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BAYESIAN ESTIMATION FOR THE PARAMETER OF AREA BIASED MAXWELL DISTRIBUTION

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Abstract

In this paper, the area biased Maxwell distribution is considered for Bayesian analysis. The expressions for Bayes estimators of the parameter have been derived under squared error, precautionary, entropy, K-loss, and Al-Bayyati's loss functions by using quasi and gamma priors.

1. Introduction

In science there are a lot of applications of Maxwell (or Maxwell-Boltzmann) distribution. It was Tyagi and Bhattacharya [1, 2] who considered the Maxwell distribution as a lifetime model. Chaturvedi and Rani [3] obtained classical and Bayes estimators for the Maxwell distribution. Rao et al. [4, 5, 6] obtained the Bayes estimators by using

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different loss functions. The probability density function of area biased Maxwell distribution (Reshi et al. [7]) is given by

$$f(x;\ \theta) = \frac{2^{-3/2}}{\Gamma(5/2)} \theta^{5/2} x^4 e^{-\theta x^2/2}; \quad x \geq 0,\ \theta > 0, \tag{1}$$

where $\Gamma(\cdot)$ is a gamma function.

The joint density function or likelihood function of (1) is given by

$$f(\underline{x};\ \theta) = \frac{2^{-3n/2}}{(\Gamma(5/2))^n} \theta^{5n/2} \left(\prod_{i=1}^{n} x_i^4 \right) e^{-\frac{\theta}{2} \sum_{i=1}^{n} x_i^2}. \tag{2}$$

The logarithm likelihood function is given by

$$\log f(\underline{x};\ \theta) = n \log \left(\frac{2^{-3/2}}{\Gamma(5/2)} \right) + \frac{5n}{2} \log \theta + \log \left(\prod_{i=1}^{n} x_i^4 \right) - \frac{\theta}{2} \sum_{i=1}^{n} x_i^2. \tag{3}$$

Differentiating (3) with respect to θ and equating to zero, we get the maximum likelihood estimator of θ which is given by

$$\hat{\theta} = \frac{5n}{\sum_{i=1}^{n} x_i^2}. \tag{4}$$

2. Bayesian Method of Estimation

The Bayesian inference procedures have been developed generally under squared error loss function

$$L(\hat{\theta},\ \theta) = (\hat{\theta} - \theta)^2. \tag{5}$$

The Bayes estimator under the above loss function, say, $\hat{\theta}_S$ is the posterior mean, i.e.,

$$\hat{\theta}_S = E(\theta). \tag{6}$$

Zellner [8] and Basu and Ebrahimi [9] have recognized that the inappropriateness of using symmetric loss function. Norstrom [10] introduced precautionary loss function is given as

$$L(\hat{\theta}, \theta) = \frac{(\hat{\theta} - \theta)^2}{\hat{\theta}}. \tag{7}$$

The Bayes estimator under precautionary loss function is denoted by $\hat{\theta}_p$ and is obtained by solving the following equation:

$$\hat{\theta}_P = [E(\theta^2)]^{1/2}. \tag{8}$$

In many practical situations, it appears to be more realistic to express the loss in terms of the ratio $\frac{\hat{\theta}}{\theta}$. In this case, Calabria and Pulcini [11] points out that a useful asymmetric loss function is the entropy loss

$$L(\delta) \propto [\delta^p - p \log_e(\delta) - 1],$$

where $\delta = \frac{\hat{\theta}}{\theta}$, and whose minimum occurs at $\hat{\theta} = \theta$. Also, the loss function $L(\delta)$ has been used in Dey et al. [12] and Dey and Liu [13], in the original form having $p = 1$. Thus $L(\delta)$ can written be as

$$L(\delta) = b[\delta - \log_e(\delta) - 1];\ b > 0. \tag{9}$$

The Bayes estimator under entropy loss function is denoted by $\hat{\theta}_E$ and is obtained by solving the following equation:

$$\hat{\theta}_E = \left[E\left(\frac{1}{\theta}\right)\right]^{-1}. \tag{10}$$

Wasan [14] proposed the K-loss function which is given as

$$L(\hat{\theta}, \theta) = \frac{(\hat{\theta} - \theta)^2}{\hat{\theta}\theta}. \tag{11}$$

Under K-loss function the Bayes estimator of θ is denoted by $\hat{\theta}_K$ and is obtained as

$$\hat{\theta}_K = \left[\frac{E(\theta)}{E(1/\theta)}\right]^{\frac{1}{2}}. \tag{12}$$

Al-Bayyati [15] introduced a new loss function using Weibull distribution which is given as

$$L(\hat{\theta}, \theta) = \theta^c (\hat{\theta} - \theta)^2. \tag{13}$$

Under Al-Bayyati's loss function the Bayes estimator of θ is denoted by $\hat{\theta}_{Al}$ and is obtained as

$$\hat{\theta}_{Al} = \frac{E(\theta^{c+1})}{E(\theta^c)}. \tag{14}$$

Let us consider two prior distributions of θ to obtain the Bayes estimators.

(i) Quasi-prior: For the situation where we have no prior information about the parameter θ, we may use the quasi density as given by

$$g_1(\theta) = \frac{1}{\theta^d}; \; \theta > 0, \; d \geq 0, \tag{15}$$

where $d = 0$ leads to a diffuse prior and $d = 1$, a non-informative prior.

(ii) Gamma prior: Generally, the gamma density is used as prior distribution of the parameter θ given by

$$g_2(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}; \quad \theta > 0. \tag{16}$$

3. Posterior Density Under $g_1(\theta)$

The posterior density of θ under $g_1(\theta)$, on using Equation (2), is given by

$$f(\theta / \underline{x}) = \frac{\frac{2^{-3n/2}}{(\Gamma(5/2))^n}\theta^{5n/2}\left(\prod_{i=1}^{n} x_i^4\right) e^{-\frac{\theta}{2}\sum_{i=1}^{n} x_i^2}\theta^{-d}}{\int_0^\infty \frac{2^{-3n/2}}{(\Gamma(5/2))^n}\theta^{5n/2}\left(\prod_{i=1}^{n} x_i^4\right) e^{-\frac{\theta}{2}\sum_{i=1}^{n} x_i^2}\theta^{-d}d\theta}$$

$$= \frac{\theta^{\frac{5n}{2}-d} e^{-\frac{\theta}{2}\sum_{i=1}^{n} x_i^2}}{\int_0^\infty \theta^{\frac{5n}{2}-d} e^{-2\frac{\theta}{2}\sum_{i=1}^{n} x_i^2} d\theta}$$

$$= \frac{\left(\frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{\frac{5n}{2}-d+1}}{\Gamma\left(\frac{5n}{2}-d+1\right)}\theta^{\frac{5n}{2}-d} e^{-\frac{\theta}{2}\sum_{i=1}^{n} x_i^2}. \tag{17}$$

Theorem 1. *On using* (17), *we have*

$$E(\theta^c) = \frac{\Gamma\left(\frac{5n}{2}-d+c+1\right)}{\Gamma\left(\frac{5n}{2}-d+1\right)}\left(\frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{-c}. \tag{18}$$

Proof. By definition,

$$E(\theta^c) = \int \theta^c f(\theta / \underline{x}) d\theta$$

$$= \frac{\left(\frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{\frac{5n}{2}-d+1}}{\Gamma\left(\frac{5n}{2}-d+1\right)}\int_0^\infty \theta^{\left(\frac{5n}{2}-d+c\right)} e^{-\frac{\theta}{2}\sum_{i=1}^{n} x_i^2} d\theta$$

$$= \frac{\left(\frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{\frac{5n}{2}-d+1}}{\Gamma\left(\frac{5n}{2}-d+1\right)} \frac{\Gamma\left(\frac{5n}{2}-d+c+1\right)}{\left(\frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{\frac{5n}{2}-d+c+1}}$$

$$= \frac{\Gamma\left(\frac{5n}{2}-d+c+1\right)}{\Gamma\left(\frac{5n}{2}-d+1\right)} \left(\frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{-c}.$$

From equation (18), for $c = 1,$ we have

$$E(\theta) = \left(\frac{5n}{2}-d+1\right)\left(\frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{-1}. \tag{19}$$

From Equation (18), for $c = 2,$ we have

$$E\left(\theta^2\right) = \left(\frac{5n}{2}-d+2\right)\left(\frac{5n}{2}-d+1\right)\left(\frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{-2}. \tag{20}$$

From Equation (18), for $c = -1,$ we have

$$E\left(\frac{1}{\theta}\right) = \frac{\sum_{i=1}^{n} x_i^2}{5n-d}. \tag{21}$$

From Equation (18), for $c = c+1,$ we have

$$E\left(\theta^{c+1}\right) = \frac{\Gamma\left(\frac{5n}{2}-d+c+2\right)}{\Gamma\left(\frac{5n}{2}-d+1\right)}\left(\frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{-(c+1)}. \tag{22}$$

4. Bayes Estimators Under $g_1(\theta)$

From Equation (6), on using (19), the Bayes estimator of θ under squared error loss function is given by

$$\hat{\theta}_S = \left(\frac{5n}{2} - d + 1\right)\left(\frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{-1}. \tag{23}$$

From Equation (8), on using (20), the Bayes estimator of θ under precautionary loss function is given by

$$\hat{\theta}_P = \left[\left(\frac{5n}{2} - d + 2\right)\left(\frac{5n}{2} - d + 1\right)\right]^{\frac{1}{2}}\left(\frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{-1}. \tag{24}$$

From Equation (10), on using (21), the Bayes estimator of θ under entropy loss function is given by

$$\hat{\theta}_E = \frac{5n - d}{\sum_{i=1}^{n} x_i^2}. \tag{25}$$

From Equation (12), on using (19) and (21), the Bayes estimator of θ under K-loss function is given by

$$\hat{\theta}_K = \left[\left(\frac{5n}{2} - d + 1\right)\left(\frac{5n}{2} - d\right)\right]^{\frac{1}{2}}\left(\frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{-1}. \tag{26}$$

From Equation (14), on using (18) and (22), the Bayes estimator of θ under Al-Bayyati's loss function is given by

$$\hat{\theta}_{Al} = \left(\frac{5n}{2} - d + c + 1\right)\left(\frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{-1}. \tag{27}$$

5. Posterior Density Under $g_2(\theta)$

Under $g_2(\theta)$, the posterior density of θ, using Equation (2), is obtained as

$$f(\theta / \underline{x}) = \frac{\frac{2^{-3n/2}}{(\Gamma(5/2))^n} \theta^{5n/2} \left(\prod_{i=1}^{n} x_i^4 \right) e^{-\frac{\theta}{2} \sum_{i=1}^{n} x_i^2} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}}{\int_0^\infty \frac{2^{-3n/2}}{(\Gamma(5/2))^n} \theta^{5n/2} \left(\prod_{i=1}^{n} x_i^4 \right) e^{-\frac{\theta}{2} \sum_{i=1}^{n} x_i^2} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} d\theta}$$

$$= \frac{\theta^{\frac{5n}{2}+\alpha-1} e^{-\left(\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2\right)\theta}}{\int_0^\infty \theta^{\frac{5n}{2}+\alpha-1} e^{-\left(\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2\right)\theta} d\theta}$$

$$= \frac{\theta^{\frac{5n}{2}+\alpha-1} e^{-\left(\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2\right)\theta}}{\Gamma\left(\frac{5n}{2}+\alpha\right) \Big/ \left(\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{\frac{5n}{2}+\alpha}}$$

$$= \frac{\left(\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{\frac{5n}{2}+\alpha}}{\Gamma\left(\frac{5n}{2}+\alpha\right)} \theta^{\frac{5n}{2}+\alpha-1} e^{-\left(\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2\right)\theta}. \tag{28}$$

Theorem 2. *On using* (28), *we have*

$$E\left(\theta^c\right) = \frac{\Gamma\left(\frac{5n}{2}+\alpha+c\right)}{\Gamma\left(\frac{5n}{2}+\alpha\right)} \left(\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{-c}. \tag{29}$$

Proof. By definition,

$$E\left(\theta^c\right) = \int \theta^c f(\theta / \underline{x}) d\theta$$

$$= \frac{\left(\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{\frac{5n}{2}+\alpha}}{\Gamma\left(\frac{5n}{2} + \alpha\right)} \int_0^\infty \theta^{\frac{5n}{2}+\alpha+c-1} e^{-\left(\beta+\frac{1}{2}\sum_{i=1}^{n} x_i^2\right)\theta} d\theta$$

$$= \frac{\left(\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{\frac{5n}{2}+\alpha}}{\Gamma\left(\frac{5n}{2} + \alpha\right)} \frac{\Gamma\left(\frac{5n}{2} + \alpha + c\right)}{\left(\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{\frac{5n}{2}+\alpha+c}}$$

$$= \frac{\Gamma\left(\frac{5n}{2} + \alpha + c\right)}{\Gamma\left(\frac{5n}{2} + \alpha\right)} \left(\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{-c}.$$

From Equation (29), for $c = 1$, we have

$$E(\theta) = (5n + 2\alpha)\left(2\beta + \sum_{i=1}^{n} x_i^2\right)^{-1}. \tag{30}$$

From Equation (29), for $c = 2$, we have

$$E\left(\theta^2\right) = \left(\frac{5n}{2} + \alpha + 1\right)\left(\frac{5n}{2} + \alpha\right)\left(\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{-2}. \tag{31}$$

From Equation (29), for $c = -1$, we have

$$E\left(\frac{1}{\theta}\right) = \frac{2\beta + \sum_{i=1}^{n} x_i^2}{5n + 2\alpha - 2}. \tag{32}$$

From Equation (29), for $c = c + 1$, we have

$$E\left(\theta^{c+1}\right) = \frac{\Gamma\left(\frac{5n}{2} + \alpha + c + 1\right)}{\Gamma\left(\frac{5n}{2} + \alpha\right)} \left(\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{-(c+1)}. \tag{33}$$

6. Bayes Estimators Under $g_2(\theta)$

From Equation (6), on using (30), the Bayes estimator of θ under squared error loss function is given by

$$\hat{\theta}_S = (5n + 2\alpha)\left(2\beta + \sum_{i=1}^{n} x_i^2\right)^{-1}. \tag{34}$$

From Equation (8), on using (31), the Bayes estimator of θ under precautionary loss function is given by

$$\hat{\theta}_P = \left[\left(\frac{5n}{2} + \alpha + 1\right)\left(\frac{5n}{2} + \alpha\right)\right]^{\frac{1}{2}} \left(\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{-1}. \tag{35}$$

From Equation (10), on using (32), the Bayes estimator of θ under entropy loss function is given by

$$\hat{\theta}_E = \frac{5n + 2\alpha - 2}{2\beta + \sum_{i=1}^{n} x_i^2}. \tag{36}$$

From Equation (12), on using (30) and (32), the Bayes estimator of θ under K-loss function is given by

$$\hat{\theta}_K = \left[\left(\frac{5n}{2} + \alpha\right)\left(\frac{5n}{2} + \alpha - 1\right)\right]^{\frac{1}{2}} \left(\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{-1}. \tag{37}$$

From Equation (14), on using (29) and (33), the Bayes estimator of θ under Al-Bayyati's loss function is given by

$$\hat{\theta}_{Al} = \left(\frac{5n}{2} + \alpha + c\right)\left(\beta + \frac{1}{2}\sum_{i=1}^{n} x_i^2\right)^{-1}. \tag{38}$$

7. Conclusion

In this paper, we have obtained a number of estimators of parameter of area biased Maxwell distribution. In Equation (4), we have obtained the maximum likelihood estimator of the parameter. In Equations (23)-(27), we have obtained the Bayes estimators under different loss functions using quasi prior. In Equations (34)-(38), we have obtained the Bayes estimators under different loss functions using gamma prior. In the above equation, it is clear that the Bayes estimators depend upon the parameters of the prior distribution.

References

[1] R. K. Tyagi and S. K. Bhattacharya, Bayes estimation of the Maxwell's velocity distribution function, Statistica, Anno XLIX 4 (1989a), 563-567.

[2] R. K. Tyagi and S. K. Bhattacharya, A note on the MVU estimation of reliability for the Maxwell failure distribution, Estadistica 41(137), (1989b), 73-79.

[3] A. Chaturvedi and U. Rani, Classical and Bayesian reliability estimation of the generalized Maxwell failure distribution, Journal of Statistical Research 32 (1998), 113-120.

[4] A. K. Rao, H. Pandey and K. L. Singh, Bayesian estimation of the parameter of the p-dimensional size-biased Rayleigh distribution, Lithuanian Journal of Statistics 56(1) (2017), 88-91.

[5] A. K. Rao and H. Pandey, Parameter estimation for the class of life-time distributions, Bulletin of Mathematics and Statistics Research 8(2) (2020), 42-48.

DOI: https://doi.org/10.33329/bomsr.8.2.42

[6] A. K. Rao and H. Pandey, Parameter estimation of Lomax distribution under weighted loss function, Scholars Journal of Physics, Mathematics and Statistics 7(7) (2020), 104-109.

DOI: https://doi.org/10.36347/sjpms.2020.v07i07.005

[7] Aijaz Ahmad Dar, A. Ahmed and J. A. Reshi, Characterization and estimation of weighted Maxwell-Boltzmann distribution, Applied Mathematics & Information Sciences 12(1) (2018), 193-302.

DOI: https://doi.org/10.18576/amis/120119

[8] A. Zellner, Bayesian estimation and prediction using asymmetric loss functions, Journal of the American Statistical Association 81(394) (1986), 446-451.

DOI: https://doi.org/10.2307/2289234

[9] A. P. Basu and N. Ebrahimi, Bayesian approach to life testing and reliability estimation using asymmetric loss function, Journal of Statistical Planning and Inference 29(1-2) (1991), 21-31.

DOI: https://doi.org/10.1016/0378-3758(92)90118-C

[10] J. G. Norstrom, The use of precautionary loss functions in risk analysis, IEEE Transactions on Reliability 45(3) (1996), 400-403.

DOI: https://doi.org/10.1109/24.536992

[11] R. Calabria and G. Pulcini, Point estimation under asymmetric loss functions for left-truncated exponential samples, Communications in Statistics: Theory and Methods 25(3) (1994), 585-600.

DOI: https://doi.org/10.1080/03610929608831715

[12] D. K. Dey, M. Ghosh and C. Srinivasan, Simultaneous estimation of parameters under entropy loss, Journal of Statistical Planning and Inference 15 (1987), 347-363.

DOI: https://doi.org/10.1016/0378-3758(86)90108-4

[13] D. K. Dey and Pei-San Liao Liu, On comparison of estimators in a generalized life model, Microelectronics Reliability 32(1/2) (1992), 207-221.

DOI: https://doi.org/10.1016/0026-2714(92)90099-7

[14] M. T. Wasan, Parametric Estimation, McGraw-Hill, New York, 1970.

[15] H. N. Al-Bayyati, Comparing Methods of Estimating Weibull Failure Models Using Simulation, Ph.D. Thesis, College of Administration and Economics, Baghdad University, Iraq, 2002.

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