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TRIVIALITY OF A SURFACE-LINK WITH MERIDIAN-BASED FREE FUNDAMENTAL GROUP

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Abstract

It is proved that every disconnected surface-link with meridian-based free fundamental group is a trivial (i.e., an unknotted-unlinked) surface-link. This result is a surface-link version of the author's recent result on smooth unknotting of a surface-knot.

1. Introduction

A *surface-link* is a closed oriented (possibly disconnected) surface F embedded in the 4-space $\mathbf{R}^4$ by a smooth (or a piecewise-linear locally flat) embedding. When F is connected, it is also called a *surface-knot*. When F is an *r* copies of the 2-sphere S^2, it is called an S^2-link with *r* components. For our argument here, a surface-link in the 4-space $\mathbf{R}^4$ is

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considered as a surface-link in the 4-sphere R^4 which is the one-point compactification $\mathbf{R}^4 \cup \{\infty\} = S^4$. Two surface-links F and F' are *equivalent* by an *equivalence* f if F is sent to F' with the orientations preserved by an orientation-preserving diffeomorphism $f : S^4 \to S^4$. A *trivial* surface-link is a surface-link F which is the boundary of the union of mutually disjoint handlebodies smoothly embedded in S^4, where a handlebody is a 3-manifold which is a 3-ball or a disk sum of some number of solid tori. A trivial surface-knot is also called an *unknotted* surface-knot. A trivial disconnected surface-link is also called an *unknotted-unlinked* surface-link. For every given closed oriented (possibly disconnected) surface F, a trivial F-link in S^4 exists uniquely up to equivalences (cf. [3]). The *exterior* of a surface-link F in S^4 is the compact 4-manifold $E = \mathrm{cl}(S^4 \setminus N(F))$ for a tubular neighbourhood $N(F)$ of F in S^4. Let q_0 be a fixed base point in the interior of E. A surface-link with *meridian-based free fundamental group* is a surface-link F in S^4 such that the fundamental group $\pi_1(E, q_0)$ is a free group with a meridian basis. Note that a trivial surface-link is a surface-link with meridian-based free fundamental group. There was an earlier method to show that a ribbon surface-knot F in $\mathbf{R}^4$ is trivial if and only if it has the infinite cyclic fundamental group. This method is first to show that it is a TOP-trivial surface-knot by [1, 2, 5] and then uses the combination result of [6, 7, 8] that any two TOP-equivalent ribbon surface-links in $\mathbf{R}^4$ are equivalent (see [8, Corollary 2.6]). In [9, 10], another method for a (not necessarily ribbon) surface-knot in S^4 is proposed by using an idea of a stable-trivial surface-knot, where it is shown that a (not necessarily ribbon) surface-knot F in S^4 is trivial if and only if it has the infinite cyclic fundamental group. The purpose of this paper is to generalize this unknotting result of a surface-knot to the unknotting-unlinking result of a surface-link that a surface-link is trivial if and only if it has a meridian-based free fundamental group. Note that there are lots of non-trivial surface-links in S^4 with non-meridian based free fundamental groups. For example, let Γ be a connected non-circular spatial graph without degree one vertices in S^3 such that the fundamental group $\pi_1(S^3 \backslash \Gamma, q_0)$ is a free group and the graph Γ has a non-trivial constituent knot ℓ in S^3. Let B_0 be a 3-ball in S^3 which is a regular neighbourhood of a maximal tree of Γ in S^3, and $B = \mathrm{cl}(S^3 \backslash B_0)$ the complementary 3-ball. Then the intersection $t = \Gamma \cap B$ is a disconnected tangle without loop components in B. By Artin's spinning construction of the tangle (B, t), we have a disconnected S^2-link $L(t)$ in S^4 with the nontrivial S^2-component obtained by Artin's spinning construction of the non-trivial knot ℓ in S^3. As it is observed in [4, p. 204], the fundamental group $\pi_1(S^4 \backslash L(t), q_0)$ is isomorphic to the fundamental group $\pi_1(B \backslash t, q_0) \cong \pi_1(S^3 \backslash \Gamma, q_0)$ which is a free group by van Kampen theorem. Thus, the S^2-link $L(t)$ in S^4 is a non-trivial S^2-link with free fundamental group. The following unknotting-unlinking result is our main theorem, answering positively the problem [11, Problem 1.55 (B)] for any S^2-link.

Theorem 1.1. *A surface-link in S^4 is a trivial surface-link if and only if it is a surface-link with meridian-based free fundamental group.*

A *stabilization* of a surface-link F in S^4 is a connected sum $\bar{F} = F \#_{k=1}^{s} T_k$ of F and a trivial torus-link $T = \bigcup_{k=1}^{s} T_k$ in S^4. By granting $T = \emptyset$, we understand that a surface-link F itself is a stabilization of F. The trivial torus-link T is called the *stabilizer* with $T_k (k = 1, 2, \dots, s)$ the *stabilizer components* on the stabilization $\bar{F}$.

A *stably trivial* surface-link is a surface-link F in S^4 such that a stabilization $\overline{F}$ of F is a trivial surface-link. In [9, Corollary 1.2] with supplement [10], it is shown that every stably trivial surface-link is a trivial surface-link. Therefore, the proof of Theorem 1.1 is completed by combining the result of [9, Corollary 1.2] with supplement [10] and the following lemma, which generalizes the result of [3, Theorem 2.10] to a surface-link with meridian-based free fundamental group.

Lemma 1.2. *Every surface-link in* S^4 *with meridian-based free fundamental group is a stably trivial surface-link in* S^4.

The proof of Lemma 1.2 is done in the next section.

2. Proof of Lemma 1.2

The proof of Lemma 1.2 is done as follows.

2.1. Proof of Lemma 1.2. Let F be a surface-link in S^4 with components $F_i (i = 1, 2, \dots, r)$. Let $N(F) = \cup_{i=1}^{r} N(F_i)$ be a tubular neighbourhood of $F = \cup_{i=1}^{r} F_i$ in S^4 which is a trivial normal disk bundle $F \times D^2$ over F, where D^2 denotes the unit disk of complex numbers of norm $\leqq 1$, and $E = \mathrm{cl}(S^4 \setminus N(F))$ the exterior of a surface-link F. The boundary $\partial E = \partial N(F) = \cup_{i=1}^{r} \partial N(F_i)$ of the exterior E is a trivial normal circle bundle over $F = \cup_{i=1}^{r} F_i$. Identify $\partial N(F_i) = F_i \times S^1$ for $S^1 = \partial D^2$ such that the composite inclusion

$$F_i \times 1 \to \partial N(F_i) \to \mathrm{cl}(S^4 \setminus N(F_i))$$

induces the zero-map in the integral first homology. Let $q_i \times 1$ be a point in $F_i \times S^1$ for every $i (i = 1, 2, \dots, r)$. Let

$$K = (\cup_{i=1}^{r} a_i) \cup (\cup_{i=1}^{r} S_i)$$

be a connected graph in E such that

(1) a_i is an edge embedded in E joining q_0 and $q_i \times 1$ such that the interiors of $a_i (i = 1, 2, \ldots, r)$ are mutually disjoint,

(2) $S_i = q_i \times S^1 (i = 1, 2, \ldots, r)$, and

(3) the inclusion $K \to E$ induces an isomorphism $\pi_1(K, q_0) \to \pi_1(E, q_0)$ such that the element $t_i = [a_i \cup S_i] \in \pi_1(E, q_0)$ is the i-th meridian generator.

By the assumption that $\pi_1(E, q_0)$ is a free group with a meridian basis, there is a graph K with properties (1), (2) and (3). The following Observation 2.2 is used for the proof of Lemma 1.2.

Observation 2.2. The composite inclusion $F_i \times 1 \to \partial N(F_i) \to E$ is null-homotopic for all i.

Proof of Observation 2.2. Since $\partial N(F_i) = F_i \times S^1$, the fundamental group elements between the factors $F_i \times 1$ and $q_i \times S^1$ are commutive. On the other hand, since $\pi_1(E, q_0)$ is a free group, the image of the homomorphism $\pi_1(a_i \cup F_i \times 1, q_0) \to \pi_1(E, q_0)$ is in the infinite cyclic group $< t_i >$ generated by t_i. The surface $F_i \times 1$ in $\partial N(F_i) = F_i \times S^1$ is chosen so that the inclusion $F_i \times 1 \to \mathrm{cl}(S^4 \setminus N(F_i))$ induces the zero-map in the integral first homology. This implies that the inclusion $F_i \times 1 \to E$ is null-homotopic. This completes the proof of Observation 2.2. □

Let

$$K^N = (\cup_{i=1}^r a_i) \cup (\cup_{i=1}^r \partial N(F_i))$$

be a polyhedron in E. Let $p : K^N \to K$ be the map defined by the projection $F_i \times S^1 \to q_i \times S^1$ sending F_i to q_i for all i. The following Observation 2.3 is used for the proof of Lemma 1.2.

Observation 2.3. The map $p : K^N \to K$ extends to a map $g : E \to K$.

Proof of Observation 2.3. Since K is a $K(\pi, 1)$-space, there is a map $f : E \to K$ inducing the inverse isomorphism $\pi_1(E, q_0) \to \pi_1(K, p_0)$ of the isomorphism $\pi_1(K, p_0) \to \pi_1(E, q_0)$. Let $j : K^N \to E$ be the inclusion map. By Observation 2.2, the map $p : K^N \to K$ and the restriction map $fj : K^N \to K$ of f induce the same homomorphism

$$p_{\#} := (fj)_{\#} : \pi_1(K^N, q_0) \to \pi_1(K, q_0).$$

Since K is a $K(\pi, 1)$-space, the map fj is homotopic to p. By the homotopy extension property in [13], there is a map $g : E \to K$ homotopic to the map f such that $gj = p$. This completes the proof of Observation 2.3. □

Replacing the map g in Observation 2.3 by a piecewise smooth approximation keeping the map p fixed to use a transverse regularity argument. Assume that the point $q_i \times 1 \in q_i \times S^1$ is a regular point for each $i (i = 1, 2, \dots, r)$. Then the preimage $V_i = g^{-1}(q_i \times r)$ is a bi-collared compact oriented 3-manifold with boundary $\partial V_i = p^{-1}(q_i \times 1) = F_i \times 1$. By discarding closed components from V_i, it is shown that there are disjoint compact connected oriented 3-manifolds $V_i (i = 1, 2, \dots, r)$ smoothly embedded in the exterior E with $\partial V_i = F_i \times 1$ for all i. Let V_i^C be a compact connected oriented smooth 3-manifold in S^4 with boundary $\partial V_i^C = F_i$ obtained from V_i by adding the boundary collar $C_i = F_i \times [0, 1]$ of V_i, where $[0, 1]$ denotes the line segment in D^2 from the origin $0 \in D^2$ to $1 \in S^1$. Let $\alpha_{ik} (k = 1, 2, \dots, n_i)$ be mutually disjoint proper arcs in V_i such that the exterior $V_i^* = \mathrm{cl}(V_i \setminus \nu_i))$ for a regular neighbourhood $\nu_i = N(\bigcup_{k=1}^{n_i} \alpha_{ik})$ of the arcs $\alpha_{ik}(k = 1, 2, \ldots, n_i)$ in V_i is a handlebody. Let $\alpha_{ik}^C(k = 1, 2, \ldots, n_i)$ be mutually disjoint proper arcs in V_i^C obtained from the arcs $\alpha_{ik}(k = 1, 2, \ldots, n_i)$ by adding $(\partial\alpha_{ik}) \times [0, 1](k = 1, 2, \ldots, n_i)$. Then the exterior $V_i^{C*} = \mathrm{cl}(V_i^C \setminus \nu(C)_i))$ for a regular neighbourhood $\nu(C)_i = N(\bigcup_{k=1}^{n_i} \alpha_{ik}^C)$ of the arcs $\alpha_{ik}^C(k = 1, 2, \ldots, n_i)$ in V_i^C is a handlebody for all i. Let β_{ik} be a simple arc in F_i with $\partial\beta_{ik} = \partial\alpha_{ik}^C$ Let $\hat{\beta}_{ik}$ be a proper arc in V_i^C obtained from a simple arc β_{ik} in F_i by pushing the interior of β_{ik} into the open collar $C_i \setminus F_i$ of V_i. Note that every loop in V_i is null-homotopic in E because the map g induces an isomorphism $g_\# : \pi_1(E, q_0) \cong \pi_1(K, q_0)$. This means that the arc α_{ik}^C is homotopic to the arc $\hat{\beta}_{ik}$ in $(S^4 \setminus F) \cup F_i$ by a homotopy relative to F_i. By an argument of [3], we see that the 1-handles $h_{ik}(i = 1, 2, \ldots, r;\ k = 1, 2, \ldots, n_i)$ on F thickening the arcs $\alpha_{ik}^C(i = 1, 2, \ldots, r;\ k = 1, 2, \ldots, n_i)$ are disjoint trivial 1-handles on F embedded in S^4, along which the surgery of F is a stabilization $\bar{F}$ of F. Since the stabilized surface-link $\bar{F}$ bounds the disjoint handlebodies $V_i^{C*}(i = 1, 2, \ldots, r)$, the surface-link F is a stably trivial surface-link. This completes the proof of Lemma 1.2. □

Thus, the proof of Theorem 1.1 is completed.

A surface link F with $r(\geq 2)$ components is a *boundary surface-link* if it bounds mutually disjoint r bounded connected oriented smooth 3-manifolds in S^4. The following corollary is contained in the proof of Lemma 1.2 whose technique is known in classical link theory (see [12]).

Corollary 2.4. *A disconnected surface link F with r components is a boundary surface-link if there is an epimorphism from the fundamental group* $\pi_1(E, q_0)$ *onto a free group of rank r sending a meridian system to a basis.*

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