\partial p^i} + Q^I \frac{\partial f}{\partial q^I} = \frac{1}{p^0} \mathcal{L}(f, f), \tag{18}$$

where

$$P^i = \left(- 2\Gamma^i_{i0} p^i - q \cdot F^{i0} - \frac{p^j g^{ii} q \cdot F_{ij}}{p^0} \right) \text{ and } Q^I = - \frac{p^i}{p^0} C^I_{LH} A^L_i q^H. \tag{19}$$

The collision operator

We recall that, there are several representations of the collision operator. However, it should be noted that, each representation depends on particles in presence; precisely, the position, the speed (the momentum) and the internal state (charge) of them. In the case of chargeless particles, one of the frequently used representation of the collision operator is provided by Glassey in [29]. Here, we note that, our Boltzmann equation model takes into account in addition to the position and the speed of particles, their internal state represented by the non-Abelian charge q. In this context, the form of the collision operator is more complex and generalizes that of the chargeless particles; see ([35]) and references therein.

We write $\mathcal{L}$ as the difference between the gain term $\mathcal{L}^+$ and the loss term $\mathcal{L}^-$:

$$\mathcal{L}(f, g)(t, \bar{p}, \tilde{q}) = \mathcal{L}^+(t, \bar{p}, \tilde{q}) - \mathcal{L}^-(t, \bar{p}, \tilde{q}),$$

where

$$\mathcal{L}^+(f, g) = \int_{\mathbb{R}^3\times\vartheta} \frac{\sqrt{|\det(g)|}}{p_*^0} d\bar{p}_* w_{\tilde{q}_*} \int_{S^2\times S^{N-2}} f(\overline{p}', \tilde{q}')\, g(\overline{p}'_*, \tilde{q}'_*)\sigma dw d\theta,$$

$$\mathcal{L}^-(f, g) = \int_{\mathbb{R}^3\times\vartheta} \frac{\sqrt{|\det(g)|}}{p_*^0} d\bar{p}_* w_{\tilde{q}_*} \int_{S^2\times S^{N-2}} f(\bar{p}, \tilde{q})\, g(\bar{p}_*, \tilde{q}_*)\sigma dw d\theta,$$

in which

- S^2 is the unit sphere of $\mathbb{R}^3$, whose element is denoted dw;

- S^{N-2} is the unit sphere of $\mathbb{R}^{N-1}$, whose element is denoted θ, and $d\theta$ his volume element;

- $|\det(g)| = h^2 r^4$, is the Bianchi type 1 space-time;

- $\sigma = \sigma(\bar{p}, \tilde{q}, \bar{p}_*, \tilde{q}_*, \overline{p}', \tilde{q}', \overline{p}'_*, \tilde{q}'_*)$ is a positive regular function called the collision kernel or the cross-section of the collision which measures the effects of interactions between particles and determines their nature, on which we require the following assumptions:

$$
(\mathrm{H}_1):\begin{cases}\exists C>0,\ 0\le\sigma\le C,\\ \Big|\sigma(\bar p_1,\tilde q_1,\bar p_*,\tilde q_*,\bar p_1',\tilde q_1',\bar p_*',\tilde q_*')\\ \qquad -\sigma(\bar p_2,\tilde q_2,\bar p_*,\tilde q_*,\bar p_2',\tilde q_2',\bar p_*',\tilde q_*')\Big|\\ \qquad \leqslant C(\|\bar p_1-\bar p_2\|+\|\tilde q_1-\tilde q_2\|),\\ (1+|\bar p|)^l\,\|\partial_{(\bar p,\tilde q)}\sigma\|_{L^1(\Omega\times\mathfrak{S})}\in L^\infty(\Omega),\ 0\le|\beta|\le m+3,\\ \qquad 0\leqslant l\le m+3,\\ (1+|\bar p|)^{|\beta|-1}\partial^\beta_{(\bar p,\tilde q)}\sigma\in L^\infty(\Omega\times\Omega\times\mathfrak{S}),\ 1\le|\beta|\le m+3,\end{cases}
$$

where $\beta\in\mathbb{N}^{m+3}$, $\Omega=\mathbb{R}^3\times\mathbb{R}^{N-1}$ and $\mathfrak{S}=S^2\times S^{N-2}$. Note that assumptions (H_1) are closed to the $\mu-N$ regularity introduced by Choquet-Bruhat and Bancel in [31, 32], and have been used in [10, 30, 33, 34]. Also note that, $\sigma=k(t)\,e^{-|\bar p_1|-|\tilde q_1|-|\bar p_*|-|\tilde q_*|-|\bar p_1'|-|\tilde q_1'|-|\bar p_*'|-|\tilde q_*'|}$, where $k(t)$ is a continuous function of t, is a simple example of collision kernel satisfying assumptions (H_1).

3.3. The Yang-Mills system in F

The Yang-Mills system (15) is a system of 4 equations for the 6 unknowns $F=(F^{0i},F_{ij})$, where F^{0i} and F_{ij} stands for the electrical and the magnetic components of the Yang-Mills field, respectively. F^{0i} and F_{ij} are raised or lowered through the metric tensor g, e.g., $F_{ij}=g_{ia}g_{jb}F^{ij}$.

Proposition 1. *We suppose that,* $[a_i,E^i]=-\int_{\mathbb{R}^3\times\vartheta}q^If_0(t,\bar p,\tilde q)\,\bar p\omega_q$. *Then the Yang-Mills system* (15) *in temporal gauge is equivalent to*:

$$
\begin{cases}\dfrac{dF^{0i}}{dt}+\Gamma^l_{l0}F^{0i}+[A_j,F^{ji}]=J^i,\\ \dfrac{dF_{ij}}{dt}=-[A_j,F_{0i}]+[A_i,F_{0j}].\end{cases}\tag{20}
$$

Proof. See [8]. □

3.4. The Yang-Mills-Boltzmann system in (A, F, f)

Now, solving the first partial differential equation (18) in f is equivalent to solve its associated characteristic system:

$$\frac{dt}{1} = \frac{dp^i}{P^i} = \frac{dq^I}{Q^I} = \frac{df}{\frac{1}{p^0}\mathcal{L}(f, f)} = ds. \tag{21}$$

Considering expression (21), (20), (19) and (10), the Yang-Mills-Boltzmann system then takes the following equivalent form:

$$(\mathbf{I}) : \begin{cases} \dfrac{dp^i}{dt} = -2\Gamma^i_{i0}p^i + q \cdot F^{0i} - \dfrac{p^j g^{ii} q \cdot F_{ij}}{p^0}, \\ \dfrac{dq^I}{dt} = -\dfrac{p^i}{p^0} C^I_{bc} A^b_i q^c, \\ \dfrac{df}{dt} = \dfrac{\mathcal{L}(f, f)}{p^0}, \\ \dfrac{dF^{0i}}{dt} = -\Gamma^l_{l0} F^{0i} + [A_j, F^{ji}] + J^i, \\ \dfrac{dF_{ij}}{dt} = -[A_j, F_{0i}] + [A_i, F_{0j}], \\ \dfrac{dA_i}{dt} = F_{0i}, \\ \qquad \text{with } i, j = 1, 2, 3; \; I = 1, 2, \ldots, N. \end{cases} \tag{22}$$

We remark that, $\bar{p}$, $\tilde{q}$, f, F^{0i}, F_{ij} and A_i are now independent variables for the differential system (**I**).

4. Function Spaces and Local Solution for the Coupled Yang-Mills-Boltzmann System

4.1. Function spaces

The function space for the distribution function f is:

Definition 1. Let $T > 0, l \in \mathbb{N}$ and $d \in \mathbb{R}^+$ be given. We set $\Omega = \mathbb{R}^3 \times \mathbb{R}^{N-1}$.

(1) $\mathbb{E}_d^l(\Omega) = \{u : \Omega \longrightarrow \mathbf{R}, (1 + |\bar{p}|)^{d+|\beta|}\partial^\beta_{(\bar{p},\, \tilde{q})} u \in L^2(\Omega), |\beta| \leq l\}$, endowed with the norm $\|u\|_{\mathbb{E}_d^l(\Omega)} = \max\limits_{0 \leq |\beta| \leq l} \left\|(1 + |\bar{p}|)^{d+|\beta|}\, \partial^\beta_{(\bar{p},\, \tilde{q})} u\right\|_{L^2(\Omega)}$.

(2) $\mathbb{E}_d^l(0, T; \Omega) = \{u : [0, T] \times \Omega \longrightarrow \mathbb{R}, u$ continuous; $u(t, .) \in \mathbb{E}_d^l(\Omega)$, $\forall t \in [0, T]\}$, endowed with the norm $\|u\|_{\mathbb{E}_d^l(0,\, T;\, \Omega)} = \sup\limits_{0 \leq t \leq T} \max\limits_{0 \leq |\beta| \leq l} \left\|(1 + |\bar{p}|)^{d+|\beta|}\, \partial^\beta_{(\bar{p},\, \tilde{q})} u(t, \cdot)\right\|_{L^2(\Omega)}$.

(3) $\mathbb{E}_{d,\, \delta}^l(0, T; \Omega) = \{u \in \mathbb{E}_d^l(0, T; \Omega), \|u\|_{\mathbb{E}_d^l(0,\, T;\, \Omega)} \leq \delta\}$, for $\delta > 0$.

Endowed with the distance induced by the norm $\|\cdot\|_{\mathbb{E}_d^l(0,\, T;\, \Omega)}$, $\mathbb{E}_{d,\, \delta}^l(0, T; \Omega)$ is a complete metric subspace of $\mathbb{E}_d^l(0, T; \Omega)$.

Remark 1. We choose as in [6, 7], $l = m + 3$ and $d = \dfrac{N + 4}{2}$ and we have $\mathbb{E}_d^{m+3}(\Omega) \hookrightarrow C_b^1(\Omega)$.

The framework in which we will refer for the determination of the components F^{0i}, F_{ij} and A_i for the Yang-Mills system is $\mathbb{R}^N$, whose norm is denoted $\|\cdot\|$ (or $\|\cdot\|_{\mathbb{R}^N}$).

Let $I = [0, T]$ a real interval, we set

$$C(I, \mathbb{R}^N) = \{v : I \longrightarrow \mathbb{R}^N, v \text{ continuous}\}, \tag{23}$$

where $C(I, \mathbb{R}^N)$ is a Banach space for the norm $\|v\|_1 = \sup\{\|v\|, t \in I\}$.

The frameworks in which we will refer for the determination of $\overline{p}$ and $\tilde{q}$ are respectively, $\mathbb{R}^3$ and $\mathbb{R}^{N-1}$, whose norm is $\|\cdot\|$ (or $\|\cdot\|_{\mathbb{R}^3}$, $\|\cdot\|_{\mathbb{R}^{N-1}}$). We set:

$$C(I, \mathbb{R}^3) = \{\overline{p} : I \longrightarrow \mathbb{R}^3, \overline{p} \text{ continuous}\},$$

and

$$C(I, \mathbb{R}^{N-1}) = \{\tilde{q} : I \longrightarrow \mathbb{R}^{N-1}, \tilde{q} \text{ continuous}\},$$

where $C(I, \mathbb{R}^3)$ and $C(I, \mathbb{R}^{N-1})$ are Banach spaces for norms $\|\overline{p}\|_1 = \sup \{\|\overline{p}\|, t \in I\}$ and $\|\tilde{q}\|_1 = \sup\{\|\tilde{q}\|, t \in I\}$, respectively.

4.1.1. Local existence of the solution for the coupled YMB system

To prove the local existence of the solution for the system $(\mathbf{I})$, we consider the Banach space

$$\mathbb{K} = \mathbb{R}^3 \times \mathbb{R}^{N-1} \times \mathbb{E}_d^{m+3}(\Omega) \times \mathbb{R}^{3N} \times \mathbb{R}^{3N} \times \mathbb{R}^{3N},$$

endowed with the norm

$$\|(\overline{p}, \tilde{q}, f, F^{0i}, F_{ij}, A_i)\|_{\mathbb{K}} = \|\overline{p}\| + \|\tilde{q}\| + \|f\|_{\mathbb{E}_d^{m+3}(\Omega)} + \|F^{0i}\| + \|F_{ij}\| + \|A_i\|. \tag{24}$$

We provide the following Cartesian product:

$$C(I, \mathbb{R}^3) \times C(I, \mathbb{R}^{N-1}) \times C(I, \mathbb{E}_d^{m+3}(\Omega)) \times C(I, \mathbb{R}^{3N}) \times C(I, \mathbb{R}^{3N}) \times C(I, \mathbb{R}^{3N}),$$

with the norm

$$\|(\bar{p}, \tilde{q}, f, F^{0i}, F_{ij}, A_i)\|_1 = \|\bar{p}\|_1 + \|\tilde{q}\|_1 + \|f\|_1 + \| F^{0i}\|_1 + \|F_{ij}\|_1 + \| A_i\|_1 , \tag{25}$$

where

$$\|f\|_1 = \sup_{t \in I}\{\|f\|_{\mathbb{E}^{m+3}_{d,\,\delta}(\Omega)}\}.$$

We apply to the system $(\mathbf{I})$ the standard theory of the first order differential systems. Therefore, we recall some useful energy estimates.

Let $T > 0,$ we consider the application defined by

$$G : [0, T]\times \mathbb{K} \longrightarrow \mathbb{K} \tag{26}$$

$$(t, X) \mapsto G(t, X), \text{ where } X = (\bar{p}, \tilde{q}, F^{0i}, F_{ij}, A_i),$$

in which G is the vector function

$$G(t, X) = (G_1, G_2, G_3, G_4, G_5, G_6)(t, X), \tag{27}$$

defined by the right hand side of system (22).

4.1.2. Energy estimates

Definition 2. Let $X_k = (\bar{p}_k, \tilde{q}_k, f_k, F_k^{0i}, F_{ij}^k, A_i^k) \in \mathbb{K},\ k = 1, 2.$ In the sequel, we set

$$\|X_1 - X_2\|_{\mathbb{K}} :\equiv \|\bar{p}_1 - \tilde{p}_2\| + \|\tilde{q}_1 - \tilde{q}_2\| + \|f_1 - f_2\|_{\mathbb{E}^{m+3}_d(\Omega)}$$

$$+ \| F_1^{0i} - F_2^{0i}\| + \|F^1_{ij} - F^2_{ij}\| + \| A_i^1 - A_i^2\|.$$

Proposition 2. *Let* $X_k = (\bar{p}_k, \tilde{q}_k, f_k, F_k^{0i}, F_{ij}^k, A_i^k) \in \mathbb{K},\ k = 1, 2,$ *then*

$$\| G_l(t, X_1) - G_l(t, X_2)\| \leq C_k \,\| X_1 - X_2\|_{\mathbb{K}}, \quad l = 1, \dots, 6; \tag{28}$$

$$\|G(t, X_1) - G(t, X_2)\| \leq C_7\|X_1 - X_2\|_{\mathbb{K}}, \tag{29}$$

where

$$\begin{cases} C_1 = 2C + e\|F_1^{0i}\| + 5g^{ii}\left(1 + \frac{h}{r} + \frac{r}{h}\right) + \|F_{ij}^2\|g^{ii}g^{jj} + eg^{ii}g^{jj}, \\ C_2 = 5Ce\|A_i^2\|\left(1 + \frac{h}{r} + \frac{r}{h}\right) + Cg^{ii}(e + \|A_i^1\|), \\ C_3 = C(\|f_1\|_{\mathbb{E}_d^{m+3}(\Omega)} + 2\|f_2\|_{\mathbb{E}_d^{m+3}(\Omega)}) + C(2h + 4r)\|f_2\|^2_{\mathbb{E}_d^{m+3}(\Omega)}, \\ C_4 = e\left(\|f_1\|_{\mathbb{E}_d^{m+3}(\Omega)} + \Theta\right) + \Theta\|f_1\|_{\mathbb{E}_d^{m+3}(\Omega)} + C[1 + g^{ii}g^{jj}(\|F_{ij}^1\| + \|A_j^2\|)], \\ C_5 = Cg_{ii}(\|F_1^{0i}\| + \|A_j^2\|) + Cg_{jj}(\|F_2^{0j}\| + \|A_i^1\|), \\ C_6 = g_{ii}, \\ C_7 = C_1 + C_2 + C_3 + C_4 + C_5 + C_6, \end{cases} \tag{30}$$

and $\Theta = \|\bar{p}(0)\| e^{Ct} + \dfrac{e\|F^{0i}\| + e\sqrt{g^{jj}}\, g^{ii}\|F_{ij}\|}{C}(e^{Ct} - 1).$

Proof. See [8]. □

4.1.3. Local existence result

In order to state the local existence theorem, we first recall this useful theorem.

Theorem 1. *Let* $t_0 \geq 0$, $(\bar{p}_{t_0}, \tilde{q}_{t_0}, f_{t_0}, F_{t_0}^{0i}, F_{ij}^{t_0}, A_i^{t_0}) \in \mathbb{K}$ *be given, with* $F_{t_0}^{0i}, F_{ij}^{t_0}$ *and* $A_i^{t_0}$ *satisfying constraints* (17) *and the hypothesis of Proposition* 1. *Then*

(1) *There exists a real number* $\eta \geq 0$ *such that the differential system* (**I**) *has a unique solution* $(\bar{p}, \tilde{q}, f, F^{0i}, F_{ij}, A_i)$, *defined in* $[t_0, t_0 + \eta]$ *with values in* $\mathbb{K}$, *satisfying*

$$(\bar{p}, \tilde{q}, f, F^{0i}, F_{ij}, A_i)(t_0) = (\bar{p}_{t_0}, \tilde{q}_{t_0}, f_{t_0}, F_{t_0}^{0i}, F_{ij}^{t_0}, A_i^{t_0}). \tag{31}$$

Moreover, f *satisfies the relation*:

$$\|f\|_{\mathbb{E}_d^{m+3}(\Omega)} \leq \|f_{t_0}\|_{\mathbb{E}_d^{m+3}(\Omega)}. \tag{32}$$

(2) *The Yang-Mills-Boltzmann system* (18)-(15) *has a unique solution* (F, A, f) *on* $[t_0, t_0 + \eta]$ *such that* $(F^{0i}, F_{ij}, A_i, f)(t_0) = (F_{t_0}^{0i}, F_{ij}^{t_0}, A_i^{t_0}, f_{t_0});$ *with*

$$\|f\|_{\mathbb{E}_d^{m+3}(\Omega)} \leq \|f_{t_0}\|_{\mathbb{E}_d^{m+3}(\Omega)}. \tag{33}$$

Proof. See [8]. We recall that, the proof of inequalities (32) and (33) is similar to the one obtained in [30].

□

We end by stating the following local existence theorem which is a particular case of Theorem 1.

Theorem 2. *Let* $(\bar{p}_0, \tilde{q}_0) \in \mathbf{R}^3 \times \mathbf{R}^{N-1}$, $f_0 \in \mathbf{E}_{d,\delta}^{m+3}(\Omega)$, E^i, Φ_{ij}, $a_i \in \mathbb{R}^N$ *be given*, *with* E^i, Φ_{ij} *and* a_i *satisfying constraints* (17) *and the hypothesis of Proposition* 1. *Then*, *there exists a real number* $T > 0$ *such that*

(1) *The differential system* $(\mathbf{I})$ *has a unique solution* $(\bar{p}, \tilde{q}, F^{0i}, F_{ij}, A_i)$ *defined on* $[0, T]$ *with values in* $\mathbb{K}$ *such that*

$$(\bar{p}, \tilde{q}, f, F^{0i}, F_{ij}, A_i)(0) = (\bar{p}_0, \tilde{q}_0, f_0, E^i, \Phi_{ij}, a_i). \tag{34}$$

Moreover, f *satisfies the relation*

$$\|f\|_{\mathbb{E}_d^{m+3}(\Omega)} \leq \|f_0\|_{\mathbb{E}_d^{m+3}(\Omega)}. \tag{35}$$

(2) *The Yang-Mills-Boltzmann system* (18)-(15) *has a unique solution* (F, A, f) *on* $[0, T]$ *satisfying*

$$(F^{0i}, F_{ij}, A_i, f)(0) = (E^i, \Phi_{ij}, a_i, f_0).$$

5. Global Existence of the Solution for the Coupled Yang-Mills-Boltzmann System

5.1. The method

Let $T > 0$ a real number, denote by $[0, T[$ the maximal existence domain of the solution of the system $(\mathbf{I})$ denote here by $\left(\overset{\circ}{\bar{p}}, \overset{\circ}{\tilde{q}}, \overset{\circ}{f}, \overset{\circ}{F}{}^{0i}, \overset{\circ}{F}_{ij}, \overset{\circ}{A}_i\right)$ and given by Theorem 2, with the initial data $(\bar{p}_0, \tilde{q}_0, f_0, E^i, \Phi_{ij}, a_i) \in \mathbf{K}$. Our goal is to prove that $T = +\infty$.

- If $T = +\infty$, the problem is solved;
- If $T < +\infty$, we are going to show that, the solution $\left(\overset{\circ}{\bar{p}}, \overset{\circ}{\tilde{q}}, \overset{\circ}{f}, \overset{\circ}{F}{}^{0i}, \overset{\circ}{F}_{ij}, \overset{\circ}{A}_i\right)$ can be extended belong T, which contradicts the maximality of T. The strategy is as follows: Let $t_0 \in [0, T]$. We show that, there exists a strictly positive number $\eta > 0$ independent of t_0, such that the system $(\mathbf{I})$ on $[t_0, t_0 + \eta]$ with the initial data $\left(\overset{\circ}{\bar{p}}_{t_0}, \overset{\circ}{\tilde{q}}_{t_0}, \overset{\circ}{F}{}^{0i}_{t_0}, \overset{\circ}{F}{}^{t_0}_{ij}, \overset{\circ}{A}{}^{t_0}_i\right)$ at $t = t_0$, has a unique solution $(\bar{p}, \tilde{q}, f, F^{0i}, F_{ij}, A_i)$ on $[t_0, t_0 + \eta]$. Then by taking t_0 such that $T < t_0 + \frac{\eta}{2}$, we can extend the solution $\left(\overset{\circ}{\bar{p}}, \overset{\circ}{\tilde{q}}, \overset{\circ}{F}{}^{0i}, \overset{\circ}{F}_{ij}, \overset{\circ}{A}_i\right)$ to $[0, t_0 + \frac{\eta}{2}]$, which contains strictly $[0, T]$, and this contradicts the maximality of T. For sake of simplicity, we will look for the number η such that, $0 < \eta < 1$.

In what follows, we take $f_0 \in \mathbb{E}^{m+3}_{d,\,\delta}(\Omega)$ which means $\|f_0\|_{\mathbb{E}^{m+3}_d(\Omega)} \leq \delta.$ Since $t_0 \in [0, T[,$ by the inequality (35), we have

$$\left\|\overset{\circ}{f}(t_0)\right\|_{\mathbb{E}^{m+3}_d(\Omega)} \leq \|f_0\|_{\mathbb{E}^{m+3}_d(\Omega)} \leq \delta. \tag{36}$$

We deduce from (35), using (36) that, any solution of the Boltzmann equation on $[t_0, t_0 + \eta]$ such that $f(t_0) = \overset{\circ}{f}(t_0),$ satisfies

$$\|f(t)\|_{\mathbb{E}^{m+3}_d(\Omega)} \leq \delta, \quad t \in [t_0, t_0 + \eta]. \tag{37}$$

We set for $R > 0$ and $I_0 = [0, T]$ a real interval

$$X_R = \{v \in \mathbb{R}^{3N}, \|v\| \leq R\}. \tag{38}$$

Endowed by the distance induced by the norm $\|\cdot\|,$ X_R is a complete metric space.

$$C(I_0, X_R) = \{v \in C(I_0, \mathbb{R}^N), v(t) \in X_R, \quad \forall\, t \in I_0\}. \tag{39}$$

Endowed with the norm $\|\cdot\|_1,$ $C(I_0, X_R)$ is a complete metric space.

5.2. Resolution

Let $t_0 \in [0, T],$ $\eta > 0$ and $\left(\check{\bar{p}}, \check{\tilde{q}}, \check{f}, \check{F}^{0i}, \check{F}_{ij}, \check{A}_i\right)$ be a given element in $C([t_0, t_0 + \eta], \mathbb{R}^3) \times C([t_0, t_0 + \eta], \mathbb{R}^{N-1}) \times \mathbb{E}^{m+3}_{d,\,\delta}([t_0, t_0 + \eta]; \Omega) \times (C([t_0, t_0 + \eta]; X_R))^3.$

Starting from the system (I), we build the following system, in which we fixed variables carrying “ ˇ ”:

$$
(2\mathrm{I}):\begin{cases}
\dfrac{dp^i}{dt} = \check{G}_1\left(t,\ \bar{p},\ \check{\tilde{q}},\ \check{f},\ \check{F}^{0i},\ \check{F}_{ij},\ \check{A}_i\right),\\[2ex]
\dfrac{dq^I}{dt} = \check{G}_2\left(t,\ \check{\bar{p}},\ \tilde{q},\ \check{f},\ \check{F}^{0i},\ \check{F}_{ij},\ \check{A}_i\right),\\[2ex]
\dfrac{df}{dt} = \check{G}_3\left(t,\ \check{\bar{p}},\ \check{\tilde{q}},\ f,\ \check{F}^{0i},\ \check{F}_{ij},\ \check{A}_i\right),\\[2ex]
\dfrac{dF^{0i}}{dt} = \check{G}_4\left(t,\ \check{\bar{p}},\ \check{\tilde{q}},\ \check{f},\ F^{0i},\ \check{F}_{ij},\ \check{A}_i\right),\\[2ex]
\dfrac{dF_{ij}}{dt} = \check{G}_5\left(t,\ \check{\bar{p}},\ \check{\tilde{q}},\ \check{f},\ \check{F}^{0i},\ F_{ij},\ \check{A}_i\right),\\[2ex]
\dfrac{dA_i}{dt} = \check{G}_6\left(t,\ \check{\bar{p}},\ \check{\tilde{q}},\ \check{f},\ \check{F}^{0i},\ \check{F}_{ij},\ A_i\right),
\end{cases}
$$

where

$$
\check{G}_1\left(t,\ \bar{p},\ \check{\tilde{q}},\ \check{f},\ \check{F}^{0i},\ \check{F}_{ij},\ \check{A}_i\right) = -\,2\Gamma^i_{i0}p^i + \check{q}\cdot\check{F}^{0i} - \frac{p^j g^{ii}\ \check{q}\cdot\check{F}_{ij}}{p^0}, \tag{40}
$$

$$
\check{G}_2\left(t,\ \check{\bar{p}},\ \tilde{q},\ \check{f},\ \check{F}^{0i},\ \check{F}_{ij},\ \check{A}_i\right) = -\frac{\check{p}^i}{p^0}\, C^I_{bc}\ \check{A}^b_i\ \check{q}^c, \tag{41}
$$

$$
\check{G}_3\left(t,\ \check{\bar{p}},\ \check{\tilde{q}},\ f,\ \check{F}^{0i},\ \check{F}_{ij},\ \check{A}_i\right) = \frac{\mathcal{L}(f,\ f,\ \check{p},\ \check{\tilde{q}})}{p^0\ (\check{p}\,)}, \tag{42}
$$

$$\check{G}_4\left(t,\ \check{\bar{p}},\ \check{\tilde{q}},\ \check{f},\ F^{0i},\ \check{F}_{ij},\ \check{A}_i\right) = -\ \Gamma^l_{l0}\check{F}^{0i} + [\check{A}_j,\ \check{F}^{ij}] + \check{J}_i, \tag{43}$$

$$\check{G}_5\left(t,\ \check{\bar{p}},\ \check{\tilde{q}},\ \check{f},\ \check{F}^{0i},\ F_{ij},\ \check{A}_i\right) = -\ [\check{A}_j,\ \check{F}_{0i}] + [\check{A}_i,\ \check{F}_{0j}], \tag{44}$$

$$\check{G}_6\left(t,\ \check{\bar{p}},\ \check{\tilde{q}},\ \check{f},\ \check{F}^{0i},\ \check{F}_{ij},\ \check{A}_i\right) = \check{F}_{0i}, \tag{45}$$

in which

$$\check{J}^i = \int_{\mathbf{R}^3\times\vartheta} \check{p}^i\ \check{q}^I\ \check{f}(t,\ p,\ q)\omega_p\omega_q.$$

Proposition 3. *Let* $t_0 \in [0, T]$, $0 < \eta < 1$ *and* $\left(\check{\bar{p}},\ \check{\tilde{q}},\ \check{f},\ \check{F}^{0i},\ \check{F}_{ij},\ \check{A}_i\right)$ $\in C([t_0, t_0+\eta]; \mathbb{R}^3) \times C([t_0, t_0+\eta], \mathbb{R}^{N-1}) \times \mathbb{E}^{m+3}_{d,\delta}([t_0, t_0+\eta]; \Omega) \times (C([t_0,$ $t_0+\eta]; X_R))^3$ *given. Then, the differential system* (2I) *has a unique solution* $(\bar{p},\ \tilde{q},\ f,\ F^{0i},\ F_{ij},\ A_i) \in C([t_0, t_0+\eta], \mathbb{R}^3) \times C([t_0, t_0+\eta], \mathbb{R}^{N-1}) \times ([t_0,$ $t_0+\eta]; \mathbb{E}^{m+3}_{d,\delta}(\Omega)) \times (C([t_0, t_0+\eta]; X_R))^3$ *such that*

$$(\bar{p},\ \tilde{q},\ f,\ F^{0i},\ F_{ij},\ A_i)(t_0) = \left(\mathring{\bar{p}}_{t_0},\ \mathring{\tilde{q}}_{t_0},\ \mathring{f}_{t_0},\ \mathring{F}^{0i}_{t_0},\ \mathring{F}^{t_0}_{ij},\ \mathring{A}^{t_0}_i\right).$$

Proof. (1) We consider the equation in $\bar{p}$ associated to system (2I). $\check{G}_1$ given by (40) is a continuous function of time. We deduce from (28) that

$$\|\check{G}_1(t,\ \bar{p}_1,\ \check{\tilde{q}},\ \check{f},\ \check{F}^{0i},\ \check{F}_{ij},\ \check{A}_i) - \check{G}_1(t,\ \bar{p}_2,\ \check{\tilde{q}},\ \check{f},\ \check{F}^{0i},\ \check{F}_{ij},\ \check{A}_i)\|$$
$$\leq C_1(\|\bar{p}_1 - \bar{p}_2\|), \tag{46}$$

where using (30), we have

$$C_1 = 2C + e\| \check{F}^{0i} \| + 5g^{ii}\left(1 + \frac{h}{r} + \frac{r}{h}\right) + \| \check{F}_{ij} \| g^{ii} g^{jj} + eg^{ii} g^{jj}.$$

For $t \in [t_0, t_0 + \eta[$, we have $t \leq t_0 + \eta \leq T + 1$, we use assumptions on the metric to increase $5g^{ii}\left(1 + \frac{h}{r} + \frac{r}{h}\right)$, and $g^{ii} g^{jj}$ by a constant dependent on r_0, h_0 and T. Furthermore, since $\check{F}^{0i}$, $\check{F}_{ij}$ and $\check{A}_i$ belong to $C([t_0, t_0 + T]; X_R)$, we finally show that

$$C_1 \leq C_1'(h_0, r_0, R, T, e). \tag{47}$$

By (46) and (47), $\check{G}_1$ is globally Lipschitzian with respect to $\bar{p}$, and the local existence and the uniqueness of a solution $\bar{p}$ for the equation in $\bar{p}$ such that $\bar{p}(t_0) = \mathring{\bar{p}}(t_0)$ is guaranteed by the standard theory of first order differential system.

Following the same way as in Lemma 1 in [8], on $[t_0, t_0 + t]$, $t \in [0, \eta[$, we have

$$\| \bar{p}(t_0 + t) \| \leq \| \check{\bar{p}}(t_0) \| e^{Ct} + \frac{1 + \sqrt{g_{t_0}^{jj} g_{t_0}^{ii}}}{C} eR(e^{Ct} - 1).$$

Since $t_0 \in [0, T]$, we have

$$\| \check{\bar{p}}(t_0) \| \leq \| \bar{p}(0) \| e^{Ct} + \frac{1 + C(h_0, r_0, T)}{C} eR(e^{Ct} - 1).$$

Then

$$\| \bar{p}(t_0 + t) \| \leq \| \bar{p}(0) \| e^{2C(T+1)} + \frac{eR}{C}\left(e^{2C(T+1)} - 1\right)(1 + C(h_0, r_0, T)), \tag{48}$$

which show that, every solution $\bar{p}$ of system (2I) satisfying $\bar{p}(t_0) = \mathring{\bar{p}}(t_0)$ and defined in $[t_0, t_0 + \eta]$ is uniformly bounded.

(2) For the equation in $\tilde{q}$ of system (2I), the function $\check{G}_2$ given by (40) is continuous in t. We deduce from (28) that

$$\|\check{G}_2(t, \check{\tilde{p}}, \tilde{q}_1, \check{f}, \check{F}^{0i}, \check{F}_{ij}, \check{A}_i) - \check{G}_2(t, \check{\tilde{p}}, \tilde{q}_2, \check{f}, \check{F}^{0i}, \check{F}_{ij}, \check{A}_i)\| \leq C_2(\|\tilde{q}_1 - \tilde{q}_2\|), \tag{49}$$

where using (30), we have

$$C_2 = 5Ce\|\check{A}_i\|\left(1 + \frac{h}{r} + \frac{r}{h}\right) + Cg^{ii}(e + \|\check{A}_i\|).$$

We show using the same arguments as before that $C_2 \leq C_2'(h_0, r_0, R, T, e)$, so that (49) becomes

$$\|\check{G}_2(t, \check{\tilde{p}}, \tilde{q}_1, \check{f}, \check{F}^{0i}, \check{F}_{ij}, \check{A}_i) - \check{G}_2(t, \check{\tilde{p}}, \tilde{q}_2, \check{f}, \check{F}^{0i}, \check{F}_{ij}, \check{A}_i)\| \leq C_2'(\|\tilde{q}_1 - \tilde{q}_2\|).$$

which means that G_2 is globally Lipschitzian with respect to $\tilde{q}$. Hence the existence of a unique solution $\tilde{q}$ such that $\tilde{q}(t_0) = \mathring{\tilde{q}}(t_0)$.

(3) We show like in [6, 7, 9] that, the equation in f associated to system (2I), admits a unique solution $f \in \mathbb{E}^{m+3}_{d,\delta}([t_0, t_0 + \eta]; \Omega)$ such that $f(t_0) = \mathring{f}(t_0)$.

(4) We consider respectively equations in F^{0i}, F_{ij}, and A_i of system (2I). $\check{G}_4$, $\check{G}_5$ and $\check{G}_6$ are continuous functions of time. We deduce from (28) that

$$\|\check{G}_4(t, \check{\tilde{p}}, \check{\tilde{q}},, \check{f}, F_1^{0i}, \check{F}_{ij}, \check{A}_i) - \check{G}_4(t, \check{\tilde{p}}, \check{\tilde{q}}, \check{f}, F_2^{0i}, \check{F}_{ij}, \check{A}_i)\| \leq C_4(\|F_1^{0i} - F_2^{0i}\|),$$

$$\| \check{G}_5(t, \check{\bar{p}}, \check{\bar{q}}, \check{f}, \check{F}^{0i}, F_{ij}^1, \check{A}_i) - \check{G}_4(t, \check{\bar{p}}, \check{\bar{q}}, \check{f}, \check{F}^{0i}, F_{ij}^2, \check{A}_i)\|$$

$$\leq C_5(\|F_{ij}^1 - F_{ij}^2\|),$$

$$\| \check{G}_6(t, \check{\bar{p}}, \check{\bar{q}}, \check{f}, \check{F}^{0i}, \check{F}_{ij}, A_i^1) - \check{G}_6(t, \check{\bar{p}}_2, \check{\bar{q}}, \check{f}, \check{F}^{0i}, \check{F}_{ij}, A_i^2)\|$$

$$\leq C_6(\|A_i^1 - A_i^2\|),$$

and using (30), we have

$$\begin{cases} C_4 = e\left(\|\check{f}_1\|_{\mathbb{E}_d^{m+3}(\Omega)} + \Theta_0\right) + \Theta_0 \|\check{f}_1\|_{\mathbb{E}_d^{m+3}(\Omega)} + C[1 + g^{ii}g^{jj}(\|\check{F}_{ij}^1\| + \|\check{A}_j^2\|)], \\ C_5 = Cg_{ii}\left(\|\check{F}^{0i}\| + \|\check{A}_j\|\right) + Cg_{jj}\left(\|\check{F}^{0j}\| + \|\check{A}_i\|\right), \\ C_6 = g_{ii}, \end{cases}$$

where Θ_0 is given by the left hand side of the inequality (48). Given the fact that, $\check{f} \in \mathbf{E}_{d,\delta}^{m+3}([t_0, t_0 + \eta]; \Omega)$, $\check{F}^{0i}$, $\check{F}_{ij}$ and $\check{A}_i$ belong to $C([t_0, t_0 + \eta]; X_R)$ and the boundedness of the metric components, we show that $C_k \leq C'_k(h_0, r_0, R, T, e)$, $k = 4, 5, 6$. Which imply that $\check{G}_4$, $\check{G}_5$, and $\check{G}_6$ are globally Lipschitzian. Hence the existence of solutions F^{0i}, F_{ij}, and A_i, respectively for the last three equation of system (2I) such as

$$F^{0i}(t_0) = \overset{\circ}{F}{}^{0i}(t_0), \quad F_{ij}(t_0) = \overset{\circ}{F}_{ij}(t_0), \quad A_i(t_0) = \overset{\circ}{A}_i(t_0),$$

which completes the proof of the Proposition 3. □

Now consider the set

$$\mathbb{Y}_\eta = C([t_0, t_0 + \eta], \mathbb{R}^3) \times C([t_0, t_0 + \eta], \mathbb{R}^{N-1})$$

$$\times \mathbb{E}^{m+3}_{d,\delta}([t_0, t_0 + \eta]; \Omega) \times (C([t_0, t_0 + \eta]; X_R))^3,$$

it is a complete metric subspace of the Banach space

$$C([t_0, t_0 + \eta], \mathbb{R}^3) \times C([t_0, t_0 + \eta], \mathbb{R}^{N-1}) \times \mathbb{E}^{m+3}_{d}([t_0, t_0 + \eta]; \Omega)$$

$$\times (C([t_0, t_0 + \eta], \mathbb{R}^{3N}))^3.$$

Let the application

$$\chi : \mathbb{Y}_\eta \longrightarrow \mathbb{Y}_\eta; \left(\check{\bar{p}}, \check{\tilde{q}}, \check{f}, \check{F}^{0i}, \check{F}_{ij}, \check{A}_i\right) \mapsto (\bar{p}, \tilde{q}, f, F^{0i}, F_{ij}, A_i), \tag{51}$$

where $(\bar{p}, \tilde{q}, f, F^{0i}, F_{ij}, A_i)$ is the solution of the system (2I) such that

$$(\bar{p}, \tilde{q}, f, F^{0i}, F_{ij}, A_i)(t_0) = \left(\check{\bar{p}}(t_0), \check{\tilde{q}}(t_0), \check{f}(t_0), \check{F}^{0i}(t_0), \check{F}_{ij}(t_0), \check{A}_i(t_0)\right).$$

χ is well defined according to the Proposition 3.

Proposition 4. *Let* $t_0 \in [0, T]$. *There is a real number* $\eta \in]0, 1[$, *independent of* t_0, *such that the system* (I) *admits a unique solution* $(\bar{p}, \tilde{q}, f, F^{0i}, F_{ij}, A_i) \in \mathbb{Y}_\eta$ *such that*

$$(\bar{p}, \tilde{q}, f, F^{0i}, F_{ij}, A_i)(t_0) = \left(\mathring{\bar{p}}_{t_0}, \mathring{\tilde{q}}_{t_0}, \mathring{F}^{0i}_{t_0}, \mathring{F}^{t_0}_{ij}, \mathring{A}^{t_0}_i\right). \tag{52}$$

Proof. We will prove that, there exists a number $\eta \in]0, 1[$, independent of t_0, such that, the map χ defined by (51) is a contraction of the complete metric space $\mathbb{Y}_\eta$ defined by (50), which will then have a unique fixed point $(\bar{p}, \tilde{q}, f, F^{0i}, F_{ij}, A_i)$ solution of the system (I) with the initial data (52) at $t = t_0$.

The differential system (2I) is equivalent to the integral system

$$
(3\mathrm{I}) : \begin{cases}
p^i(t_0+t) = \check{p}^i(t_0) + \displaystyle\int_{t_0}^{t_0+t} \check{G}_1\left(\tau,\, \bar{p},\, \check{\tilde{q}},\, \check{f},\, \check{F}^{0i},\, \check{F}_{ij},\, \check{A}_i\right) d\tau, \\
q^I(t_0+t) = \check{q}^I(t_0) + \displaystyle\int_{t_0}^{t_0+t} \check{G}_2\left(\tau,\, \check{\bar{p}},\, \tilde{q},\, \check{f},\, \check{F}^{0i},\, \check{F}_{ij},\, \check{A}_i\right) d\tau, \\
f(t_0+t) = \check{f}(t_0) + \displaystyle\int_{t_0}^{t_0+t} \check{G}_3\left(\tau,\, \check{\bar{p}},\, \check{\tilde{q}},\, f,\, \check{F}^{0i},\, \check{F}_{ij},\, \check{A}_i\right) d\tau, \\
F^{0i}(t_0+t) = \check{F}^{0i}(t_0) + \displaystyle\int_{t_0}^{t_0+t} \check{G}_4\left(\tau,\, \check{\bar{p}},\, \check{\tilde{q}},\, \check{f},\, F^{0i},\, \check{F}_{ij},\, \check{A}_i\right) d\tau, \\
F_{ij}(t_0+t) = \check{F}_{ij}(t_0) + \displaystyle\int_{t_0}^{t_0+t} \check{G}_5\left(\tau,\, \check{\bar{p}},\, \check{\tilde{q}},\, \check{f},\, \check{F}^{0i},\, F_{ij},\, \check{A}_i\right) d\tau, \\
A_i(t_0+t) = \check{A}_i(t_0) + \displaystyle\int_{t_0}^{t_0+t} \check{G}_6\left(\tau,\, \check{\bar{p}},\, \check{\tilde{q}},\, \check{f},\, \check{F}^{0i},\, \check{F}_{ij},\, A_i\right) d\tau, \\
t \in [0,\, \eta[.
\end{cases}
$$

Let $\left(\check{\bar{p}}_k,\, \check{\tilde{q}}_k,\, \check{f}_k,\, \check{F}^{0i}{}_k,\, \check{F}_{ij}^{\,k},\, \check{A}_i^k\right) \in \mathbb{Y}_\eta,\ k = 1,\, 2.$ According to the Proposition 6, they exist $\left(\bar{p}_k,\, \tilde{q}_k,\, f_k,\, F_k^{0i},\, F_{ij}^k,\, A_i^k\right),\ k = 1,\, 2$ solutions of the system (3I).

We are writing the system (3I) for $k = 1$ and $k = 2$ then, making the difference, we get

$$
(p_1^i - p_2^i)(t_0+t) = \int_{t_0}^{t_0+t} \Big[\check{G}_1\left(\tau,\, \bar{p}_1,\, \check{\tilde{q}}_1,\, \check{f}_1,\, \check{F}_1^{0i},\, \check{F}_{ij}^1,\, \check{A}_i^1\right) - \check{G}_1\left(\tau,\, \bar{p}_2,\, \check{\tilde{q}}_2,\, \check{f}_2,\, \check{F}_2^{0i},\, \check{F}_{ij}^2,\, \check{A}_i^2\right) \Big] d\tau, \tag{53}
$$

$$(q_1^I - q_2^I)(t_0 + t) = \int_{t_0}^{t_0+t} \left[\check{G}_2 \left(\tau,\ \check{\bar{p}}_1,\ \tilde{q}_1,\ \check{f}_1,\ \check{F}_1^{0i},\ \check{F}_{ij}^1,\ \check{A}_i^1 \right) - \check{G}_2 \left(\tau,\ \check{\bar{p}},\ \tilde{q}_2,\ \check{f}_2,\ \check{F}_2^{0i},\ \check{F}_{ij}^2,\ \check{A}_i^2 \right) \right] d\tau, \tag{54}$$

$$(f_1 - f_2)(t_0 + t) = \int_{t_0}^{t_0+t} \left[\check{G}_3 \left(\tau,\ \check{\bar{p}}_1,\ \check{\tilde{q}}_1,\ f_1,\ \check{F}_1^{0i},\ \check{F}_{ij}^1,\ \check{A}_i^1 \right) - \check{G}_3 \left(\tau,\ \check{\bar{p}}_2,\ \check{\tilde{q}}_2,\ f_2,\ \check{F}_2^{0i},\ \check{F}_{ij}^2,\ \check{A}_i^2 \right) \right] d\tau, \tag{55}$$

$$(F_1^{0i} - F_2^{0i})(t_0 + t) = \int_{t_0}^{t_0+t} \left[\check{G}_4 \left(\tau,\ \check{\bar{p}}_1,\ \check{\tilde{q}}_1,\ \check{f}_1,\ F_1^{0i},\ \check{F}_{ij}^1,\ \check{A}_i^1 \right) - \check{G}_4 \left(\tau,\ \check{\bar{p}}_2,\ \check{\tilde{q}}_2,\ \check{f}_2,\ F_2^{0i},\ \check{F}_{ij}^2,\ \check{A}_i^2 \right) \right] d\tau, \tag{56}$$

$$(F_{ij}^1 - F_{ij}^2)(t_0 + t) = \int_{t_0}^{t_0+t} \left[\check{G}_5 \left(\tau,\ \check{\bar{p}}_1,\ \check{\tilde{q}}_1,\ \check{f}_1,\ \check{F}_1^{0i},\ F_{ij}^1,\ \check{A}_i^1 \right) - \check{G}_5 \left(\tau,\ \check{\bar{p}}_2,\ \check{\tilde{q}}_2,\ \check{f}_2,\ \check{F}_2^{0i},\ F_{ij}^2,\ \check{A}_i^2 \right) \right] d\tau, \tag{57}$$

$$(A_i^1 - A_i^2)(t_0 + t) = \int_{t_0}^{t_0+t} \left[\check{G}_6 \left(\tau, \check{\bar{p}}_1, \check{\tilde{q}}_1, \check{f}^1, \check{F}_1^{0i}, \check{F}_{ij}^1, A_i^1 \right) - \check{G}_6 \left(\tau, \check{\bar{p}}_2, \check{\tilde{q}}_2, \check{f}_2, \check{F}_2^{0i}, \check{F}_{ij}^2, A_i^2 \right) \right] d\tau. \tag{58}$$

Since $\left(\check{\bar{p}}_k, \check{\tilde{q}}_k, \check{f}_k, \check{F}^{0i}{}_k, \check{F}_{ij}{}^k, \check{A}_i{}^k \right) \in \mathbb{Y}_\eta$, we get from the Proposition 2

$$\| \check{G}_1 \left(\tau, \bar{p}_1, \check{\tilde{q}}_1, \check{f}_1, \check{F}_1^{0i}, \check{F}_{ij}^1, \check{A}_i^1 \right) - \check{G}_1 \left(\tau, \bar{p}_2, \check{\tilde{q}}_2, \check{f}_2, \check{F}_2^{0i}, \check{F}_{ij}^2, \check{A}_i^2 \right) \|$$

$$\leq C_1' \left(\| \bar{p}_1 - \bar{p}_2 \| + \| \check{\tilde{q}}_1 - \check{\tilde{q}}_2 \| + \| \check{f}_1 - \check{f}_2 \| + \| \check{F}_1^{0i} - \check{F}_2^{0i} \| + \| \check{F}_{ij}^1 - \check{F}_{ij}^2 \| + \| \check{A}_i^1 - \check{A}_i^2 \| \right), \tag{59}$$

$$\| \check{G}_2 \left(\tau, \check{\bar{p}}_1, \tilde{q}_1, \check{f}_1, \check{F}_1^{0i}, \check{F}_{ij}^1, \check{A}_i^1 \right) - \check{G}_2 \left(\tau, \check{\bar{p}}_2, \tilde{q}_2, \check{f}_2, \check{F}_2^{0i}, \check{F}_{ij}^2, \check{A}_i^2 \right) \|$$

$$\leq C_2' \left(\| \check{\bar{p}}_1 - \check{\bar{p}}_2 \| + \| \tilde{q}_1 - \tilde{q}_2 \| + \| \check{f}_1 - \check{f}_2 \| + \| \check{F}_1^{0i} - \check{F}_2^{0i} \| + \| \check{F}_{ij}^1 - \check{F}_{ij}^2 \| + \| \check{A}_i^1 - \check{A}_i^2 \| \right), \tag{60}$$

$$\| \check{G}_3 \left(\tau, \check{\bar{p}}_1, \check{\tilde{q}}_1, f_1, \check{F}_1^{0i}, \check{F}_{ij}^1, \check{A}_i^1 \right) - \check{G}_3 \left(\tau, \check{\bar{p}}_2, \check{\tilde{q}}_2, f_2, \check{F}_2^{0i}, \check{F}_{ij}^2, \check{A}_i^2 \right) \|$$

$$\leq C_3' \left(\| \check{\bar{p}}_1 - \check{\bar{p}}_2 \| + \| \check{\tilde{q}}_1 - \check{\tilde{q}}_2 \| + \| f_1 - f_2 \| + \| \check{F}_1^{0i} - \check{F}_2^{0i} \| + \| \check{F}_{ij}^1 - \check{F}_{ij}^2 \| + \| \check{A}_i^1 - \check{A}_i^2 \| \right), \tag{61}$$

$$\|\check{G}_4\left(\tau,\ \check{\bar{p}}_1,\ \check{\tilde{q}}_1,\ \check{f}_1,\ F_1^{0i},\ \check{F}_{ij}^1,\ \check{A}_i^1\right) - \check{G}_4\left(\tau,\ \check{\bar{p}}_2,\ \check{\tilde{q}}_2,\ \check{f}_2,\ F_2^{0i},\ \check{F}_{ij}^2,\ \check{A}_i^2\right)\|$$

$$\leq C_4'\left(\|\check{\bar{p}}_1 - \check{\bar{p}}_2\| + \|\check{\tilde{q}}_1 - \check{\tilde{q}}_2\| + \|\check{f}_1 - \check{f}_2\| + \|F_1^{0i} - F_2^{0i}\| + \|\check{F}_{ij}^1 - \check{F}_{ij}^2\| + \|\check{A}_i^1 - \check{A}_i^2\|\right), \tag{62}$$

$$\|\check{G}_5\left(\tau,\ \check{\bar{p}}_1,\ \check{\tilde{q}}_1,\ \check{f}_1,\ \check{F}_1^{0i},\ F_{ij}^1,\ \check{A}_i^1\right) - \check{G}_5\left(\tau,\ \check{\bar{p}}_2,\ \check{\tilde{q}}_2,\ \check{f}_2,\ \check{F}_2^{0i},\ F_{ij}^2,\ \check{A}_i^2\right)\|$$

$$\leq C_5'\left(\|\check{\bar{p}}_1 - \check{\bar{p}}_2\| + \|\check{\tilde{q}}_1 - \check{\tilde{q}}_2\| + \|\check{f}_1 - \check{f}_2\| + \|\check{F}_1^{0i} - \check{F}_2^{0i}\| + \|F_{ij}^1 - F_{ij}^2\| + \|\check{A}_i^1 - \check{A}_i^2\|\right), \tag{63}$$

$$\|\check{G}_6\left(\tau,\ \check{\bar{p}}_1,\ \check{\tilde{q}}^1,\ \check{f}^1,\ \check{F}_1^{0i},\ \check{F}_{ij}^1,\ A_i^1\right) - \check{G}_6\left(\tau,\ \check{\bar{p}}_2,\ \check{\tilde{q}}_2,\ \check{f}_2,\ \check{F}_2^{0i},\ \check{F}_{ij}^2,\ A_i^2\right)\|$$

$$\leq C_6'\left(\|\check{\bar{p}}_1 - \check{\bar{p}}_2\| + \|\check{\tilde{q}}_1 - \check{\tilde{q}}_2\| + \|\check{f}_1 - \check{f}_2\| + \|\check{F}_1^{0i} - \check{F}_2^{0i}\| + \|\check{F}_{ij}^1 - \check{F}_{ij}^2\| + \|A_i^1 - A_i^2\|\right), \tag{64}$$

where constants $C_l' = C_l'(h_0,\ r_0,\ R,\ T,\ e)$, $l = 1, \ldots, 6$ are independents of t_0 and have been present in the proof of Proposition 3.

From inequalities (59) to (64), setting $\|f_1 - f_2\| = \|f_1 - f_2\|_{\mathbb{E}_d^{m+3}(\Omega)}$, we deduce from relations (53) to (58) using $\|\cdot\|_1$ and the fact that $t \in [0,\ \eta]$:

$$\|\bar{p}_1 - \bar{p}_2\|_1 \leq C_1'\eta$$

$$\times\left(\|\bar{p}_1 - \bar{p}_2\|_1 + \|\check{\tilde{q}}_1 - \check{\tilde{q}}_2\|_1 + \|\check{f}_1 - \check{f}_2\| + \|\check{F}_1^{0i} - \check{F}_2^{0i}\|_1 + \|\check{F}_{ij}^1 - \check{F}_{ij}^2\|_1 + \|\check{A}_i^1 - \check{A}_i^2\|_1\right), \tag{65}$$

$$\| \tilde{q}_1 - \tilde{q}_2 \|_1 \leq C_2' \eta$$

$$\times \left(\| \check{\tilde{p}}_1 - \check{\tilde{p}}_2 \|_1 + \| \tilde{q}_1 - \tilde{q}_2 \|_1 + \| \check{f}_1 - \check{f}_2 \| + \| \check{F}_1^{0i} - \check{F}_2^{0i} \|_1 + \| \check{F}_{ij}^1 - \check{F}_{ij}^2 \|_1 + \| \check{A}_i^1 - \check{A}_i^2 \|_1 \right), \tag{66}$$

$$\| f_1 - f_2 \|_1 \leq C_3' \eta$$

$$\times \left(\| \check{\tilde{p}}_1 - \check{\tilde{p}}_2 \|_1 + \| \check{\tilde{q}}_1 - \check{\tilde{q}}_2 \|_1 + \| f_1 - f_2 \| + \| \check{F}_1^{0i} - \check{F}_2^{0i} \|_1 + \| \check{F}_{ij}^1 - \check{F}_{ij}^2 \|_1 + \| \check{A}_i^1 - \check{A}_i^2 \|_1 \right), \tag{67}$$

$$\| F_1^{0i} - F_2^{0i} \|_1 \leq C_4' \eta$$

$$\times \left(\| \check{\tilde{p}}_1 - \check{\tilde{p}}_2 \|_1 + \| \check{\tilde{q}}_1 - \check{\tilde{q}}_2 \|_1 + \| \check{f}_1 - \check{f}_2 \| + \| F_1^{0i} - F_2^{0i} \|_1 + \| \check{F}_{ij}^1 - \check{F}_{ij}^2 \|_1 + \| \check{A}_i^1 - \check{A}_i^2 \|_1 \right), \tag{68}$$

$$\| F_{ij}^1 - F_{ij}^2 \|_1 \leq C_5' \eta$$

$$\times \left(\| \check{\tilde{p}}_1 - \check{\tilde{p}}_2 \|_1 + \| \check{\tilde{q}}_1 - \check{\tilde{q}}_2 \|_1 + \| \check{f}_1 - \check{f}_2 \| + \| \check{F}_1^{0i} - \check{F}_2^{0i} \|_1 + \| F_{ij}^1 - F_{ij}^2 \|_1 + \| \check{A}_i^1 - \check{A}_i^2 \|_1 \right), \tag{69}$$

$$\| A_i^1 - A_i^2 \|_1 \leq C_6' \eta$$

$$\times \left(\| \check{\tilde{p}}_1 - \check{\tilde{p}}_2 \|_1 + \| \check{\tilde{q}}_1 - \check{\tilde{q}}_2 \|_1 + \| \check{f}_1 - \check{f}_2 \| + \| \check{F}_1^{0i} - \check{F}_2^{0i} \|_1 + \| \check{F}_{ij}^1 - \check{F}_{ij}^2 \|_1 + \| A_i^1 - A_i^2 \|_1 \right). \tag{70}$$

Adding member to member inequalities (65) to (70), we get

$$\| \check{\bar{p}}_1 - \check{\bar{p}}_2 \|_1 + \| \tilde{q}_1 - \tilde{q}_2 \|_1 + \| f_1 - f_2 \| + \| F_1^{0i} - F_2^{0i} \|_1 + \| F_{ij}^1 - F_{ij}^2 \|_1 + \| A_i^1 - A_i^2 \|_1$$

$$\leq \left(\sum_{l=1}^{6} C'_l \right) \eta \, (\| \check{\bar{p}}_1 - \check{\bar{p}}_2 \|_1 + \| \tilde{q}_1 - \tilde{q}_2 \|_1 + \| f_1 - f_2 \| + \| F_1^{0i} - F_2^{0i} \|_1$$

$$+ \| F_{ij}^1 - F_{ij}^2 \|_1 + \| A_i^1 - A_i^2 \|_1)$$

$$+ \left(\sum_{l=1}^{6} C'_l \right) \eta \, (\| \check{\bar{p}}_1 - \check{\bar{p}}_2 \|_1 + \| \check{\tilde{q}}_1 - \check{\tilde{q}}_2 \|_1 + \| \check{f}_1 - \check{f}_2 \| + \| \check{F}_1^{0i} - \check{F}_2^{0i} \|_1$$

$$+ \| \check{F}_{ij}^1 - \check{F}_{ij}^2 \|_1 + \| \check{A}_i^1 - \check{A}_i^2 \|_1). \tag{71}$$

Then, if we take η such that

$$0 < \eta < \inf \{ 1, \frac{1}{12(C'_1 + C'_2 + C'_3 + C'_4 + C'_5 + C'_6)} \}, \tag{72}$$

implies in particular $0 < \eta (C'_1 + C'_2 + C'_3 + C'_4 + C'_5 + C'_6) < \frac{1}{6}$. From which we deduce, by sending the first term of the right hand side of (71) to the left hand side

$$\frac{11}{12} (\| \bar{p}_1 - \bar{p}_2 \|_1 + \| \tilde{q}_1 - \tilde{q}_2 \|_1 + \| f_1 - f_2 \|_1 + \| F_1^{0i} - F_2^{0i} \|_1$$

$$+ \| F_{ij}^1 - F_{ij}^2 \|_1 + \| A_i^1 - A_i^2 \|_1)$$

$$\leq \frac{1}{12} \Bigg(\| \check{\bar{p}}_1 - \check{\bar{p}}_2 \|_1 + \| \check{\tilde{q}}_1 - \check{\tilde{q}}_2 \|_1 + \| \check{f}_1 - \check{f}_2 \| + \| \check{F}_1^{0i} - \check{F}_2^{0i} \|_1$$

$$+ \| \check{F}_{ij}^1 - \check{F}_{ij}^2 \|_1 + \| \check{A}_i^1 - \check{A}_i^2 \|_1 \Bigg),$$

which gives

$$(\| \bar{p}_1 - \bar{p}_2 \|_1 + \| \tilde{q}_1 - \tilde{q}_2 \|_1 + \| f_1 - f_2 \|_1 + \| F_1^{0i} - F_2^{0i} \|_1$$

$$+ \| F_{ij}^1 - F_{ij}^2 \|_1 + \| A_i^1 - A_i^2 \|_1)$$

$$\leq \frac{1}{11} \Big(\| \check{\bar{p}}_1 - \check{\bar{p}}_2 \|_1 + \| \check{\tilde{q}}_1 - \check{\tilde{q}}_2 \|_1 + \| \check{f}_1 - \check{f}_2 \|_1 + \| \check{F}_1^{0i} - \check{F}_2^{0i} \|_1$$

$$+ \| \check{F}_{ij}^1 - \check{F}_{ij}^2 \|_1 + \| \check{A}_i^1 - \check{A}_i^2 \|_1 \Big).$$

Then, the map χ defined by (51) is a contracting map in the complete metric space $\mathbb{Y}_\eta$ which then has a unique fixed point $(\bar{p}, \tilde{q}, f, F^{0i}, F_{ij}, A_i)$, solution of the system (3I) and hence of the differential system (I) such that

$$(\bar{p}, \tilde{q}, f, F^{0i}, F_{ij}, A_i)(t_0) = \left(\mathring{\bar{p}}_{t_0}, \mathring{\tilde{q}}_{t_0}, \mathring{f}_{t_0}, \mathring{F}_{t_0}^{0i}, \mathring{F}_{ij}^{t_0}, \mathring{A}_i^{t_0} \right).$$

This completes the proof of Proposition 4. □

Based on the method presented in Subsection 5.1, we have proved the following result.

Theorem 3. *Let* $\bar{p}_0 \in \mathbb{R}^3$, $\tilde{q}_0 \in \mathbb{R}^{N-1}$, $f_0 \in \mathbb{E}_{d,\delta}^{m+3}(\Omega)$ *and* E^i, Φ_{ij}, $a_i \in X_R$. *Then*

(1) *The differential system* (I) *has a unique global solution* $(\bar{p}, \tilde{q}, f, F^{0i}, F_{ij}, A_i)$ *defined all over the interval* $[0, +\infty[$, *and such that*

$$(\bar{p}, \tilde{q}, f, F^{0i}, F_{ij}, A_i)(0) = (\bar{p}_0, \tilde{q}_0, f, E^i, \Phi_{ij}, a_i).$$

(2) *The Yang-Mills-Boltzmann system* (15)-(16) *has a unique solution* (f, F, A) *defined all over the interval* $[0, +\infty[$ *and satisfying*

$$f(0) = f_0, \quad F^{0i}(0) = E^i, \quad F_{ij}(0) = \Phi_{ij}, \quad A_i(0) = a_i.$$

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