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HYDROGEN ATOM MODEL IN GEOMETRIC ALGEBRA TERMS

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Abstract

Geometric Algebra formalism opens the door to developing a theory replacing conventional quantum theory (Mathematics Subject Classification, item 81). Generalizations, stemming from changing of complex numbers by geometrically feasible objects in three dimensions, followed by unambiguous definition of states, observables, measurements, bring into reality clear explanations of weird quantum mechanical features, for example, primitively considering atoms as a kind of planetary system, very familiar from macroscopic experience but recklessly used in a physically very different situation. In the current work the three-sphere becomes the playground of the torsion kind states eliminating abstract Hilbert space vectors. The states as $\mathbb{S}^3$ points evolve, governed by updated Schrodinger equation, and act as operators on observables in measurements.

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1. Introduction: Experimental Reasons of the Planetary Model of Atoms

The Rutherford scattering experiments [1] demonstrated that positively charged part of atoms was very small, less than 10^{-14} m, and only one of hundred thousand positively charged alpha particles got deflected by angles greater than 90 degrees when bombarding gold leaf of only a few atoms thick. The logically reasonable conclusion was that positively charged nucleus concentrating majority of mass occupied very small regions.

Scattering results were explained using the retractive force between positively charged alpha particles and nucleus. Calculation of the force followed from Coulomb potential that was only experimentally justified for macroscopic point like objects but recklessly used in a physically very different situation.

So it happened that the planetary model appeared from viewing atoms as something very familiar from macroscopic experience: planets (electrons) rotating around the Sun (nucleus). Mathematical formalism was also taken from available mathematics at hand: mostly the Hilbert spaces [2].

The approach suggested in this article uses different point of view and assumes that actual weirdness of all conventional quantum mechanics comes from logical inconsistence of what is meant by basic quantum mechanical definitions and has nothing to do with the phenomena scale. The theory should speak about proper separation of measurement process arrangement into operator, three-sphere $\mathbb{S}^3$ element, acting on observable, and operand, measured observable.

Unambiguous definition of states and observables, does not matter are we in “classical” or “quantum” frame, should follow the general paradigm, [3], [4], [5], [6]:

– Measurement of observable $O(\mu)$ by state[1] $S(\lambda)$ is a map:

$$(S(\lambda),\, O(\mu)) \to O(v),$$

where $O(\mu)$ is an element of the set of observables, $S(\lambda)$ is element of, generally though not necessarily, another set, set of states.

– The result (value) of a measurement of observable $O(\mu)$ by state $S(\lambda)$ is the result of a map sequence:

$$(S(\lambda),\, O(\mu)) \to O(v) \to V(B),$$

where V is a set of (Boolean) algebra subsets identifying possible results of measurements.

State and observable are different things. Evolution of a state should be considered separately, and then action of modified state will be applied to an observable in measurements.

The option to expand, to lift the space where physical processes are considered, may have critical consequences to a theory. A kind of expanding is the core of the suggested formulation aimed at the theory deeper than conventional quantum mechanics. States as Hilbert space complex valued vectors with formal imaginary unit are lifted to a torsion kind object identified by points on sphere $\mathbb{S}^3$.

2. Working with *g*-Qubits Instead of Qubits

A theory that is an alternative to conventional quantum mechanics has been under development for a while, see, [3], [4], [7], [6], [8].

Its novel features are:

– Replacing complex numbers by elements of even subalgebra of geometric algebra in three dimensions, that's by elements of the form "scalar plus bivector".

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[1]One should say "by a state". State is operator acting on observable.

– Elementary physical objects bear the structure: position in space plus explicitly defined object as the G_3, geometric algebra in three dimensions, elements.

– Operators acting on those objects are identified as direct sums of position translation and points on the three-sphere $\mathbb{S}^3$. All those points are connected, due to Hedgehog theorem, by parallel (Clifford) translations.

– Evolution of the $\mathbb{S}^3$ part of operators by Clifford translations is governed by generalization of the Schrodinger equation with unit bivectors in three dimensions instead of formal imaginary unit.

In the following the $\mathbb{S}^3$ part of the operators will only be considered.

The even subalgebra G_3^+ is subalgebra of elements of the form $M_3 = \alpha + I_S\beta$, where α and β are (real)[2] scalars and I_S is some unit bivector arbitrary placed in three-dimensional space. Elements of G_3^+ can be depict as in Figure 2.1.

[2]In the current formalism scalars can only be real numbers. “Complex” scalars make no sense anymore, see, for example, [4], [8].

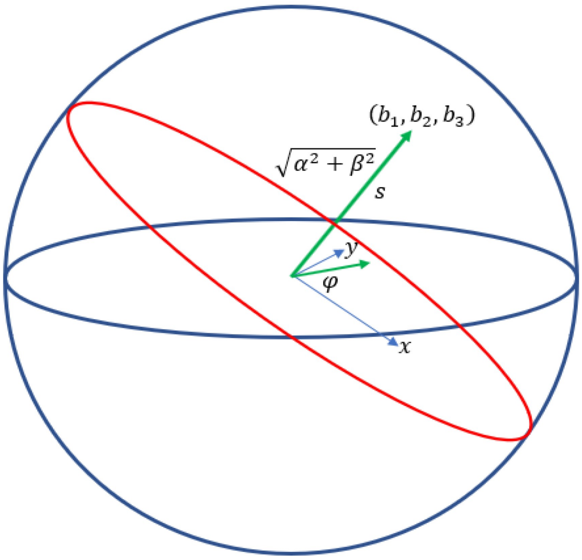

Figure 2.1. An element of G_3^+.

In the G_3^+ multiplication is more complicated than in Hilbert space C^2. It reads

$$g_1 g_2 = (\alpha_1 + I_{s_1}\beta_1)(\alpha_2 + I_{s_2}\beta_2) = \alpha_1\alpha_2 + I_{s_1}\alpha_2\beta_1 + I_{s_2}\alpha_1\beta_2 + I_{s_1}I_{s_2}\beta_1\beta_2.$$

It is not commutative due to the not commutative product of bivectors $I_{s_1}I_{s_2}$. Indeed, taking vectors to which I_{s_1} and I_{s_2} are dual: $s_1 = -I_3 I_{s_1}$, $s_2 = -I_3 I_{s_2}$, we have

$$I_{s_1}I_{s_2} = -s_1 \cdot s_2 - I_3(s_1 \times s_2),\ I_3 \text{ is oriented unit value volume.}$$

Then,

$$g_1 g_2 = \alpha_1\alpha_2 - (s_1 \cdot s_2)\beta_1\beta_2 + I_{s_1}\alpha_2\beta_1 + I_{s_2}\alpha_1\beta_2 - I_3(s_1 \times s_2)\beta_1\beta_2,$$

and

$$g_2 g_1 = \alpha_1\alpha_2 - (s_1 \cdot s_2)\beta_1\beta_2 + I_{s_1}\alpha_2\beta_1 + I_{s_2}\alpha_1\beta_2 + I_3(s_1 \times s_2)\beta_1\beta_2.$$

We see that if $I_{s_1} = I_{s_2} = I_s$, then $s_1 \cdot s_2 = 1$, $s_1 \times s_2 = 0$, so

$$g_1 g_2 = g_2 g_1 = \alpha_1\alpha_2 - \beta_1\beta_2 + I_s(\alpha_2\beta_1 + \alpha_1\beta_2),$$

that is the same, up to replacing i by I_s, as for complex numbers.

Unit value elements of G_3^+, when $\alpha^2 + \beta^2 = 1$, will be called *g-qubits*. The wave functions, states, implemented as g-qubits store much more information than qubits, see Figure 2.2.

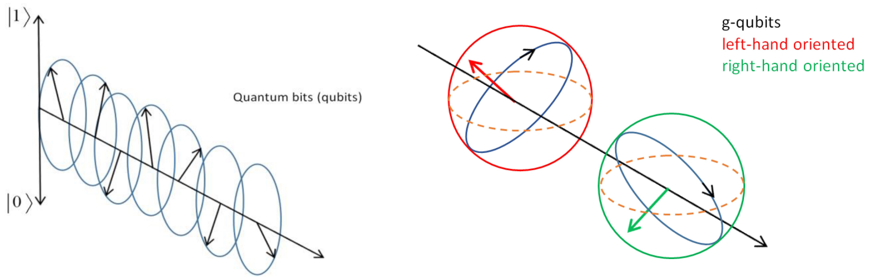

Figure 2.2. Geomectrically picted qubits and g-qubits.

3. Implementation of the Definitions from the Introduction in the *g*-Qubit State Case

General definition of measurement in the suggested approach is based on:

– the set of observables, particularly elements of G_3^+,

– the set of states, normalized elements of G_3^+, g-qubits,

– special case of measurement of a G_3^+ observable $C = C_0 + C_1 B_1 + C_2 B_2 + C_3 B_2$ by g-qubit (wave function) $\alpha + I_s\beta = \alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3$ is defined as

$$(\alpha - I_s\beta) C (\alpha + I_s\beta),$$

with the result:

$$
\begin{aligned}
& C_0 + C_1 B_1 + C_2 B_2 + C_3 B_3 \xrightarrow{\alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3} C_0 \\
& + (C_1[(\alpha^2 + \beta_1^2) - (\beta_2^2 + \beta_3^2)] + 2C_2(\beta_1\beta_2 - \alpha\beta_3) + 2C_3(\alpha\beta_2 + \beta_1\beta_3))B_1 \\
& + (2C_1(\alpha\beta_3 + \beta_1\beta_2) + C_2[(\alpha^2 + \beta_2^2) - (\beta_1^2 + \beta_3^2)] + 2C_3(\beta_2\beta_3 - \alpha\beta_1))B_2 \\
& + (2C_1(\beta_1\beta_3 - \alpha\beta_2) + 2C_2(\alpha\beta_1 + \beta_2\beta_3) + C_3[(\alpha^2 + \beta_3^2) - (\beta_1^2 + \beta_2^2)])B_3.
\end{aligned}
\tag{3.1}
$$

Since g-qubit (state, wave function) is normalized, the measurement can be written in exponential form:

$$e^{-I_S\varphi} C e^{I_S\varphi},$$

where $\varphi = \cos^{-1}\alpha$. The above is updated variant of quantum mechanical formula $\langle\psi|A|\psi\rangle$.

The lift from C^2 to G_3^+ needs a $\{B_1, B_2, B_3\}$ reference frame of unit value bivectors. This frame, as a solid, can be arbitrary rotated in three dimensions. In that sense we have principal fiber bundle $G_3^+ \to C^2$ with the standard fiber as group of rotations which is also effectively identified by elements of G_3^+. Probabilities of the results of measurements are measures of the $\mathbb{S}^3$ states giving considered results.

4. Evolution of *g*-Qubit States

Measurement of an observable C by a state $e^{I_S\varphi}$ is defined as $e^{-I_S\varphi} C e^{I_S\varphi}$. Evolution of a state is its movement on surface of $\mathbb{S}^3$.

Consider necessary formalism:

Multiplication of two geometric algebra exponents reads, see Subsection 1.2 of [8]:

$$e^{I_{s_1}\alpha}e^{I_{s_2}\beta} = (\cos\alpha + I_{s_1}\sin\alpha)(\cos\beta + I_{s_2}\sin\beta)$$

$$= \cos\alpha\cos\beta + I_{s_1}\sin\alpha\cos\beta + I_{s_2}\cos\alpha\sin\beta$$

$$+ I_{s_1}I_{s_2}\sin\alpha\sin\beta.$$

It follows from the formula for bivector multiplication

$$g_1g_2 = \alpha_1\alpha_2 - (s_1 \cdot s_2)\beta_1\beta_2 + I_{s_1}\alpha_2\beta_1 + I_{s_2}\alpha_1\beta_2 - I_3(s_1 \times s_2)\beta_1\beta_2,$$

with vectors to which the unit bivectors I_{s_1} and I_{s_2} are duals: $s_1 = -I_3I_{s_1}$ and $s_2 = -I_3I_{s_2}$. In the current case,

$$\alpha_1 = \cos\alpha, \quad \alpha_2 = \cos\beta, \quad \beta_1 = \sin\alpha, \quad \beta_2 = \sin\beta,$$

and we get above formula for $e^{I_{s_1}\alpha}e^{I_{s_2}\beta}$.

The product of two exponents is again an exponent, because generally $|g_1g_2| = |g_1||g_2|$ and $\left|e^{I_{s_1}\alpha}e^{I_{s_2}\beta}\right| = \left|e^{I_{s_1}\alpha}\right|\left|e^{I_{s_2}\beta}\right| = 1$, see Subsection 1.3 of [8].

Multiplication of an exponent by another exponent is often called *Clifford translation*. Using the term *translation* follows from the fact that Clifford translation does not change distances between the exponents it acts upon when we identify exponents as points on unit sphere $\mathbb{S}^3$:

$$\cos\alpha + I_s\sin\alpha = \cos\alpha + b_1\sin\alpha\, B_1 + b_2\sin\alpha\, B_2 + b_3\sin\alpha\, B_3$$

$$\Leftrightarrow \{\cos\alpha,\ b_1\sin\alpha,\ b_2\sin\alpha,\ b_3\sin\alpha\}$$

$$(\cos\alpha)^2 + (b_1\sin\alpha)^2 + (b_2\sin\alpha)^2 + (b_3\sin\alpha)^2 = 1.$$

This result follows again from $|g_1 g_2| = |g_1||g_2|$:

$$|e^{I_s \alpha}(g_1 - g_2)| = |e^{I_s \alpha}||g_1 - g_2| = |g_1 - g_2|.$$

Assume the angle α in Clifford translation is a variable one. Then in the case $I_{s_1} = const$:

$$\frac{\partial}{\partial \alpha} e^{I_{s_1} \alpha} = I_{s_1} e^{I_{s_1} \alpha}.$$

If I_{s_1} is dual to some unit vector H, $I_{s_1} = -I_3 H$ (this is the case of the matrix Hamiltonian map to G_3^+, see [5]), then $e^{I_{s_1} \alpha} = e^{-I_3 H \alpha} \equiv \psi(H, \alpha)$ and

$$\frac{\partial}{\partial \alpha} \psi(H, \alpha) = -I_3 H \psi(H, \alpha),$$

that is obviously Geometric Algebra generalization of the Schrodinger equation.

5. Model of the Alpha Particle Deflection by Nucleus

An observable will be from the set of duals, bivectors, of parallel vectors coming from a plane parallel to a plane passing through the nucleus. Section of the incoming particles by an arbitrary plane passing through the central line looking at the nucleus and parallel to incoming vectors is shown in Figure 5.1:

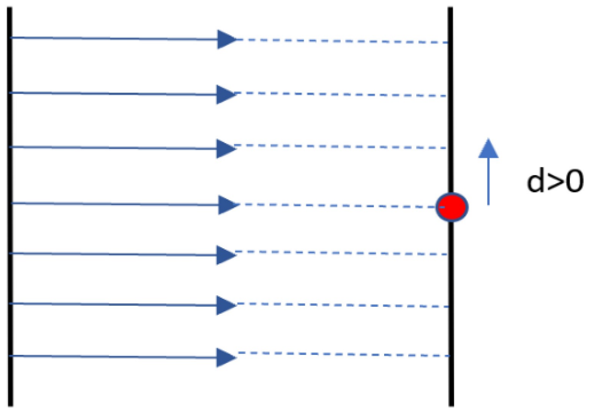

Figure 5.1. Incoming alpha particles.

An alpha particle moving along the vector passing exactly through the nucleus should deflect, bounce, in the opposite direction. Assuming that deflection happens as torsion in plane I_s, defined by bivector parallel to incoming vectors, and denoting an arbitrary incoming vector as $v(d)$ we get dual bivector of deflection of $v(0)$ (looking in the opposite direction):

$$e^{-I_s(\pi-0)} I_3 v(0) e^{I_s(\pi-0)},$$

or deflecting vector itself

$$I_3(e^{-I_s(\pi-0)} I_3 v(0) e^{I_s(\pi-0)}).$$

If we denote maximum value of at which deflection takes place by $D_{\max}$ then for arbitrary $-D_{\max} \leq d \leq D_{\max}$ the deflection vector is, see Figure 5.2:

$$I_3\left(e^{-I_s(sign(d)\pi - \pi\frac{d}{D_{\max}})} I_3 v(d) e^{I_s(sign(d)\pi - \pi\frac{d}{D_{\max}})} \right).$$

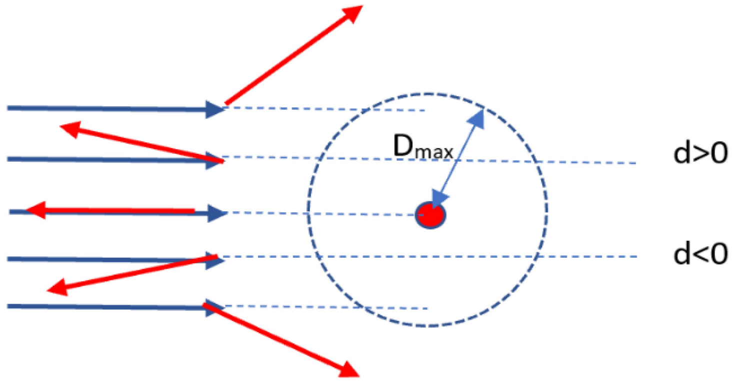

Figure 5.2. Results of deflection.

If we know the value of $d_{\pi/2}$ (assuming $d > 0$)) for particles deflected by angle equal to $\frac{\pi}{2}$ the nucleus effective radius $D_{\max}$ can be calculated, see Figure 5.3:

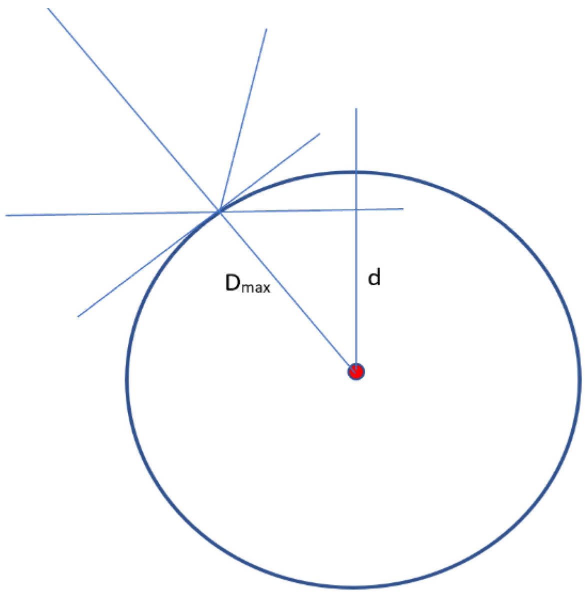

Figure 5.3. For the calculation of the effective radius.

From this figure, angle of deflection is

$$\delta = \pi - 2\left(\frac{\pi}{2} - \cos^{-1}\frac{D_{\max} - d_{\pi/2}}{D_{\max}}\right).$$

Then

$$\frac{\pi}{4} = \cos^{-1}\frac{D_{\max} - d_{\pi/2}}{D_{\max}},$$

and

$$D_{\max} = \frac{d_{\pi/2}}{1 - \frac{\sqrt{2}}{2}}. \quad (5.1)$$

So, whilst the nucleus occupies very small volume, it is extended up to an effective radius (5.1).

6. Movement of an Electron Relative to Nucleus

Let a state is of Hamiltonian type[3]:

$$\psi(H(t), t) = e^{-I_3(\frac{H(t)}{|H(t)|})|H(t)|}, \quad (6.1)$$

where $H(t)$ is vector in three dimensions.

An observable it will act upon is something of a torsion kind, $|r|e^{I_{s(t)}\omega t}$. Thus, at instant of time t we have the following result of action of state (6.1):

$$e^{I_3(\frac{H(t)}{|H(t)|})|H(t)|}|r|e^{I_{s(t)}\omega t}e^{-I_3(\frac{H(t)}{|H(t)|})|H(t)|}. \quad (6.2)$$

The Hamiltonian type of wave function (6.1) bears its origin from proton, while the observable $|r|e^{I_s\omega t}$ represents electron.

[3]Not critical, just for resembling traditional form of states.

The geometric algebra existence of the hydrogen atom can only follow from stable sequence of measurement results (6.2) with appropriate combination(s) of $H(t)$ and ω. Let

$$H(t) = -h_1(t)I_3B_1 - h_2(t)I_3B_2 - h_3(t)I_3B_3.$$

Then $|H(t)| = \sqrt{h_1^2(t) + h_2^2(t) + h_3^2(t)}$, bivector part of (6.1) is $\frac{\sin(|H(t)|t)}{|H(t)|}$ $(h_1(t)B_1 + h_2(t)B_2 + h_3(t)B_3)$, and the scalar part of the wave function (6.1) is $\cos(|H(t)|t)$.

If initial bivector plane of observable is $c_1B_1 + c_2B_2 + c_3B_3$, $c_1^2 + c_2^2 + c_3^2 = 1$, the scalar part is then $|r|\cos\omega t$, thus $|r|e^{I_s\omega t} = |r|\cos\omega t + |r|\sin\omega t$ $(c_1(t)B_1 + c_2(t)B_2 + c_3(t)B_3)$. Let us denote the plane $-I_3\left(\frac{H(t)}{|H(t)|}\right)$ as $I_{H(t)}$, $-I_3\left(\frac{H(t)}{|H(t)|}\right) \equiv I_{H(t)}$.

Assuming $I_{s(t)} = I_{H(t)}$, that's the torsion of electron instantly follows the action of wave function, then we get the following result of action of the state (6.1) on observable $|r|e^{I_s\omega t}$:

$$|r|e^{-I_{H(t)}^{|H(t)|}} e^{I_{s(t)}\omega t} e^{I_{H(t)}|H(t)|} = |r|e^{I_{H(t)}^{(\omega t + 2|H(t)|)}}.$$

That means that at every single instant of time the torsion angle ωt is additionally increased by $2|H(t)|$. It follows particularly that, for synchronization, after rotation by $2\pi n$, $n = 1, 2, \ldots,$ when $t = \frac{2\pi n}{\omega}$, the addition by $2\left|H\left(\frac{2\pi n}{\omega}\right)\right|$ should be $l\omega\frac{2\pi n}{\omega}$, $n = 1, 2, \ldots,$ $l = 0, 1, \ldots, n-1$. Thus, the equation connecting $H(t)$ and ω is

$$\left|H\left(\frac{2\pi n}{\omega}\right)\right| = \text{ln}, \; n = 1, 2, \ldots, \; l = 0, 1, \ldots, n-1.$$

The numbers l and n correspond to common quantum mechanics numbers, angular quantum number and principal quantum number.

7. Conclusion

It was demonstrated that the geometric algebra formalism along with generalization of complex numbers and subsequent lift of the two-dimensional Hilbert space valued qubits to geometrically feasible elements of even subalgebra [6] of geometric algebra in three dimensions allows, particularly, to explain what actually means senseless "find system in state". The approach particularly allows to eliminate primitive Bohr's planetary model of the hydrogen atom and explains the atom structure in pure geometrical terms.

Some other highly impressive perspectives of the approach comprise, particularly, explanations of the double-slit experiment, collapse of wave functions [9], possibility to modify blockchain information back and forth in time, see Subsection 6.3 of [8]. All that supports feasibility of the suggested approach to replace formalism of conventional quantum theory.

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