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THE GENERALIZATIONS OF LOCAL FRACTIONAL HILBERT-TYPE INEQUALITIES

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Abstract

The main objective of this paper is to prove local fractional Hilbert-type inequalities with a general homogeneous kernel. Special attention is given to conditions under which the constant involved in inequalities are the best possible. Some particular cases of local fractional Hilbert-type inequalities are presented.

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1. Introduction

Suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(\geq 0) \in L_p(\mathbb{R}_+), g(\geq 0) \in L_q(\mathbb{R}_+),$ we have the celebrated Hilbert inequality and its equivalent form (see [5]) as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy \leq \frac{\pi}{\sin \frac{\pi}{p}} \left[\int_0^\infty f^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty g^q(y)dy\right]^{\frac{1}{q}}, \tag{1}$$

$$\int_0^\infty \left[\int_0^\infty \frac{f(x)}{x+y} dx\right] dy \leq \left(\frac{\pi}{\sin \frac{\pi}{p}}\right)^p \int_0^\infty f^p(x)dx, \tag{2}$$

where the constants $\pi / \sin \frac{\pi}{p}$ and $[\pi / \sin \frac{\pi}{p}]^p$ are optimal. Although there have been many results on the study of inequalities (1) and (2), these inequalities are still topic of interest to numerous authors. For a starting development of inequalities (1) and (2) the reader can be referred to [5, 7], while some recent results are found in [1, 2, 8].

In recent years, the fractal theory has attracted the attention of many researchers, local fractional calculus (also called fractal calculus) has applied to solve some problems not only in mathematics but also in physics and engineers [3, 4, 6, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19].

The aim of this paper is to present some new Hilbert-type inequalities via local fractional integrals established by Yang [18]. In the beginning, we give basic definitions and properties of the local fractional calculus (see [18] and [19]). First, we recall Yang's fractal set Ω^α, where the set Ω is called base set of fractional set, and α denotes the dimension of cantor set, $0 < \alpha \leq 1$. The α-type set of integers $\mathbb{Z}^\alpha$ is defined by

$$\mathbb{Z}^\alpha := \{0^\alpha\} \cup \{\pm m^\alpha : m \in \mathbb{N}\}.$$

The α-type set of rational numbers $\mathbb{Q}^\alpha$ is defined by

$$\mathbb{Q}^\alpha := \{q^\alpha : q \in \mathbb{Q}\} = \left\{\left(\frac{m}{n}\right)^\alpha : m \in \mathbb{Z},\ n \in \mathbb{N}\right\}.$$

The α-type set of irrational numbers $\mathbb{J}^\alpha$ is defined by

$$\mathbb{J}^\alpha := \{r^\alpha : r \in \mathbb{J}\} = \left\{r^\alpha \neq \left(\frac{m}{n}\right)^\alpha : m \in \mathbb{Z},\ n \in \mathbb{N}\right\}.$$

The α-type set of real line numbers $\mathbb{R}^\alpha$ is defined by

$$\mathbb{R}^\alpha := \mathbb{Q}^\alpha \cup \mathbb{J}^\alpha.$$

Recall basic operation rules on $\mathbb{R}^\alpha$: If $a^\alpha, b^\alpha, c^\alpha \in \mathbb{R}^\alpha$, then

(1) $a^\alpha + b^\alpha \in \mathbb{R}^\alpha$, $a^\alpha b^\alpha \in \mathbb{R}^\alpha$.

(2) $a^\alpha + b^\alpha = b^\alpha + a^\alpha = (a+b)^\alpha = (b+a)^\alpha$.

(3) $a^\alpha + (b^\alpha + c^\alpha) = (a+b)^\alpha + c^\alpha$.

(4) $a^\alpha b^\alpha = b^\alpha a^\alpha = (ab)^\alpha = (ba)^\alpha$.

(5) $a^\alpha(b^\alpha c^\alpha) = (a^\alpha b^\alpha)c^\alpha$.

(6) $a^\alpha(b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$.

(7) $a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha$ and $a^\alpha 1^\alpha = 1^\alpha a^\alpha = a^\alpha$.

(8) For each $a^\alpha \in \mathbb{R}^\alpha$, its inverse element $(-a^\alpha)$ may be written as $-a^\alpha$; for each $b^\alpha \in \mathbb{R}^\alpha \setminus \{0^\alpha\}$, its inverse element $(1/b)^\alpha$ may be written as $1^\alpha / b^\alpha$ but not as $1/b^\alpha$.

(9) $a^\alpha < b^\alpha$ if and only if $a < b$.

(10) $a^\alpha = b^\alpha$ if and only if $a = b$.

Further, we give a brief overview of the local fractional derivative and integral.

Definition 1. A non-differentiable function $f(x)$ is said to be local fractional continuous at $x = x_0$ if for each $\varepsilon > 0$, there exists for $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon^{\alpha},$$

holds for $0 < |x - x_0| < \delta$. If a function f is local continuous on the interval (a, b), we denote $f \in C_{\alpha}(a, b)$.

Definition 2. Let $f(x) \in C_{\alpha}[a, b]$. Local fractional derivative of the function $f(x)$ at $x = x_0$ is given by

$$f(x_0) = \left.\frac{d^{\alpha} f(x)}{dx^{\alpha}}\right|_{x=x_0} = \lim_{x \to x_0} \frac{\Gamma(1+\alpha)(f(x) - f(x_0))}{(x - x_0)^{\alpha}}.$$

Definition 3. Let $f(x) \in C_{\alpha}[a, b]$ and let $P = \{t_0, t_1, \dots, t_N\}$, $N \in \mathbb{N}$, be a partition of interval $[a, b]$ such that $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$. Further, for this partition P, let $\Delta t_j = t_{j+1} - t_j$, $j = 0, \dots, N-1$, and $\Delta t = \max\{\Delta t_1, \Delta t_2, \dots, \Delta t_{N-1}\}$. Then the local fractional integral of f on the interval $[a, b]$ of order α (denoted by ${}_aI_b^{\alpha} f(x)$) is defined by

$${}_aI_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^{\alpha} = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^{\alpha}.$$

The above definition implies that ${}_aI_b^{(\alpha)} f(x) = 0$ if $a = b$, and ${}_aI_b^{(\alpha)} f(x) = -{}_bI_a^{(\alpha)} f(x)$ if $a < b$.

At the end of this overview, we give some useful formulas:

(1) $$\frac{d^{\alpha}x^{k\alpha}}{dx^{\alpha}} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)}x^{(k-1)\alpha}.$$

(2) $$\frac{d^{\alpha}E_{\alpha}((cx)^{\alpha})}{dx^{\alpha}} = c^{\alpha}E_{\alpha}((cx)^{\alpha}),$$

where $E_{\alpha}(\cdot)$ denotes the Mittag-Leffler function given by

$$E_{\alpha}(x^{\alpha}) = \sum_{k=0}^{\infty}\frac{x^{k\alpha}}{\Gamma(1+k\alpha)}.$$

(3) $$\frac{1}{\Gamma(1+\alpha)}\int_a^b E_{\alpha}(x^{\alpha})(dx)^{\alpha} = E_{\alpha}(b^{\alpha}) - E_{\alpha}(a^{\alpha}).$$

(4) $$\frac{1}{\Gamma(1+\alpha)}\int_a^b x^{k\alpha}(dx)^{\alpha} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)}(b^{(k+1)\alpha} - a^{(k+1)\alpha}).$$

Throughout the paper, we denote by ${}_aI_b^{(\alpha)}f(x)$ and ${}_aI_b^{(\alpha)}[{}_aI_b^{(\alpha)}h(x, y)]$ local fractional integrals

$${}_aI_b^{(\alpha)}f(x) = \frac{1}{\Gamma(1+\alpha)}\int_a^b f(x)(dx)^{\alpha},$$

and

$${}_aI_b^{(\alpha)}[{}_aI_b^{(\alpha)}h(x, y)] = \frac{1}{\Gamma^2(1+\alpha)}\int_a^b\int_a^b h(x, y)(dx)^{\alpha}(dy)^{\alpha}.$$

For the reader's convenience, from now on we use the following abbreviations:

$$\mathbf{x} = (x_1, x_2, \dots, x_n), \quad (dx)^{\alpha} = \prod_{i=1}^{n}(dx_i)^{\alpha}.$$

Recall that the function $K : \mathbb{R}^n \to \mathbb{R}^\alpha$ is homogeneous of degree $-\alpha\lambda$, $\lambda > 0$, if $K(t\mathbf{x}) = t^{-\alpha\lambda} K(\mathbf{x})$ holds for all $t > 0$. Further, for $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, we define

$$k_i(\mathbf{a}) = {}_0I_\infty^{((n-1)\alpha)} K(\hat{\mathbf{u}}^i) \prod_{j=1,\ j\neq i}^{n} u_j^{\alpha a_j},\ i = 1, 2, \dots, n, \tag{3}$$

where $\hat{\mathbf{u}}^i = (u_1, \dots, u_{i-1}, 1, u_{i+1}, \dots, u_n)$. If nothing else is explicitly stated, we assume that the integral defined by (3) converges for all considered values.

Our result will be based on the following result of Krnić and Vuković from [10].

Theorem A. *Let* $\sum_{i=1}^n \frac{1}{p_i} = 1$, $p_i > 1$, $i = 1, 2, \dots, n$. *Further, suppose that* A_{ij}, $i, j = 1, 2, \dots, n$, *are real parameters such that* $\sum_{i=1}^n A_{ij} = 0$, *for* $j = 1, 2, \dots, n$, *and let* $\beta_i := \sum_{j=1}^n A_{ij}$, *for* $i = 1, 2, \dots, n$. *If* $f_i \in C_\alpha(\mathbb{R}_+)$, $i = 1, 2, \dots, n$, *are non-negative functions and* $K \in C_\alpha(\mathbb{R}_+^n)$ *is a non-negative homogeneous function of degree* $-s$, $s > 0$, *then holds the inequality*

$$ {}_0I_\infty^{(n\alpha)} K(\mathbf{x}) \prod_{i=1}^n f_i(x_i)$$

$$\leq \prod_{i=1}^n k_i^{\frac{1}{p_i}}(p_i \mathbf{A}_i) \prod_{i=1}^n \left[{}_0I_\infty^{(\alpha)} x_i^{(n-1)\alpha - \alpha s + \alpha p_i \beta_i} f_i^{p_i}(x_i) \right]^{\frac{1}{p_i}}, \tag{4}$$

where $p_i \mathbf{A}_i = (p_i A_{i1}, p_i A_{i2}, \dots, p_i A_{in})$, $i = 1, 2, \dots, n$, *and* $k_i(\cdot)$ *is defined by* (3).

2. Main Results

By applying Theorem A, we get the following result:

Theorem 1. *Let* $K \in C_\alpha(\mathbb{R}_+^n)$ *and* $A_{ij}, \beta_i, i, j = 1, 2, \ldots, n,$ *be as in Theorem A. Suppose that for every* $i = 1, 2, \ldots, n,$ $u_i : (a_i, b_i) \to \mathbb{R}_+,$ *is a strictly increasing differentiable function such that* $u_i(a_i) = 0$ *and* $u_i(b_i) = \infty$. *If* $f_i \in C_\alpha(\mathbb{R}_+)$, $i = 1, 2, \ldots, n$ *are non-negative functions, then holds the inequality*

$$\frac{1}{\Gamma^n(1+\alpha)}\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} K(u_1(t_1), \ldots, u_n(t_n))\prod_{i=1}^n f_i(t_i)(dt_1)^\alpha \ldots (dt_n)^\alpha$$

$$\leq \prod_{i=1}^n k_i^{\frac{1}{p_i}}(p_i\mathbf{A}_i)\prod_{i=1}^n$$

$$\times\left(\frac{1}{\Gamma(1+\alpha)}\int_{a_i}^{b_i}(u_i(t_i))^{\alpha(n-1)-\alpha s+\alpha p_i\beta_i} \times [u_i'(t_i)]^{\alpha(1-p_i)} f_i^{p_i}(t_i)(dt_i)^\alpha\right)^{\frac{1}{p_i}}, \tag{5}$$

where $p_i\mathbf{A}_i = (p_iA_{i1}, p_iA_{i2}, \ldots, p_iA_{in}), i = 1, 2, \ldots, n,$ *and* $k_i(\cdot)$ *is defined by* (3).

Proof. The proof follows directly from Theorem A. Namely, setting the functions $h_i : \mathbb{R}_+ \to \mathbb{R}^\alpha$, $i = 1, 2, \ldots, n,$ such that $f_i(t_i) = h_i(u_i(t_i))[u_i'(t_i)]^\alpha$, the inequality (4) with the functions h_i takes the following form:

$${}_0I_\infty^{(n\alpha)}K(\mathbf{x})\prod_{i=1}^n h_i(x_i)$$

$$\leq \prod_{i=1}^n k_i^{\frac{1}{p_i}}(p_i\mathbf{A}_i)\prod_{i=1}^n\left[{}_0I_\infty^{(\alpha)}x_i^{\alpha(n-1)-\alpha s+\alpha p_i\beta_i}h_i^{p_i}(x_i)\right]^{\frac{1}{p_i}}. \tag{6}$$

By using the substitutions $x_i = u_i(t_i)$, $(dx_i)^\alpha = [u_i'(t_i)]^\alpha (dt_i)^\alpha$, $i = 1, 2, \ldots, n$, the left-hand side of the inequality (6) becomes

$$\frac{1}{\Gamma^n(1+\alpha)} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} K(u_1(t_1), \ldots, u_n(t_n))$$

$$\times \prod_{i=1}^{n} h_i(u_i(t_i)) [u_i'(t_i)]^\alpha (dt_1)^\alpha (dt_2)^\alpha \ldots (dt_n)^\alpha, \tag{7}$$

where we used the facts $u_i(a_i) = 0$ and $u_i(b_i) = \infty$, $i = 1, 2, \ldots, n$.

Similarly, the right-hand side of the inequality (6) becomes

$$J = \prod_{i=1}^{n} k_i^{\frac{1}{p_i}} (p_i \mathbf{A}_i) \prod_{i=1}^{n} \Bigg(\frac{1}{\Gamma(1+\alpha)} \int_{a_i}^{b_i} (u_i(t_i))^{\alpha(n-1)-\alpha s+\alpha p_i \beta_i}$$

$$\times [u_i'(t_i)]^{\alpha(1-p_i)} h_i^{p_i}(u_i(t_i)) [u_i'(t_i)]^{\alpha p_i} (dt_i)^\alpha \Bigg)^{\frac{1}{p_i}}. \tag{8}$$

Now, from (6), (7), (8) and the fact $f_i(t_i) = h_i(u_i(t_i))[u_i'(t_i)]^\alpha$, follows the inequality (5). □

In the following, we analyze the conditions which yield the best possible constants in obtained inequalities. More precisely, we introduce the following conditions on the parameters A_{ij} :

$$p_j A_{ij} = s - n - p_i(\beta_i - A_{ii}), \; i, \; j = 1, 2, \ldots, n, \quad i \neq j, \tag{9}$$

where $\beta_i = \sum_{j=1}^{n} A_{ij}$. In that case the constant $\prod_{i=1}^{n} k_i^{\frac{1}{p_i}} (p_i \mathbf{A}_i)$ from Theorem 1 can be transformed to the form:

$$L^* := k_1(\widetilde{\mathbf{A}}), \tag{10}$$

where $\widetilde{\mathbf{A}} = (\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_n)$,

$$\widetilde{A}_i = p_1 A_{1i}, \quad \text{for} \quad i \neq 1, \quad \text{and} \quad \widetilde{A}_1 = p_n A_{n1}.$$

Using (10), the inequality (5) becomes

$$\frac{1}{\Gamma^n(1+\alpha)}\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n}K(u_1(t_1), \ldots, u_n(t_n))\prod_{i=1}^{n}f_i(t_i)(dt_1)^{\alpha}\ldots(dt_n)^{\alpha}$$

$$\leq L^*\prod_{i=1}^{n}\left(\frac{1}{\Gamma(1+\alpha)}\int_{a_i}^{b_i}(u_i(t_i))^{-\alpha-\alpha p_i\widetilde{A}_i}\times[u_i'(t_i)]^{\alpha(1-p_i)}f_i^{p_i}(t_i)(dt_i)^{\alpha}\right)^{\frac{1}{p_i}}. \tag{11}$$

Further, we can prove that, if the parameters A_{ij} satisfy the condition (9), then one obtains the best possible constant.

Theorem 2. *Let* $K \in C_{\alpha}(\mathbb{R}_+^n)$ *and* $u_i : (a_i, b_i) \to \mathbb{R}_+$, $i = 1, 2, \ldots, n$, *be as in Theorem* 1. *If the parameters* A_{ij}, $i = 1, 2, \ldots, n$, *satisfy conditions* (9), *then the constant* L^* *is the best possible in inequality* (11).

Proof. Let $h_i : \mathbb{R}_+ \to \mathbb{R}^{\alpha}$, $i = 1, 2, \ldots, n$, be the functions such that $f_i(t_i) = h_i(u_i(t_i))[u_i'(t_i)]^{\alpha}$. By using the substitutions $x_i = u_i(t_i)$, $i = 1, 2, \ldots, n$, the inequality (11) with the functions h_i defined above becomes

$${}_0I_{\infty}^{(n\alpha)}K(\mathbf{x})\prod_{i=1}^{n}h_i(x_i) \leq L^*\prod_{i=1}^{n}\left[{}_0I_{\infty}^{(\alpha)}x_i^{-\alpha-\alpha p_i\widetilde{A}_i}h_i^{p_i}(x_i)\right]^{\frac{1}{p_i}}, \tag{12}$$

where the constant L^* is defined by (10).

Suppose that the constant factor L^* is not the best possible in the inequality (12). Let $0 < L_1 < L^*$ such that the inequality (12) is still valid when L^* is replaced by L_1. Specially, we define

$$\widetilde{h}_i(x_i) = \begin{cases} 0, & x \in (0, 1) \\ x_i^{\alpha\widetilde{A}_i - \frac{\alpha\varepsilon}{p_i}}, & x \in [1, \infty) \end{cases}, \; i = 1, 2, \ldots, n,$$

where $\varepsilon > 0$ is small enough. Setting these functions in the inequality (12), the right-hand side of the inequality (12) becomes

$$L_1 \prod_{i=1}^{n} \left[\int_0^\infty x_i^{-\alpha-\alpha\varepsilon} (dx_i)^\alpha \right]^{\frac{1}{p_i}} = \frac{L_1}{\varepsilon^\alpha \Gamma(1+\alpha)}. \tag{13}$$

By using the substitution $u_i = \frac{x_i}{x_1}$, $i = 2, 3, \dots, n$, and the Fubini theorem, we find that the left-hand side of inequality (12) takes form

$$I := \frac{1}{\Gamma^n(1+\alpha)} \int_{[1,\infty)^n} K(\mathbf{x}) \prod_{i=1}^{n} x_i^{\alpha\tilde{A}_i - \frac{\varepsilon\alpha}{p_i}} (d\mathbf{x})^\alpha$$

$$= \frac{1}{\Gamma(1+\alpha)} \int_1^\infty x_1^{-\alpha-\varepsilon\alpha} \left(\frac{1}{\Gamma^{n-1}(1+\alpha)} \int_{[1/x_1,\infty)} K(\hat{\mathbf{u}}^1) \prod_{i=2}^{n} u_i^{\alpha\tilde{A}_i - \frac{\varepsilon\alpha}{p_i}} (\hat{d}^1\mathbf{u})^\alpha \right) (dx_1)^\alpha.$$

It is evident that the following inequality holds:

$$I \geq \frac{1}{\Gamma(1+\alpha)} \int_1^\infty x_1^{-\alpha-\varepsilon\alpha} \left(\frac{1}{\Gamma^{n-1}(1+\alpha)} \int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^1) \prod_{i=2}^{n} u_i^{\alpha\tilde{A}_i - \frac{\varepsilon\alpha}{p_i}} (\hat{d}^1\mathbf{u})^\alpha \right) (dx_1)^\alpha$$

$$- \frac{1}{\Gamma(1+\alpha)} \int_1^\infty x_1^{-\alpha-\varepsilon\alpha} \sum_{j=2}^{n} I_j(x_1) (dx_1)^\alpha, \tag{14}$$

where $I_j(x_1)$ and D_j, $j = 2, 3, \dots, n$, are defined by

$$I_j(x_1) = \frac{1}{\Gamma^{n-1}(1+\alpha)} \int_{D_j} K(\hat{\mathbf{u}}^1) \prod_{i=2}^{n} u_i^{\alpha\tilde{A}_i - \frac{\varepsilon\alpha}{p_i}} (\hat{d}^1\mathbf{u})^\alpha,$$

and $D_j = \{(u_2, u_3, \dots, u_n) : 0 < u_j \leq \frac{1}{x_1}, u_i > 0, i \neq j\}$. By using the integral formula (3), the above inequality can be rewritten as

$$I \geq \frac{1}{\varepsilon^\alpha \Gamma(1+\alpha)} k_1(\tilde{\mathbf{A}} - \varepsilon\alpha\mathbf{1}/\mathbf{p})$$

$$- \frac{1}{\Gamma(1+\alpha)} \int_1^\infty x_1^{-\alpha-\varepsilon\alpha} \sum_{j=2}^{n} I_j(x_1) (dx_1)^\alpha, \tag{15}$$

where $\frac{\mathbf{1}}{\mathbf{p}} = (\frac{1}{p_1}, \ldots, \frac{1}{p_n})$. Without losing generality, here we only estimate the integral $I_2(x_1)$. Since $1^\alpha - u_2^{\alpha\varepsilon} \to 1^\alpha (u_2 \to 0^+)$, there exists $M \geq 0$ such that $1^\alpha - u_2^{\alpha\varepsilon} \leq M^\alpha (u_2 \in (0, 1])$. By using the Fubini's theorem, it follows that

$$\begin{aligned}
0 &\leq \frac{\varepsilon^\alpha}{\Gamma(1+\alpha)} \int_1^\infty x_1^{-\alpha-\alpha\varepsilon} I_2(x_1)(dx_1)^\alpha \\
&= \frac{\varepsilon^\alpha}{\Gamma(1+\alpha)} \int_1^\infty x_1^{-\alpha-\alpha\varepsilon} \left(\frac{1}{\Gamma^{n-1}(1+\alpha)} \int_{\mathbb{R}_+^{n-2}} \int_0^{1/x_1} K(\hat{\mathbf{u}}^1) \right. \\
&\quad \left. \times \prod_{i=2}^n u_i^{\alpha\widetilde{A}_i - \frac{\varepsilon\alpha}{p_i}} (\hat{d}^1\mathbf{u})^\alpha \right) (dx_1)^\alpha \\
&= \frac{\varepsilon^\alpha}{\Gamma^{n-1}(1+\alpha)} \int_{\mathbb{R}_+^{n-2}} \int_0^1 K(\hat{\mathbf{u}}^1) \prod_{i=2}^n u_i^{\alpha\widetilde{A}_i - \frac{\varepsilon\alpha}{p_i}} \\
&\quad \times \left(\frac{1}{\Gamma(1+\alpha)} \int_0^{1/u_2} x_1^{-\alpha-\alpha\varepsilon} (dx_1)^\alpha \right) (\hat{d}^1\mathbf{u})^\alpha \\
&= \frac{\varepsilon^\alpha}{\Gamma^{n-1}(1+\alpha)} \int_{\mathbb{R}_+^{n-2}} \int_0^1 K(\hat{\mathbf{u}}^1) \prod_{i=2}^n u_i^{\alpha\widetilde{A}_i - \frac{\varepsilon\alpha}{p_i}} \left(\frac{1}{\varepsilon} (1^\alpha - u_2^\alpha) \right) (\hat{d}^1\mathbf{u})^\alpha \\
&\leq \frac{M^\alpha}{\Gamma^{n-1}(1+\alpha)} \int_{\mathbb{R}_+^{n-2}} \int_0^1 K(\hat{\mathbf{u}}^1) \prod_{i=2}^n u_i^{\alpha\widetilde{A}_i - \frac{\varepsilon\alpha}{p_i}} (\hat{d}^1\mathbf{u})^\alpha \\
&\leq \frac{M^\alpha}{\Gamma^{n-1}(1+\alpha)} \int_{\mathbb{R}_+^{n-1}} K(\hat{\mathbf{u}}^1) \prod_{i=2}^n u_i^{\alpha\widetilde{A}_i - \frac{\varepsilon\alpha}{p_i}} (\hat{d}^1\mathbf{u})^\alpha \\
&= M^\alpha k_1(\widetilde{\mathbf{A}} - \varepsilon\mathbf{1}/\mathbf{p}) < \infty.
\end{aligned}$$

Hence, by (15), one has

$$I \geq \frac{1}{\varepsilon^\alpha \Gamma(1+\alpha)} k_1(\widetilde{\mathbf{A}} - \varepsilon 1/\mathbf{p}) - O(1). \tag{16}$$

In view of (13) and (16), we conclude that $L^* \le L_1$ when $\varepsilon \to 0^+$, which is an obvious contradiction. Hence, the constant L^* in the inequality (11) is the best possible. □

3. Applications

As applications, we will build some new inequalities. First, we proceed with a function $K_1(x_1, \dots, x_n) = (\sum_{i=1}^{n} x_i^{s(\lambda-1)})^{\alpha} / (\sum_{i=1}^{n} x_i^{\lambda})^{\alpha s}$, where $s > 0$, and $\lambda > 1$. It is easy to see that $K_1 \in C_{\alpha}(\mathbb{R}_+^n)$ and its degree of homogeneity is $-\alpha s$.

To obtain the Hilbert-type inequality with the kernel K_1, we need an extension of the usual Gamma function. The local fractional Gamma function $\Gamma_{\alpha}(\cdot)$, $0 < \alpha \le 1$, can be expressed as

$$\Gamma_{\alpha}(x) = {}_0I_{\infty}^{(\alpha)} E_{\alpha}(-t^{\alpha}) t^{\alpha(x-1)}.$$

First, we need some technical lemmas.

Lemma 1 (see [10]). *If* $n \in \mathbb{N}$, $r_i > 0$, $i = 1, 2, \dots, n$, *then holds the relation* (17)

$$\frac{1}{\Gamma^{n-1}(1+\alpha)} \int_{\mathbb{R}_+^{n-1}} \frac{\prod_{i=1}^{n-1} x_i^{\alpha(r_i - 1)}}{(1 + \sum_{i=1}^{n-1} x_i)^{\alpha \sum_{i=1}^{n} r_i}} (dx_1)^{\alpha} \dots (dx_{n-1})^{\alpha} = \frac{\prod_{i=1}^{n} \Gamma_{\alpha}(r_i)}{\Gamma_{\alpha}(s)}.$$

Applying Lemma 1 we obtain the next result.

Lemma 2. *Suppose that* $n \in \mathbb{N}$, $s, \lambda > 0$. *If* $\gamma_i > -1$, $i = 1, 2, \ldots, n-1$, *and* $\sum_{i=1}^{n-1} \gamma_i < \lambda s - n + 1$, *then*

$$I := \frac{1}{\Gamma^{n-1}(1+\alpha)} \int_{\mathbb{R}_+^{n-1}} \frac{\prod_{i=1}^{n-1} t_i^{\alpha\gamma_i}}{(1 + \sum_{i=1}^{n-1} t_i^{\lambda})^{\alpha s}} (dt_1)^{\alpha} \ldots (dt_{n-1})^{\alpha}$$

$$= \frac{1}{\Gamma_{\alpha}(s)\lambda^{\alpha(n-1)}} \left(\prod_{i=1}^{n-1} \Gamma_{\alpha}\left(\frac{\gamma_i + 1}{\lambda}\right) \right) \Gamma_{\alpha}\left(s - \frac{1}{\lambda} \sum_{i=1}^{n-1} (\gamma_i + 1) \right). \tag{18}$$

Proof. The substitution $u_i = t_i^{\lambda}$, $i = 1, 2, \ldots, n-1$, yields

$$I = \frac{1}{\lambda^{\alpha(n-1)}\Gamma^{n-1}(1+\alpha)} \int_{\mathbb{R}_+^{n-1}} \frac{\prod_{i=1}^{n-1} u_i^{\alpha\left(\frac{\gamma_i+1}{\lambda} - 1\right)}}{(1 + \sum_{i=1}^{n-1} u_i)^{\alpha s}} (du_1)^{\alpha} \ldots (du_{n-1})^{\alpha}.$$

By using Lemma 1, we get

$$I = \frac{1}{\Gamma_{\alpha}(s)\lambda^{\alpha(n-1)}} \prod_{i=1}^{n} \Gamma_{\alpha}(r_i),$$

where $r_i = \frac{\gamma_i + 1}{\lambda}$, $i = 1, 2, \ldots, n-1$, and $r_n = s - \frac{1}{\lambda} \sum_{i=1}^{n-1} (\gamma_i + 1)$. In this way we obtained (18). □

Further, we define the parameters A_{ij}, $i, j = 1, 2, \ldots, n$, by

$$A_{ij} = \frac{s - p_j}{p_i p_j}, \; i \neq j, \text{ and } A_{jj} = \frac{(s - p_j)(1 - p_j)}{p_j^2}. \tag{19}$$

Then,

$$\sum_{i=1}^{n} A_{ij} = \sum_{i \neq j} \frac{s - p_j}{p_i p_j} + \frac{(s - p_j)(1 - p_j)}{p_j^2} = \frac{s - p_j}{p_j} \left(\sum_{i=1}^{n} \frac{1}{p_j} - 1 \right) = 0,$$

for $j = 1, 2, \ldots, n$. Similarly, we obtain $\beta_i = 1 - \frac{n}{p_i}$, $i = 1, 2, \ldots, n$. Now, it is easy to see that the parameters A_{ij} defined by (19) satisfy the condition (9).

Our next result is a consequence of Theorem 2.

Corollary 1. *Let* $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, $p_i > 1$, $i = 1, 2, \ldots, n$, *and let* $\lambda > 1$ *and* $s > 0$. *If* $a > 1$ *and* $f_i \in C_\alpha(\mathbb{R})$, $i = 1, 2, \ldots, n$, *are non-negative functions, then holds the inequality*

$$\frac{1}{\Gamma^{n-1}(1+\alpha)} \int_{\mathbb{R}^n} \frac{\left(\sum_{i=1}^{n} a^{s(\lambda-1)t_i}\right)^{\alpha}}{\left(\sum_{i=1}^{n} a^{\lambda t_i}\right)^{\alpha s}} \prod_{i=1}^{n} f_i(t_i)\,(dt_1)^{\alpha} \ldots (dt_n)^{\alpha}$$

$$\leq M^* \prod_{i=1}^{n} \left(\frac{1}{\Gamma(1+\alpha)} \int_{\mathbb{R}} a^{-\alpha s t_i} f_i^{p_i}(t_i)\,(dt_i)^{\alpha} \right)^{\frac{1}{p_i}}, \tag{20}$$

where the constant

$$M^* = \frac{(\lambda \ln a)^{\alpha(1-n)}}{\Gamma_\alpha(s)} \sum_{j=1}^{n} \left[\left(\prod_{i=1,\, i \neq j}^{n} \Gamma_\alpha\left(\frac{s}{p_i \lambda}\right) \right) \cdot \Gamma_\alpha\left(\frac{s p_j(\lambda-1)+s}{p_j \lambda} \right) \right] \tag{21}$$

is the best possible.

Proof. We put $K_1(x_1, \ldots, x_n)$, the parameters A_{ij} defined by (19) and $u_i(t_i) = a^{t_i}$, $i = 1, 2, \ldots, n$, in Theorem 2. By using the definition of $\widetilde{A}_i$, we have $\widetilde{A}_i = \frac{s - p_i}{p_i}$, $i = 1, 2, \ldots, n$. Taking into account $u_i(t_i) = a^{t_i}$, we obtain

$$(u_i(t_i))^{-\alpha - \alpha p_i \widetilde{A}_i} [u_i'(t_i)]^{\alpha(1-p_i)} = (\ln a)^{\alpha(1-p_i)} a^{-\alpha s t_i}.$$

Now, it is enough to calculate the constant $(\ln a)^{\alpha(1-n)} \cdot M$, where

$$M := k_1\left(1, \frac{s-p_2}{p_2}, \dots, \frac{s-p_n}{p_n}\right).$$

Applying the definition of function $k_1(\cdot)$ given by (3), we find that

$$M = \frac{1}{\Gamma^{n-1}(1+\alpha)} \int_{\mathbb{R}_+^{n-1}} \frac{\left(1+\sum_{j=2}^{n} t_j^{s(\lambda-1)}\right)^{\alpha}}{\left(1+\sum_{j=2}^{n} t_j^{\lambda}\right)^{\alpha s}} \prod_{i=2}^{n} t_i^{\alpha\left(\frac{s}{p_i}-1\right)} (dt_2)^{\alpha} \dots (dt_n)^{\alpha} = \sum_{k=1}^{n} I_k, \tag{22}$$

where

$$I_1 = \frac{1}{\Gamma^{n-1}(1+\alpha)} \int_{\mathbb{R}_+^{n-1}} \frac{\prod_{i=2}^{n} t_i^{\alpha\left(\frac{s}{p_i}-1\right)}}{\left(1+\sum_{j=2}^{n} t_j^{\lambda}\right)^{\alpha s}} (dt_2)^{\alpha} \dots (dt_n)^{\alpha},$$

and

$$I_k = \frac{1}{\Gamma^{n-1}(1+\alpha)} \int_{\mathbb{R}_+^{n-1}} \frac{t_k^{\alpha s(\lambda-1)+\alpha\left(\frac{s}{p_k}-1\right)} \prod_{i=2,\, i\neq k}^{n} t_i^{\alpha\left(\frac{s}{p_i}-1\right)}}{\left(1+\sum_{j=2}^{n} t_j^{\lambda}\right)^{\alpha s}} (dt_2)^{\alpha} \dots (dt_n)^{\alpha},$$

for $k = 2, 3, \dots, n$. By using Lemma 2, we obtain

$$I_1 = \frac{1}{\Gamma_\alpha(s)\lambda^{\alpha(n-1)}} \Gamma_\alpha\left(\frac{sp_1(\lambda-1)+s}{p_1\lambda}\right) \prod_{i=2}^{n} \Gamma_\alpha\left(\frac{s}{p_i\lambda}\right),$$

and similarly

$$I_k = \frac{1}{\Gamma_\alpha(s)\lambda^{\alpha(n-1)}} \Gamma_\alpha\left(\frac{sp_{k+1}(\lambda-1)+s}{p_k\lambda}\right) \prod_{i=1,\, i\neq k}^{n} \Gamma_\alpha\left(\frac{s}{p_i\lambda}\right),$$

for $k = 2, \dots, n$. Finally, from (22) we get (21).

□

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