Research and Communications in Mathematics and Mathematical Sciences
Vol. 13, Issue 1, 2021, Pages 49-62
ISSN 2319-6939
Published Online on April 08, 2021

http://jyotiacademicpress.org

A CLASS OF SYSTEMS OF LINEAR FREDHOLM INTEGRAL EQUATIONS OF THE THIRD KIND WITH MULTIPOINT SINGULARITIES IN THE SEMIAXIS

RUHIDIN ASANOV and AVYT ASANOV

Kyrgyz State Technical University
Bishkek, 720044
Kyrgyzstan
e-mail: ruhidin_asanov@yahoo.com

Kyrgyz-Turkish Manas University
Bishkek, 720044
Kyrgyzstan
e-mail: avyt.asanov@manas.edu.kg
avyt.asanov@mail.ru

Abstract

A new approach is used to show that the solution for one class of systems of linear Fredholm integral equations of the third kind with multipoint singularities in the semiaxis is equivalent to the solution of systems of linear Fredholm integral equations of the second kind in the semiaxis with additional conditions. The existence, nonexistence, uniqueness and nonuniqueness of solutions to systems of linear Fredholm integral equations of the third kind with multipoint singularities in the semiaxis are analyzed.

2020 Mathematics Subject Classification: 45B05, 45A05, 45G10.

Keywords and phrases: linear, system, integral equations, Fredholm, third kind, multipoint singularities, semiaxis.

Communicated by Haik G. Ghazaryan.

Received February 8, 2021; Revised February 23, 2021

1. Introduction

Consider the system of linear integral equations of the third kind

$$p_i(x)u_i(x) = \lambda \sum_{j=1}^{n} \int_a^\infty k_{ij}(x, y)u_j(y)dy + f_i(x), \ x \in [a, \infty), \tag{1}$$

where $i = 1, 2, \dots, n$, $p_i(x)$ and $f_i(x)$ are given continuous functions on $[a, \infty)$, $k_{ij}(x, y)$ are given continuous functions in $G = [a, \infty) \times [a, \infty)$, $u_i(x)$ are the sought functions on $[a, \infty)$, $i, j = 1, 2, \dots, n$; and λ is a real parameter. There exists $t \in \{1, 2, \dots, n\}$ such that, for all $i = t, t+1, \dots, n$ and $l = 1, 2, \dots, m(i)$, $p_i(x_{il}) = 0$, where $x_{il} \in [a, \infty)$ and for all $i = 1, 2, \dots, t-1$, $p_i(x) = 1$ for all $x \in [a, \infty)$.

Various issues concerning the theory of integral equations were studied in [1-14]. Specifically, in [12], Lavrent'ev constructed regularizing operators for solving linear Fredholm integral equations of the first kind. In [7], uniqueness theorems were proved for systems of nonlinear Volterra integral equations of the third kind and regularizing operators in the sense of Lavrent'ev were constructed. In [9], a new approach was used to analyze the existence and uniqueness of solutions to systems (1) in the case where $x_{il} = a$ for all $i = t, \dots, n$ and $l = 1, 2, \dots, m(i)$.

Here, a new approach, we prove that the solution of system (1) in the space $L_{2,n}(a, \infty)$ is equivalent to the solution of systems of linear integral equations of the second kind with the some integral conditions. Here $L_{2,n}(a, \infty)$ denote the space of all n-dimensional vector-functions with elements of $L_2(a, \infty)$.

Let $C_n[a, \infty)$ denote the space of all n-dimensional vector functions with elements from $C[a, \infty)$. Here $C[a, \infty)$ denote the space of all continuous functions on $[a, \infty)$. For vectors $u = (u_1, \ldots, u_n)^T$ and $v = (v_1, \ldots, v_n)^T \in R^n$, the inner product is defined by the formula

$$\langle u, v\rangle = u_1v_1 + \ldots + u_nv_n.$$

Throughout this paper, we assume that

$$p_i(x) = \prod_{l=1}^{m(i)} p_{i,l}(x),\ p_{i,l}(x_{il}) = 0,\ p_{i,l}(x) \in C[a, \infty), \tag{2}$$

$$i = t, \ldots, n,\ l = 1, \ldots, m(i).$$

$p_{i,l}(x) \neq 0$ for $x \in [a, \infty)$ and $x \neq x_{il} \in [a, \infty)$, where $m(i) \in N$.

Setting $x = x_{i1}$, we find from (1) that

$$\lambda \sum_{j=1}^{n} \int_a^\infty k_{ij}(x_{i1}, y)u_j(y)dy + f_i(x_{i1}) = 0. \tag{3}$$

$$i = t, t+1, \ldots, n.$$

Subtracting (3) from (1) yields

$$p_i(x)u_i(x) = \lambda \sum_{j=1}^{n} \int_a^\infty [k_{ij}(x, y) - k_{ij}(x_{i1}, y)]u_j(y)dy + f_i(x) - f_i(x_{i1}).$$

$$x \in [a, \infty),\ i = t, t+1, \ldots, n. \tag{4}$$

Assume that the following conditions hold.

(a) For all $i = 1, 2, \ldots, t-1,\ t \leq n,\ j = 1, 2, \ldots, n,\ k_{ij}(x, y) \in C(G) \cap L_2(G)$.

(b) For all $i = t, t+1, \ldots, n$, $j = 1, 2, \ldots, n$, $l = 1, 2, \ldots, m(i)$, $k_{ij,l}(x, y) \in C(G)$, for fixed $x \in [a, \infty)$, $k_{ij,l}(x, y) \in L_2(a, \infty)$, $k_{ij,m(i)}(x, y) \in L_2(G)$, where $k_{ij,0}(x, y) = k_{ij}(x, y)$.

$$k_{ij,l}(x, y) = \frac{1}{p_{i,l}(x)}[k_{ij,l-1}(x, y) - k_{ij,l-1}(x_{il}, y)], \ (x, y) \in G.$$

(c) For all $i = 1, 2, \ldots, t-1$, $t \leq n$, $f_i(x) \in C[a, \infty) \cap L_2(a, \infty)$, for all $i = t, t+1, \ldots, n$, $l = 1, 2, \ldots, m(i)$, $F_{i,l}(x) \in C[a, \infty)$, $F_{i,m(i)}(x) \in L_2(a, \infty)$, where $F_{i,0}(x) = f_i(x)$,

$$F_{i,l}(x) = \frac{1}{p_{i,l}(x)}[f_{i,l-1}(x) - f_{i,l-1}(x_{il})], \ x \in (a, \infty).$$

Theorem. *Let conditions* (2), (a), (b) *and* (c) *satisfied. Then the solution of linear integral equations* (1) *in* $L_{2,n}(a, \infty) \cap C_n[a, \infty)$ *is equivalent to the solution of the following system of linear integral equations of the second kind*:

$$\begin{cases} u_i(x) = \lambda \sum_{j=1}^{n} \int_a^{\infty} k_{ij}(x, y) u_j(y) dy + f_i(x), \ i = 1, 2, \ldots, t-1, \\ u_i(x) = \lambda \sum_{j=1}^{n} \int_a^{\infty} k_{ij,m(i)}(x, y) u_j(y) dy + F_{i,m(i)}(x), \\ i = t, t+1, \ldots, n, \ x \in [a, \infty), \end{cases} \tag{5}$$

with conditions

$$\lambda \int_a^{\infty} \langle k_{i,l-1}(x_{il}, y) u(y) \rangle dy + F_{i,l-1}(x_{il}) = 0, \tag{6}$$

where $i = t, t+1, \ldots, n$, $l = 1, 2, \ldots, m(i)$, *and*

$$k_{i,l-1}(x_{il}, y) = (k_{i1,l-1}(x_{il}, y), \ldots, k_{in,l-1}(x_{il}, y))^T.$$

Proof. First, let $u(t) = (u_1(t), \ldots, u_n(t))^T \in C_n[a, \infty) \cap L_{2,n}(a, \infty)$ be a solution of system (1). Then identities (3) and (4) hold. Taking into account (2) and conditions (b) and (c) we find from (4) that

$$\prod_{i=2}^{m(i)} p_{i,l}(x)u_i(x) = \lambda\sum_{j=1}^{n}\int_a^{\infty} k_{ij,1}(x,\ y)u_j(y)dy + F_{i,1}(x),$$

$$x \in [a, \infty),\ i = t,\ t+1, \ldots, n. \tag{7}$$

In system (7), for i satisfying $m(i) = 1,$ we have

$$\prod_{i=2}^{m(i)} p_{i,l}(x) = 1,\ x \in [a, \infty).$$

Furthermore, in system (7), for equations for which $m(i) \geq 2,$ setting $x = x_{i2},$ we have

$$\lambda\sum_{j=1}^{n}\int_a^{\infty} k_{ij,1}(x_{i2},\ y)u_j(y)dy + F_{i,1}(x_{i2}) = 0, \tag{8}$$

$$i = t,\ t+1, \ldots, n,\ m(i) \geq 2.$$

Subtracting (8) from (7) and taking into account conditions (b) and (c), we obtain

$$\prod_{i=3}^{m(i)} p_{i,l}(x)u_i(x) = \lambda\sum_{j=1}^{n}\int_a^{\infty} k_{ij,2}(x,\ y)u_j(y)dy + F_{i,2}(x),\ x \in [a, \infty), \tag{9}$$

where $i = t, t+1, \ldots, n.$ and $m(i) \geq 2.$ In system (9), for i such that $m(i) = 2,$ we assume that

$$\sum_{l=3}^{m(i)} p_{i,l}(x) = 1,\ x \in [a, \infty).$$

Continuing this process, we see that the vector function $u(x)$ solves systems (5) with conditions (6).

Conversely, let $u(t) = (u_1(t), \ldots, u_n(t))^T \in C_n[a, \infty) \cap L_{2,n}(a, \infty)$ be a solution of system (5) with conditions (6). In system (5), we consider the i-th equation at $i = t, t+1, \ldots, n.$ Multiplying it by $p_{i,m(i)}(x)$ and taking into account condition (6) at $l = m(i),$ we obtain

$$p_{i,m(i)}(x)u_i(x) = \lambda \sum_{j=1}^{n} \int_a^\infty k_{ij,m(i)-1}(x, y)u_j(y)dy + F_{i,m(i)-1}(x),$$

$$x \in [a, \infty),\ i = t, t+1, \ldots, n. \tag{10}$$

Multiplying the i-th equation in system (10) by $p_{i,m(i)-1}(x)$ and taking into account condition (6), for $l = m(i) - 1,$ we have

$$p_{i,m(i)-1}(x)p_{i,m(i)}(x)u_i(x) = \lambda \sum_{j=1}^{n} \int_a^\infty k_{ij,m(i)-2}(x, y)u_j(y)dy$$

$$+ F_{i,m(i)-2}(x),\ x \in [a, \infty),$$

$$i = t, t+1, \ldots, n,\ m(i) \geq 2. \tag{11}$$

Continuing this process with respect to (11) and taking into account condition (6), we see that $u(t)$ solves system (1). Theorem is proved.

Corollary 1. *Let conditions* (2), (a), (b) *and* (c) *be satisfied, and* $\frac{1}{\lambda}$ *be a real number that is not an eigenvalue of the matrix kernel* $k(x, y),$ *where*

$$k(x, y) = \begin{pmatrix} k_{11}(x, y) & k_{12}(x, y) & \ldots k_{1n}(x, y) \\ \ldots & \ldots & \ldots \\ k_{t-1,1}(x, y) & k_{t-1,2}(x, y) & \ldots k_{t-1,n}(x, y) \\ k_{t1,m(t)}(x, y) & k_{t2,m(t)}(x, y) & \ldots k_{tn,m(t)}(x, y) \\ \ldots & \ldots & \ldots \\ k_{n1,m(n)}(x, y) & k_{n2,m(n)}(x, y) & \ldots k_{nn,m(n)}(x, y) \end{pmatrix}.$$

Then the following assertions hold:

(i) *The solution of system* (1) *is unique in* $L_{2,n}(a, \infty) \cap C_n[a, \infty)$.

(ii) *The solution of system* (5) *can be written as*

$$u(x) = F(x) + \int_a^{\infty} R(x, y, \lambda)F(y)dy, \; x \in [a, \infty), \tag{13}$$

where $F(x) = (f_1(x), \ldots, f_{t-1}(x), F_{t, m(t)}(x), \ldots, F_{n, m(n)}(x))^T$.

$u(x) = (u_1(x), u_2(x), \ldots, u_n(x))^T$, $R(x, y, \lambda)$ *is the matrix resolvent of the matrix kernel* $\lambda k(x, y)$, *and* $k(x, y)$ *is defined by formula* (12). *In this case, the vector function* $u(x)$ *defined by* (13) *is a solution of system* (1) *if and only if* $u(x)$ *satisfies condition* (6).

Corollary 2. *Let conditions* (2), (a), (b) *and* (c) *be satisfied, and* $\frac{1}{\lambda}$ *be a real number that is an eigenvalue of the matrix kernel* $k(x, y)$, *and the vector functions* $\varphi_1(x), \varphi_2(x), \ldots, \varphi_q(x)$ *and* $\psi_1(x), \psi_2(x), \ldots, \psi_q(x)$ *be the eigenfunctions of the matrix kernels* $k(x, y)$ *and* $(k(y, x))^T$ *corresponding to the eigenvalue* $\frac{1}{\lambda}$, *where* $k(x, y)$ *is defined by* (12). *Then the following assertions hold*:

(i) *If there exists* $i \in \{1, 2, \ldots, q\}$ *such that*

$$\int_a^{\infty} \langle \psi_i(x), F(x) \rangle dx \neq 0,$$

the system (1) *has no solution in* $L_{2,n}(a, \infty) \cap C_n[a, \infty)$.

(ii) *If for all* $i \in \{1, 2, \ldots, q\}$

$$\int_a^{\infty} \langle \psi_i(x), F(x) \rangle dx = 0,$$

and $r(A) \neq r(B)$, *where* A *is an* $s \times q$ *matrix and* $s = \sum_{i=t}^{n} m(i)$,

$$A = \begin{pmatrix} A_t \\ \cdots \\ A_n \end{pmatrix}, \; Q = \begin{pmatrix} b_t \\ \cdots \\ b_n \end{pmatrix}, \; B = (A, Q), \; A_i = \begin{pmatrix} a_{i1,1} \dots a_{i1,q} \\ \cdots\cdots\cdots\cdots\cdots \\ a_{im(i),1} \dots a_{im(i),q} \end{pmatrix}, \tag{14}$$

$$b_i = \begin{pmatrix} b_{i1} \\ \cdots \\ b_{im(i)} \end{pmatrix}, \; a_{il,j} = \lambda \int_a^{\infty} \langle k_{i,l-1}(x_{il}, y), \varphi_j(y) \rangle dy, \; i = t, \dots, n,$$

$$l = 1, \dots, m(i), \; b_{il} = -F_{i,l-1}(x_{il}) - \lambda \int_a^{\infty} \langle k_{i,l-1}(x_{il}, y), \varphi_0(y) \rangle dy, \; j = 1, \dots, q,$$

$r(A)$ *is the rank of the matrix* A *and* $\varphi_0(x)$ *is a partial solution of system* (5), *then system* (1) *has no solution in* $L_{2,n}(a, \infty) \cap C_n[a, \infty)$.

(iii) *If for all* $i \in \{1, 2, \dots, q\}$

$$\int_a^{\infty} \langle \psi_i(x), F(x) \rangle dx = 0,$$

and $r(A) = r(B) = q$, *then system* (1) *has a unique solution in* $L_{2,n}(a, \infty) \cap C_n[a, \infty)$ *and this solution can be represented as*

$$u(x) = \varphi_0(x) + \sum_{j=1}^{q} c_j \varphi_j(x). \tag{15}$$

Here, $c = (c_1, c_2, \dots, c_q)^T$ *is the only vector satisfying the system*

$$Ac = Q, \tag{16}$$

where the matrices A *and* Q *are defined by formula* (14).

(iv) *If for all* $i \in \{1, 2, \ldots, q\}$

$$\int_a^{\infty} \langle \psi_i(x), F(x) \rangle dx = 0,$$

and $r = r(A) = r(B) < q,$ *then system* (1) *has a solution in* $L_{2,n}(a, \infty) \cap C_n[a, \infty)$ *and this solution is given by* (15), *where the vector* $c = (c_1, c_2, \ldots, c_n)^T$ *depends on* $q - r$ *arbitrary constants and satisfies system* (16).

Proof. In case (i), by the Fredholm alternative, system (5) has no solution in $L_{2,n}(a, \infty)$. Therefore, system (1) also has no solution in $L_{2,n}(a, \infty)$. In cases (ii)-(iv), by the Fredholm alternative, system (5) has a solution representable as (15), where $c_1, c_2, \ldots, c_q$ are arbitrary constants. Substituting (15) into (6) gives system (16). Applying the Kronecker-Capelli theorem to system (16), we prove assertions (ii)-(iv) in Corollary 2 to Theorem.

Example 1. Consider the system

$$\begin{cases} u_1(x) = \lambda \displaystyle\int_1^{\infty} \left[\frac{1}{xy} u_1(y) + \frac{2}{(x+1)y} u_2(y) \right] dy + \frac{\alpha_1}{x} + \frac{\beta_1}{x+1}, \\ (x-1)(x-2)(x-3) u_2(x) = \lambda \displaystyle\int_1^{\infty} \left[\frac{(x-1)(x-3)}{y} u_1(y) + \frac{x}{y+1} u_2(y) \right] dy \\ \qquad\qquad + \alpha_2 (x-1)(x-3) + \dfrac{\beta_2}{x}, \end{cases} \tag{17}$$

where $x \in [1, \infty)$ and $\lambda, \alpha_1, \beta_1, \alpha_2, \beta_2$ are parameters. It is easy to see that system (17) satisfies conditions (2), (a), (b) and (c) for $n = 2$, $a = 1$, $t = 2$, $p_{2,1}(x) = x - 3$, $p_{2,2}(x) = x - 1$, $p_{2,3}(x) = x - 2$, $x_{21} = 3$, $x_{22} = 1$, $x_{23} = 2$, $k_{11}(x, y) = \frac{1}{xy}$, $k_{12}(x, y) = \frac{2}{(x+1)y}$, $k_{21}(x, y) = \frac{(x-1)(x-3)}{y}$, $k_{22}(x, y) = \frac{x}{y+1}$, $k_{21,1}(x, y) = \frac{x-1}{y}$, $k_{22,1}(x, y) = \frac{1}{y+1}$, $k_{21,2}(x, y) = \frac{1}{y}$, $k_{22,2}(x, y) = 0,$

$k_{21,3}(x, y) = 0$, $k_{22,3}(x, y) = 0$ for $(x, y) \in [1, \infty) \times [1, \infty)$, $f_1(x) = \frac{\alpha_1}{x} + \frac{\beta_1}{x+1}$,

$f_2(x) = \alpha_2(x-1)(x-3) + \frac{\beta_2}{x}$, $F_{2,1}(x) = \alpha_2(x-1) - \frac{\beta_2}{3x}$, $F_{2,2}(x) = \alpha_2 + \frac{\beta_2}{3x}$,

$F_{2,3}(x) = -\frac{\beta_2}{6x}$ for $x \in [1, \infty)$.

Then, for system (17), system (5) and conditions (6) are written as

$$\begin{cases} u_1(x) = \lambda \int_1^\infty \left[\frac{1}{xy} u_1(y) + \frac{2}{(x+1)y} u_2(y) \right] dy + \frac{\alpha_1}{x} + \frac{\beta_1}{x+1}, \\ u_2(x) = -\frac{\beta_2}{6x}, \ x \in [1, \infty), \end{cases} \tag{18}$$

with the conditions

$$\begin{cases} \lambda \int_1^\infty \frac{3}{y+1} u_2(y) dy + \frac{\beta_2}{3} = 0, \\ \lambda \int_1^\infty \frac{1}{y+1} u_2(y) dy - \frac{\beta_2}{3} = 0, \\ \lambda \int_1^\infty \frac{1}{y} u_1(y) dy + \alpha_2 + \frac{\beta_2}{6} = 0. \end{cases} \tag{19}$$

From (19), we have

$$\begin{cases} \lambda \int_1^\infty \frac{1}{y+1} u_2(y) dy = 0, \\ \beta_2 = 0, \\ \lambda \int_1^\infty \frac{1}{y} u_1(y) dy + \alpha_2 = 0. \end{cases} \tag{20}$$

(I) If $\beta_2 \neq 0$, then from (18) and (20) it follows that system (17) has no solution in $L_{2,2}(1, \infty)$.

(II) Let $\beta_2 = 0$, It is easy to see that $\frac{1}{\lambda} = 1$ is a unique eigenvalue of the matrix kernel $k(x, y)$, where

$$k(x, y) = \begin{pmatrix} \frac{1}{xy} & \frac{2}{(x+1)y} \\ 0 & 0 \end{pmatrix}, \ (x, y) \in [1, \infty) \times [1, \infty).$$

(i) Let $\lambda \neq 1$, $\alpha_2 = \frac{\lambda}{\lambda - 1}(\alpha_1 + \beta_1 \ln 2)$. Then system (17) has a unique solution in $L_{2,2}(1, \infty) \cap C_2[1, \infty)$ and this solution is given by

$$u_1(x) = \frac{1}{1-\lambda}(\alpha_1 + \lambda\beta_1 \ln 2)\frac{1}{x} + \frac{\beta_1}{x+1},\ u_2(x) = 0,\ x \in [1, \infty).$$

(ii) Let $\lambda \neq 1$, $\alpha_2 \neq \frac{\lambda}{\lambda - 1}(\alpha_1 + \beta_1 \ln 2)$. Then system (17) has no solution in $L_{2,2}(1, \infty)$.

(iii) Let $\lambda = 1$, $\alpha_1 \neq -\beta_1 \ln 2$. Then system (17) has no solution in $L_{2,2}(1, \infty)$.

(iv) Let $\lambda = 1$, $\alpha_1 = -\beta_1 \ln 2$. Then system (17) has a unique solution in $L_{2,2}(1, \infty) \cap C_2[1, \infty)$ and this solution is given by

$$u_1(x) = (-\alpha_2 - \beta_1 \ln 2)\frac{1}{x} + \frac{\beta_1}{x+1},\ u_2(x) = 0,\ x \in [1, \infty).$$

Example 2. Consider the system

$$\begin{cases} (x-1)u_1(x) = \lambda\int_0^\infty \left[\frac{(x-1)}{(x+1)(y+2)}u_1(y) + \frac{1}{y+1}u_2(y)\right]dy \\ \qquad\qquad + \frac{\alpha(x-1)}{x+1} + \frac{\beta(x-1)}{x+2}, \\ xu_2(x) = \lambda\int_0^\infty \left[\frac{1}{(y+2)}u_1(y) + \frac{3}{(y+1)}u_2(y)\right]dy + \frac{\gamma x}{x+1}, \end{cases} \tag{21}$$

where $x \in [0, \infty)$ and $\lambda, \alpha, \beta, \gamma$ are real parameters. It is easy to see that system (21) satisfies conditions (2), (a), (b) and (c) for $n = 2$, $a = 0$, $t = 1$, $p_{1,1}(x) = x - 1$, $p_{2,1}(x) = x$, $x_{11} = 1$, $x_{21} = 0$, $k_{11}(x, y) = \frac{x-1}{(x+1)(y+2)}$,

$k_{12}(x, y) = \frac{1}{y+1}$, $k_{21}(x, y) = \frac{1}{y+2}$, $k_{22}(x, y) = \frac{3}{y+1}$, $k_{11,1}(x, y) = \frac{1}{(x+1)(y+2)}$,
$k_{12,1}(x, y) = 0$, $k_{21,1}(x, y) = 0$, $k_{22,1}(x, y) = 0$ for $(x, y) \in [0, \infty) \times [0, \infty)$,

$$f_1(x) = \frac{\alpha(x-1)}{x+1} + \frac{\beta(x-1)}{x+2},\ f_2(x) = \frac{\gamma x}{x+1},$$

$$F_{1,1}(x) = \frac{\alpha}{x+1} + \frac{\beta}{x+2},\ F_{2,1}(x) = \frac{\gamma}{x+1},$$

for $x \in [0, \infty)$. Then, for system (21), system (5) and conditions (6) are written as

$$\begin{cases} u_1(x) = \lambda \int_0^{\infty} \frac{u_1(y)}{(x+1)(y+2)}\,dy + \frac{\alpha}{x+1} + \frac{\beta}{x+2}, \\ u_2(x) = \frac{\gamma}{x+1},\ x \in [0, \infty), \end{cases} \tag{22}$$

with the conditions

$$\begin{cases} \lambda \int_0^{\infty} \frac{u_2(y)}{y+1}\,dy = 0, \\ \lambda \int_0^{\infty} \frac{u_1(y)}{y+2}\,dy = 0. \end{cases} \tag{23}$$

It is easy to see that $\frac{1}{\lambda} = \ln 2$ is the only eigenvalue of the matrix kernel $k(x, y)$, where

$$k(x, y) = \begin{pmatrix} \frac{1}{(x+1)(y+2)} & 0 \\ 0 & 0 \end{pmatrix},\ (x, y) \in [0, \infty) \times [0, \infty).$$

(i) Let $\gamma \neq 0$. Then system (21) has no solution in $L_{2,2}(0, \infty)$.

(ii) Let $\lambda \neq \frac{1}{\ln 2}$ and $\lambda \neq 0$. Then system (22) has a unique solution in $L_{2,2}(0, \infty)$ given by the formula

$$\begin{cases} u_1(x) = \frac{1}{1-\lambda \ln 2}\left(\alpha + \frac{\lambda\beta}{2}\right)\frac{1}{x+1} + \frac{\beta}{x+2}, \\ u_2(x) = \frac{\gamma}{x+1},\ x \in [0, \infty). \end{cases} \tag{24}$$

Combining (24) with (23) yields.

$$\gamma = 0, \ \alpha = -\frac{\beta}{2\ln 2}. \tag{25}$$

Thus, the functions defined by (24) are a solution (21) if and only if conditions (25) hold. Then system (21) has a unique solution in $L_{2,2}(0, \infty)$ given by formula

$$\begin{cases} u_1(x) = \dfrac{1}{1-\lambda\ln 2}\left(-\dfrac{\beta}{2\ln 2} + \dfrac{\lambda\beta}{2}\right)\dfrac{1}{x+1} + \dfrac{\beta}{x+2}, \\ u_2(x) = 0, \ x \in [0, \infty). \end{cases} \tag{26}$$

(iii) Let $\lambda = 0$. Then system (21) has a unique solution in $L_{2,2}(0, \infty)$ given by the formula

$$\begin{cases} u_1(x) = \dfrac{\alpha}{x+1} + \dfrac{\beta}{x+2}, \\ u_2(x) = \dfrac{\gamma}{x+1}, \ x \in [0, \infty). \end{cases}$$

(iv) Let $\lambda = \frac{1}{\ln 2}$, $\alpha \neq -\frac{\beta}{2\ln 2}$, $\gamma = 0$. Then system (21) has no solution in $L_{2,2}(0, \infty)$.

(v) Let $\lambda = \frac{1}{\ln 2}$, $\alpha = -\frac{\beta}{2\ln 2}$, $\gamma = 0$. Then system (21) has a unique solution in $L_{2,2}(0, \infty)$ given by formula

$$\begin{cases} u_1(x) = -\dfrac{\beta}{2(x+1)\ln 2} + \dfrac{\beta}{x+2}, \\ u_2(x) = 0, \ x \in [0, \infty). \end{cases} \tag{27}$$

References

[1] A. S. Apartsyn, Nonclassical Linear Volterra Equations of the First Kind, Utrecht-Boston, The Netherlands: VSP, 2003, 168 p.

[2] A. Asanov, Regularization Uniqueness and Existence of Solutions of Volterra Equations of the First Kind, Utrecht, The Netherlands: VSP, 1998, 282 p.

[3] Avyt Asanov, Kalyskan B. Matanova and Ruhidin Asanov, A class of linear and nonlinear Fredholm integral equations of the third kind, Kuwait Journal of Science 44(1) (2017), 17-28.

[4] A. L. Bukhgeim, Volterra Equations and Inverse Problems, Utrecht, The Netherlands: VSP, 1999, 204 p.

[5] A. M. Denisov, Elements of the Theory of Inverse Problems, Utrecht-Boston, The Netherlands: VSP, 1999, 272 p.

[6] M. I. Imanaliev and A. Asanov, Solutions of systems of nonlinear Volterra integral equations of the first kind, Doklady Akademii Nauk SSSR 309(5) (1989), 1052-1055. (in Russian)

[7] M. I. Imanaliev and A. Asanov, Regularization and uniqueness of solutions to systems of nonlinear Volterra integral equations of the third kind, Doklady Mathematics 76(1) (2007), 490-493.

DOI: https://doi.org/10.1134/S1064562407040035

[8] M. I. Imanaliev and A. Asanov, Solutions to systems of linear Fredholm integral equations of the third kind, Doklady Mathematics 81(1) (2010), 115-118.

DOI: https://doi.org/10.1134/S1064562410010321

[9] M. I. Imanaliev, A. Asanov and R. A. Asanov, A class of systems of linear Fredholm integral equations of the third kind, Doklady Mathematics 83(2) (2011), 227-231.

DOI: https://doi.org/10.1134/S1064562411020293

[10] M. I. Imanaliev, A. Asanov and R. A. Asanov, Solutions to systems of linear Fredholm integral equations of the third kind with multipoint singularities, Doklady Mathematics 95(3) (2017), 235-239.

DOI: https://doi.org/10.1134/S1064562417030140

[11] Ismat Beg, A. Saha, Anamika Ganguly and Debashis Dey, Random fixed point of Gregus mapping and its application to nonlinear stochastic integral equations, Kuwait Journal of Science 41(2) (2014), 1-14.

[12] M. M. Lavrent'ev, The integral equations of the first kind, Doklady Akademii Nauk SSSR 127(1) (1959), 31-33. (in Russian)

[13] M. M. Lavrent'ev, V. G. Romanov and S. P. Shishatskii, Ill-Posed Problems of Mathematical Physics and Analysis, Providence, R. I. American Mathematical Society, 1986.

[14] S. P. Shishatskii, A. Asanov and E. R. Atamanov, Uniqueness Problems for Degenerating Equations and Nonclassical Problems, Utrecht-Boston, The Netherlands: VSP, 2001, 178 p.

■