Transnational Journal of Mathematical Analysis and Applications
Vol. 10, Issue 2, 2022, Pages 115-143
ISSN 2347-9086
Published Online on January 20, 2023

http://jyotiacademicpress.org

ESTIMATION OF THE DOMAIN OF LOCATION OF A LIMIT CYCLE OF THE NONSYMMETRIC LIÉNARD SYSTEM

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Abstract

The Liénard system $\frac{dx}{dt} = y, \ \frac{dy}{dt} = -f(x)y - g(x)$ is considered. Under some assumptions on functions $f(x)$ and $g(x)$, we estimate the domain of location of the unique stable limit cycle of the Liénard system. This estimation has the form $\alpha_2 < x < \alpha_1$, where α_1 and α_2 are respectively, the positive and the negative roots of the equation $\int_0^\alpha \left[\int_0^x f(s)ds\right] g(x)dx = 0$. We use the above result for evaluating the amplitude of the limit cycle of the van der Pol equation $\ddot{x}(t) + \mu[x^2(t) - 1]\dot{x}(t) + x(t) = 0$.

2020 Mathematics Subject Classification: 34C05, 34C07, 34C25, 34C15, 34A26.
Keywords and phrases: nonsymmetric Liénard system, location of a limit cycle.
Communicated by Francisco Bulnes.
Received December 8, 2022

1. Introduction

One of the most difficult problems connected with the study of nonlinear systems of the form

$$\dot{x} = X(x, y), \quad \dot{y} = Y(x, y) \tag{1}$$

is the problem of the existence of limit cycles. Limit cycles, or isolated periodic solutions, are the most common form of solution observed when modelling physical systems in the plane. Early investigations were concerned with mechanical and electronic systems, but periodic behaviour is evident in all branches of science. In mathematics, in the study of dynamical systems with two-dimensional phase space, a limit cycle is a closed trajectory in phase space having the property that other neighbouring trajectories spirals into it either as time approaches infinity or as time approaches negative infinity. If all neighbouring trajectories approach the limit cycle as time approaches infinity, we say that the limit cycle is stable or attracting [1]. Limit cycles are an inherently nonlinear phenomenon; they cannot occur in linear systems. If functions $X(x, y)$ and $Y(x, y)$ in system (1) are linear, then system (1) cannot have a limit cycle.

Most of the early history in the theory of limit cycles was stimulated by practical problems displaying periodic behaviour. For example, the differential equation derived by Rayleigh [2] in 1877, related to the oscillation of a violin string, is given by

$$\frac{d^2y}{dt^2} + \mu\left[\frac{1}{3}\left(\frac{dy}{dt}\right)^3 - \left(\frac{dy}{dt}\right)\right] + y = 0. \tag{2}$$

In 1927, the Dutch scientist van der Pol described self-excited oscillations in an electrical circuit with a triode tube with resistive properties that change with the current. The equation derived by van der Pol has the following form [3]:

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0. \tag{3}$$

In dynamics, the van der Pol oscillator is a non-conservative oscillator with non-linear damping where μ is a positive parameter. Equations (2) and (3) are equivalent, as can be seen by differentiating Equation (2) with respect to t and putting $dy/dt = x$.

In 1928, the French physicist and engineer Alfred-Marie Liénard [4, 5] gave a criterion for the uniqueness of a periodic solution for a general class of equations of the form

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + x = 0, \tag{4}$$

for which the van der Pol equation is a special case. Setting $dx/dt = z$, Liénard wrote Equation (4) in the following form of the system of differential equations of first order:

$$\frac{dx}{dt} = z, \quad \frac{dz}{dt} = -x - f(x)z. \tag{5}$$

But in his proof of the uniqueness of a periodic solution of Equation (4), Liénard used other system of differential equations which is equivalent to system (5). For this, in system (5) he changed the variable $z = y - F(x)$, where

$$F(x) = \int_0^x f(\xi)d\xi, \tag{6}$$

and obtained the system

$$\frac{dx}{dt} = y - F(x), \; \frac{dy}{dt} = -x. \tag{7}$$

Equation (4) is referred to as a Liénard equation, and both systems of Equations (5) and (7) are called Liénard systems.

In 1942, Levinson and Smith [6] considered the following differential equation:

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0, \tag{8}$$

which is a generalization of Equation (4). Equation (8) as well as Equation (4) most of authors call the Liénard equation[1]. The qualitative properties of solutions of differential equation (8) have been studied in many papers [7, 8, 9, 10, 11, 12]. Equation (8) can be written in the form of the system of ordinary differential equations

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -f(x)y - g(x). \tag{9}$$

This system can be used to model mechanical systems, where $f(x)$ is known as the damping term and $g(x))$ is called the restoring force or stiffness. System (9) is also used to model resistor inductor capacitor circuits with nonlinear circuit elements.

One of important properties of system (9) is the existence of periodic solutions. An important case of a periodic solution is a limit cycle. In papers [13, 14, 15, 16, 19], the authors obtained conditions, under which system (9) or the equivalent system

$$\frac{dx}{dt} = z - F(x), \quad \frac{dz}{dt} = -g(x) \tag{10}$$

has a limit cycle. In papers [17, 18], the authors studied the location of a limit cycle of symmetric Liénard system (10) if $f(x)$ is even and $g(x)$ is odd. The aim of this paper is the estimation of the domain of location of a limit cycle of system (9) without assumption that $f(x)$ is even and $g(x)$ is odd.

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[1]Some authors call Equation (8) the generalized Liénard equation.

2. On the Existence of Periodic Solutions of System (10)

Let us find the conditions that ensure the existence of periodic solutions of system (9). Note that the periodic solution of system (9) exists if and only if there is a periodic solution of system (10). The following theorem gives sufficient conditions for the existence of periodic solutions of system (10).

Theorem 2.1. *Suppose that* $F(x)$ *is continuously differentiable,* $g(x)$ *is locally Lipschitz, and besides*

- $xg(x) > 0$ *for* $x \neq 0$;

- *the equation* $F(x) = 0$ *has three real roots:* $x = b_1 > 0$, $x = b_2 < 0$, *and* $x = 0$; $F(x) > 0$ *for* $x \in (b_2, 0) \bigcup (b_1, +\infty)$; $F(x) < 0$ *for* $x \in (-\infty, b_2) \bigcup (0, b_1)$;

- $F(x)$ *monotonically increases in the intervals* $(-\infty, b_2)$ *and* $(b_1, +\infty)$; $F(x) \to +\infty$ *as* $x \to +\infty$, $F(x) \to -\infty$ *as* $x \to -\infty$.

Then system (10) *has a unique nontrivial* (*nonzero*) *periodic solution.*

Proof. As has been shown in [11, 21], any solution of system (10) is a clockwise rotation around the origin, i.e., any solution that starts on the positive semiaxis of ordinate Oz, sequentially passes the first quadrant, then the fourth, third, second, first again, and so on. Consider the trajectory $x(t)$, $z(t)$ of system (10) in the plane Oxz starting at the point H with the coordinates $(0, z_H)$ at the zero moment of time t (see Figure 1).

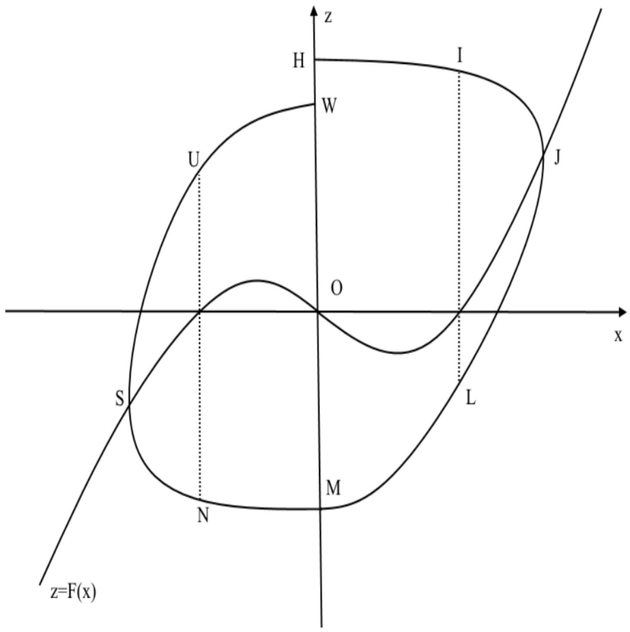

Figure 1.

Denote by J and S the points of intersection of this trajectory with the curve $z = F(x)$, by I and L the points of intersection of the trajectory with the straight line $x = b_1$, by U and N the points of intersection of the trajectory with the straight line $x = b_2$, and, finally, by W and M the points of intersection of the trajectory $x(t)$, $z(t)$ with the axis Oz.

Obviously, the solution $x(t)$, $z(t)$ is periodic if and only if the points H and W coincide, i.e., $z_H = z_W$.

Denote $G(x) := \int_0^x g(\xi)d\xi$. Consider the function $v(x, z) = \frac{z^2}{2} + G(x)$. Its derivative along solutions of system (10) is equal to

$$\frac{dv(x(t), z(t))}{dt} = -z(t)g(x(t)) + g(x(t))[z(t) - F(x(t))] = -g(x(t))F(x(t)). \tag{11}$$

The change of the function v from point H to point W is equal to

$$\Delta v = v(0,\, z_W) - v(0,\, z_H) = \int_0^{\tau} \frac{dv(x(t),\, z(t))}{dt}\, dt = -\int_0^{\tau} g(x(t))F(x(t))dt, \tag{12}$$

where τ is moment of time when the trajectory $x(t), z(t)$ reaches the point W. Assume that $x_J > b_1, x_S < b_2$. Let us show that Δv is a decreasing function of z_H. To do this, we break the trajectory between H and W into 6 pieces, where the first piece is a segment of the trajectory between points H and I, the second piece is a segment of the trajectory between points I and L, the third piece is the segment of the trajectory between the points L and M, the fourth piece is the segment of the trajectory between the points M and N, the fifth piece is a segment of the trajectory between the points N and U, the sixth piece is a segment of the trajectory between the points U and W. So Δv can be represented in the form $\Delta v = \sum_{i=1}^{6} \Delta v_i$, where Δv_i is the change of the function v on i-th piece of the trajectory. On the first, third, fourth and sixth pieces, z can be represented as a function of a variable x, because on these pieces $x(t)$ either monotonically increases or monotonically decreases; hence, the change of variable $dt = \dfrac{dx}{z - F(x)}$ is quite correct.

On the second and fifth pieces we use the substitution $dt = -\dfrac{dz}{g(x)}$. We want to argue that Δv is a monotonically decreasing function of z_H. So consider two trajectories starting at $t = 0$ from points $(0,\, z_H)$ and $(0,\, z_H + \Delta z_H)$, where $\Delta z_H > 0$. We denote the trajectories of system (10), starting at $t = 0$ from the points $(0,\, z_H)$ and $(0,\, z_H + \Delta z_H)$ by symbols $T1$ and $T2$, respectively. By virtue of the conditions of the theorem of existence and uniqueness of solutions of system (10), trajectories $T1$ and $T2$ have no common points, hence the trajectory $T2$ is located outside of the trajectory $T1$, i.e., any ray emerging from the origin, first intersects the trajectory $T1$ and then the trajectory $T2$. Let us discover how changes the expression for $\Delta v_i (i = 1, \dots, 6)$ in the transition from the trajectory $T1$ to the trajectory $T2$

$$\Delta v_1 = \int_0^{b_1} \frac{g(x)[-F(x)]}{z(x) - F(x)} dx = \int_0^{b_1} \frac{g(x)|F(x)|}{|z(x) - F(x)|} dx.$$

The value for $z(x)$ on the trajectory $T2$ is more then the value for $z(x)$ on $T1$, hence, $\Delta v_1(T2) < \Delta v_1(T1)$. Here and below $\Delta v_i(T2)$ and $\Delta v_i(T1)$ denote the values of Δv_i on trajectories $T2$ and $T1$, respectively.

$$\Delta v_2 = -\int_{z_I}^{z_L} g(x)F(x)\left[-\frac{dz}{g(x)}\right] = -\int_{z_L}^{z_I} F(x(z))dz.$$

Taking into account that on this piece $F(x)$ is positive and monotonically increasing and $x(z)|_{T2} > x(z)|_{T1}$, we obtain that $\Delta v_2(T2) < \Delta v_2(T1)$.

$$\Delta v_3 = \int_{b_1}^{0} \frac{g(x)[-F(x)]}{z(x) - F(x)} dx = \int_0^{b_1} \frac{g(x)|F(x)|}{|z(x) - F(x)|} dx.$$

In this case we also have $\Delta v_3(T2) < \Delta v_3(T1)$.

$$\Delta v_4 = \int_0^{b_2} \frac{[-g(x)]F(x)}{z(x) - F(x)} dx = \int_{b_2}^{0} \frac{[-g(x)]F(x)}{F(x) - z(x)} dx,$$

whence $\Delta v_4(T2) < \Delta v_4(T1)$.

$$\Delta v_5 = -\int_{z_N}^{z_U} g(x)F(x)\left[-\frac{dz}{g(x)}\right] = \int_{z_N}^{z_U} F(x(z))dz.$$

On this piece $F(x)$ is negative. Since $x(z)|_{T2} < x(z)|_{T1}$, then

$$F(x(z))|_{x(z)\in T2} < F(x(z))|_{x(z)\in T1},$$

hence $\Delta v_5(T2) < \Delta v_5(T1)$.

$$\Delta v_6 = -\int_{b_2}^{0} \frac{g(x)F(x)}{z(x) - F(x)}\, dx = \int_{b_2}^{0} \frac{[-g(x)]F(x)}{z(x) - F(x)}\, dx.$$

Here $z(x)|_{T2} > z(x)|_{T1}$, therefore $\Delta v_6(T2) < \Delta v_6(T1)$. Thus, it has been proved that $\Delta v_i (i = 1, \dots, 6)$ decrease if z_H increase, hence Δv also decreases with increasing z_H.

In cases $x_J \le b_1$ or $x_S \ge b_2$ (or both $x_J \le b_1$, $x_S \ge b_2$) the second or fifth pieces respectively (or both) of the trajectory *HJMSW* are absent but the proof that Δv is a decreasing function of z_H is the same.

Let us show that $\lim_{z_H \to +\infty} \Delta v = -\infty$. To do this, it is enough to prove that $\lim_{z_H \to +\infty} \Delta v_2 = -\infty$. We will show that z_I increases indefinitely with unlimited increase of the value z_H. Getting rid of t in system (10) and passing to the argument x, we write the differential equation which describes the orbit *HIJ*

$$\frac{dz}{dx} = -\frac{g(x)}{z - F(x)}. \tag{13}$$

According to the condition of the theorem $F(x) < 0$ for $x \in (0, b_1)$, hence

$$\frac{g(x)}{z - F(x)} < \frac{g(x)}{z} \text{ for } x \in (0, b_1). \tag{14}$$

From Equation (13) and inequality (14) it follows

$$-\frac{dz}{dx} < \frac{g(x)}{z} \text{ for } x \in (0, b_1).$$

Separating variables and integrating, we obtain

$$\frac{1}{2} z^2(b_1) - \frac{1}{2} z_H^2 > -\int_0^{b_1} g(x) dx,$$

whence bearing in mind that $z(b_1) = z_I$, we get that $z_I \to +\infty$ if $z_H \to +\infty$.

Let $c \in (b_1, x_J)$. Let us designate the ordinates of the intersection points of the trajectory $T1$ and the line $x = c$ on pieces IJ and JL, respectively z^* and z^{**}. Taking into account that L is the intersection point of the trajectory $T1$ and the line $x = b_1$, we conclude that $z_L < 0$ (see Figure 1). Bearing in mind the continuity of the trajectory $T1$, the value $c \in (b_1, x_J)$ we choose so close to the value of b_1 that $z^{**} < 0$.

Let $z(x)$ be the solution of Equation (13) such that $z(0) = z_H$. We shall show that $z(c) \to +\infty$ if $z_H \to +\infty$. The inequality $z - F(x) > z - F(c)$ holds on the interval (b_1, c) because the function $F(x)$ monotonically increases on this interval. Hence Equation (13) yields

$$-\frac{dz}{dx} = \frac{g(x)}{z - F(x)} < \frac{g(x)}{z - F(c)}.$$

Separating variables and integrating, we obtain

$$-\left[\frac{1}{2}z^2 - F(c)z\right]_{z_I}^{z(c)} < \int_{b_1}^{c} g(x)dx,$$

whence (taking into account that $z(c) = z^* > 0$) it follows the inequality

$$z(c) > F(c) + \sqrt{[z_I - F(c)]^2 - 2\int_{b_1}^{c} g(x)dx}.$$

Since $z_I \to +\infty$ if $z_H \to +\infty$, then $z^* = z(c) \to +\infty$ if $z_H \to +\infty$.

Bearing in mind that $F(x)$ increases for $x > b_1$, we have

$$\Delta v_2 = -\int_{z_L}^{z_I} F(x(z))dz < -F(c)(z^* - z^{**})$$
$$< -F(c)\left[F(c) + \sqrt{[z_I - F(c)]^2 - 2\int_{b_1}^{c} g(x)dx}\right]. \tag{15}$$

The obtained inequality implies that $\Delta v_2 \to -\infty$ if $z_H \to +\infty$.

If we choose z_H small enough, such that the entire trajectory between points H and W is located in the domain $x \in (b_2, b_1)$, then obviously that $\Delta v > 0$. Taking into account that Δv decreases when z_H increases, and tends to $-\infty$ when $z_H \to +\infty$, one can conclude that there exists the unique value $z_H > 0$ such that $\Delta v = 0$. This means that there exists a unique periodic solution of system (10). The proof is complete.

Corollary 2.1. *If conditions of Theorem* 2.1 *are satisfied*, *then there is a unique periodic solution of system* (9).

3. The Estimation of the Domain of Location of a Stable Limit Cycle

Let us find the conditions under which system (9) has a unique stable limit cycle, and evaluate the domain of location of this limit cycle.

Theorem 3.1. *Suppose that* $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ *are such that conditions of Theorem* 2.1 *are fulfilled and besides*:

• *there exist* $a_1 > 0$ *and* $a_2 < 0$ *such that* $f(x) < 0$ *for* $x \in (a_2, a_1)$, $f(x) > 0$ *for* $x \in (-\infty, a_2) \bigcup (a_1, +\infty)$;

• *the function* $-\dfrac{g(x)}{f(x)}$ *monotonically increases in intervals* $(-\infty, a_2)$, (a_2, a_1), *and* $(a_1, +\infty)$;

• *the equation* $\Phi(x) = 0$, *where* $\Phi(x) = \int_0^x g(s)F(s)ds$ *has one positive root* $x = \alpha_1$ *and one negative root* $x = \alpha_2$;

• *the relation*

$$\int_{\alpha_2}^{\alpha_1} [g(x) + f(x)F(x)]dx = 0 \tag{16}$$

holds.

Then system (9) *has a unique stable limit cycle*, *and this limit cycle is located in the strip* $\alpha_2 < x < \alpha_1$.

Before proving the theorem, we note some properties of system (9). From conditions of the theorem it follows that $b_1 > a_1$, $b_2 < a_2$, and the function $f(x)$ is continuous. As has been shown in [11], the continuity of $f(x)$ and locally Lipschitz of $g(x)$ are sufficient for the existence and uniqueness of solutions of system (9).

Property 1. System (9) has the unique position of equilibrium

$$x = 0, \; y = 0, \tag{17}$$

which is unstable. Moreover, for every $0 < \varepsilon < \min\{a_1, |a_2|\}$ and for any trajectory of system (9) with initial values $x(0)$, $y(0)$ such that $0 < y^2(0)/2 + G(x(0)) < \delta$ for any arbitrarily small δ, there exists such moment of time $t_1 > 0$ that $y^2(t_1)/2 + G(x(t_1)) = \varepsilon$.

To prove the uniqueness of the equilibrium position, let us solve the next system of algebraic equations

$$y = 0, \quad - f(x)y - g(x) = 0. \tag{18}$$

Taking into account that $g(x) = 0$ only when $x = 0$, we see that system (18) has the unique solution (17). Consider the auxiliary function

$$V(x, \; y) = \frac{y^2}{2} + G(x). \tag{19}$$

Function $V(x, y)$ is positive definite. Consider its derivative along solutions of system (9)

$$\frac{dV}{dt} = y[- f(x)y - g(x)] + g(x)y = - f(x)y^2.$$

Function $- f(x)y^2$ is nonnegative for $|x| < \min\{a_1, |a_2|\}$ and can vanish only on the set $y = 0$. Bearing in mind that the set $y = 0$ is not invariant for system (9), using the reasoning used in the proof of the Barbashin-Krasovskii instability theorem ([20], Theorem 15.1), we can prove Property 1. We omit a verbatim repetition of this proof.

Property 2 ([11, 21])**.** Any trajectory of system (9), the beginning of which is located on the positive semi-axis of ordinate, sequentially passes the first quadrant, then the fourth, third, second, first again, and so on (i.e., the motion is clockwise around the origin).

Proof of Theorem 3.1. Choose a point A with coordinates $(x_A, 0)$ on the positive semi-axis of abscissa such that $x_A = \alpha_1$. From the conditions of the theorem it follows that if $x \in (0, b_1)$, then $g(x) > 0$, $F(x) < 0$, whence we obtain that $\Phi(x) < 0$ for $x \in (0, b_1]$. Taking into account that $\alpha_1 > 0, \Phi(\alpha_1) = 0$, we see that $x_A = \alpha_1 > b_1$. Let γ_A^- and γ_A^+ be respectively negative and positive semi-orbits, passing through the point A. Let us denote the first intersection of the negative semi-orbit with the positive semi-axis of ordinates by letter $B(0, y_B)$, and the first crossing of the positive semi-orbit with the negative semi-axis of ordinates by letter $C(0, y_C)$, where $y_B > 0$, $y_C < 0$.

Consider the segment of the trajectory of system (9), leaving the point B at null moment of time, passing through the point A and ending at the point C. Let us suppose that

$$y_B \geq -y_C. \tag{20}$$

Condition (20) holds iff

$$y_B^2 \geq y_C^2. \tag{21}$$

Let us find values of function $V(x, y)$ at points B and C

$$V_B = V(0, y_B) = \frac{y_B^2}{2}, \quad V_C = V(0, y_C) = \frac{y_C^2}{2}.$$

Dividing $\frac{dy}{dt}$ by $\frac{dx}{dt}$ and $-f(x)y - g(x)$ by y in system (9), we obtain the differential equation

$$\frac{dy}{dx} = -f(x) - \frac{g(x)}{y}. \tag{22}$$

This differential equation describes the orbits BA and CA for $x \in [0, x_A)$. Let $y = y_1(x)$ and $y = y_2(x)$ be functions, graphs of which are respectively orbits BA and CA, hence their initial values are equal to $y_1(0) = y_B$, $y_2(0) = y_C$.

We find $\frac{d}{dx} V(x, y_i(x))$, $i = 1, 2$:

$$\frac{d}{dx} V(x, y_i(x)) = \frac{d}{dx}\left[\frac{y_i^2(x)}{2} + G(x)\right] = y_i(x)\left[-f(x) - \frac{g(x)}{y_i(x)}\right]$$

$$+ g(x) = -f(x)y_i(x). \tag{23}$$

Equality (23) yields

$$\left[\frac{y_i^2(x)}{2} + G(x)\right]_0^{x_A} = -\int_0^{x_A} f(x)y_i(x)dx, \quad i = 1, 2. \tag{24}$$

Integrating by parts, we transform the integral on the right-hand side of (24)

$$\int_0^{x_A} f(x)y_i(x)dx = y_i(x)F(x)\Big|_0^{x_A} + \int_0^{x_A} f(x)F(x)dx + \int_0^{x_A} \frac{g(x)F(x)}{y_i(x)}dx,$$

$$i = 1, 2.$$

Bearing in mind that $F(0) = 0$, $y_i(x_A) = 0$, $i = 1, 2$, we obtain

$$\int_0^{x_A} f(x)y_i(x)dx = \int_0^{x_A} f(x)F(x)dx + \int_0^{x_A} \frac{g(x)F(x)}{y_i(x)}dx, \quad i = 1, 2,$$

whence from (24) it follows

$$\frac{y_i^2(0)}{2} = G(x_A) + \int_0^{x_A} f(x)F(x)dx + \int_0^{x_A} \frac{g(x)F(x)}{y_i(x)}dx, \quad i = 1, 2. \tag{25}$$

Lemma 3.1. $y_1(x) > y_1(b_1)$ *for* $x \in [0, b_1)$.

Proof. The function $y = -\frac{g(x)}{f(x)}$ has the following properties: $y(0) = 0$, $y > 0$ for $x \in (0, a_1)$ and $\lim_{x \to a_1^-}\left[-\frac{g(x)}{f(x)}\right] = +\infty$. The function $y_1(x)$ is continuous and differentiable in the interval $(0, x_A)$; $y_1(x) > 0$ for $x \in [0, x_A)$, $y_1(x_A) = 0$. The function $y_1(x)$ increases in the interval $(0, x_D)$, where x_D is the abscissa of the point D, in which the curves $y = y_1(x)$ and $y = -\frac{g(x)}{f(x)}$ intersect. Obviously, $x_D < a_1$.

Let us show that for $x \in (0, a_1)$, curves $y = y_1(x)$ and $y = -\frac{g(x)}{f(x)}$ intersect at a single point D (no other intersection points). Indeed, for $x_D < x < a_1$ the function $-\frac{g(x)}{f(x)}$ increases and $y_1(x)$ decreases (since $\frac{dy_1}{dx} < 0$ for $x > x_D$), so there are no other points of intersections (except the point D). Consequently, in the interval $(0, x_D)$ the function $y_1(x)$ increases and in (x_D, b_1) it decreases.

Let us show that $y_1(0) > y_1(b_1)$. To do this, consider the function $y_1(x) + F(x)$. Its derivative is equal to

$$\frac{d}{dx}[y_1(x) + F(x)] = -f(x) - \frac{g(x)}{y_1(x)} + f(x) = -\frac{g(x)}{y_1(x)} < 0. \tag{26}$$

From inequality (26) it follows that

$$y_1(0) + F(0) > y_1(b_1) + F(b_1). \tag{27}$$

Bearing in mind that $F(0) = F(b_1) = 0,$ from (27) we obtain

$$y_1(0) > y_1(b_1). \tag{28}$$

Since $y_1(x)$ in $(0, x_D)$ increases and in (x_D, b_1) decreases, from (28) it follows that $y_1(x) > y_1(b_1)$ for $x \in [0, b_1).$ This completes the proof of lemma.

From this lemma it follows that

$$-\frac{g(x)F(x)}{y_1(x)} < -\frac{g(x)F(x)}{y_1(b_1)} \text{ for } x \in (0, b_1),$$

or the same

$$\frac{g(x)F(x)}{y_1(x)} > \frac{g(x)F(x)}{y_1(b_1)} \text{ for } x \in (0, b_1). \tag{29}$$

The function $y_1(x)$ decreases in $x \in [b_1, x_A),$ therefore $y_1(b_1) > y_1(x)$ for $x \in (b_1, x_A).$ This yields

$$\frac{g(x)F(x)}{y_1(x)} > \frac{g(x)F(x)}{y_1(b_1)} \text{ for } x \in (b_1, x_A). \tag{30}$$

From equality (25) and inequalities (29) and (30), we obtain

$$\frac{y_1^2(0)}{2} > G(x_A) + \int_0^{x_A} f(x)F(x)dx + \frac{1}{y_1(b_1)}\int_0^{x_A} g(x)F(x)dx.$$

Since the point A has been chosen from the condition $x_A = \alpha_1,$ where α_1 is the root of the equation $\Phi(x) = 0,$ this inequality can be written in the form

$$\frac{y_1^2(0)}{2} > G(x_A) + \int_0^{x_A} f(x)F(x)dx. \tag{31}$$

Lemma 3.2. *The curve* $y = y_2(x)$ *has not points of intersection with the branch of the curve* $y = -\dfrac{g(x)}{f(x)}$ *which lies in the fourth quadrant of the plane* Oxy.

Proof. Assume the contrary: let these curves have one or more points of intersection. If we move along the curve $y = y_2(x)$ from the point C towards the point A, then let P be the point of intersection of this curve with the curve $y = -\dfrac{g(x)}{f(x)}$ closest to the point A. Taking into account that the function $-\dfrac{g(x)}{f(x)}$ increases for $x > a_1$, the slope of the tangent to the curve $y = y_2(x)$ at the point P is positive (see Figure 2).

On the other hand, since the equality $y = -\dfrac{g(x)}{f(x)}$ holds at the point P, then the slope tangent to the curve $y = y_2(x)$ is equal to zero. This contradiction proves the lemma.

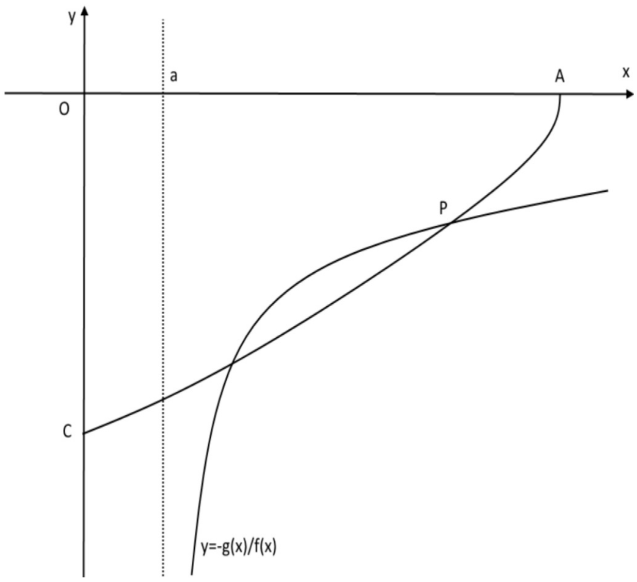

Figure 2.

Since according to Lemma 3.2 the curves $y = y_2(x)$ and $y = -\frac{g(x)}{f(x)}$ do not intersect, then conditions

$$y_2(x) < -\frac{g(x)}{f(x)} \text{ for } x \in (0, a_1),$$

$$y_2(x) > -\frac{g(x)}{f(x)} \text{ for } x \in (a_1, x_A) \tag{32}$$

are valid.

Consider the function $y = y_2(x)$. For $x \in (0, a_1)$, we have $-f(x) > 0$, $y_2(x) < 0$, whence it follows

$$\frac{dy_2}{dx} = -f(x) - \frac{g(x)}{y_2(x)} > 0. \tag{33}$$

For $x \in (a_1, x_A)$, the conditions $f(x) > 0$ and (32) also imply inequality (33), whence it follows that $y_2(x)$ increases for $x \in (0, x_A)$, therefore $y_2(x) < y_2(b_1)$ for $x \in (0, b_1)$.

We have $F(x) < 0,\ y_2(x) < 0$ in the interval $(0, b_1)$, whence we obtain $\frac{g(x)[-F(x)]}{-y_2(x)} > 0$ for $x \in (0, b_1)$. Since the function $y_2(x)$ monotonically increases, then $-y_2(x)$ monotonically decreases, hence $-y_2(x) > -y_2(b_1) > 0$ for $x \in (0, b_1)$, whence we have

$$\frac{g(x)F(x)}{y_2(x)} < \frac{g(x)F(x)}{y_2(b_1)} \text{ for } x \in (0, b_1).$$

Consider the function $\frac{g(x)F(x)}{y_2(x)}$ in (b_1, x_A):

$$\frac{g(x)F(x)}{y_2(x)} = -\left[\frac{g(x)F(x)}{-y_2(x)}\right],$$

whence by increasing $y_2(x)$ in (b_1, x_A), we have

$$\frac{g(x)F(x)}{-y_2(x)} > \frac{g(x)F(x)}{-y_2(b_1)},$$

or the same

$$\frac{g(x)F(x)}{y_2(x)} < \frac{g(x)F(x)}{y_2(b_1)}. \tag{34}$$

Hence inequality (34) holds for $x \in (0, b_1) \bigcup (b_1, x_A)$. From equality (25) for $i = 2$ and inequality (34), we obtain

$$\frac{y_2^2(0)}{2} < G(x_A) + \int_0^{x_A} f(x)F(x)dx + \frac{1}{y_2(b_1)} \int_0^{x_A} g(x)F(x)dx.$$

Taking into account that

$$\int_0^{x_A} g(x)F(x)dx = 0, \tag{35}$$

we get the inequality

$$\frac{y_2^2(0)}{2} < G(x_A) + \int_0^{x_A} f(x)F(x)dx. \tag{36}$$

Bearing in mind that $y_1^2(0) = y_B^2,\ y_2^2(0) = y_C^2$, from (31) and (36) it follows

$$\frac{y_C^2}{2} < G(x_A) + \int_0^{x_A} f(x)F(x)dx < \frac{y_B^2}{2}. \tag{37}$$

Thus, we see that inequality (21) holds under condition (35).

Now choose a point E with coordinates $(x_E, 0)$ on the negative semi-axis of abscissa such that $x_E = \alpha_2$. Conditions of the theorem imply that inequalities $g(x) < 0,\ F(x) > 0$ hold for $x \in (b_2, 0)$, whence it follows that $\Phi(x) > 0$ for $x \in [b_2, 0)$. Taking into account that $\alpha_2 < 0,\ \Phi(\alpha_2) = 0$, we obtain that $x_E = \alpha_2 < b_2$. Let γ_E^- and γ_E^+ be respectively negative and positive semi-orbits of system (9), passing through the point E. Denote by letter $K(0, y_K)$, the first intersection of the negative semi-orbit γ_E^- with the negative semi-axis of ordinates, and denote by letter $Q(0, y_Q)$ the first intersection of the positive semi-orbit γ_E^+ with the positive semi-axis of ordinates. Here $y_K < 0$, $y_Q > 0$.

Consider now the segment of the trajectory of system (9), leaving the point K at null moment of time, passing through the point E, and ending at the point Q. Assume that

$$-y_K \geq y_Q. \tag{38}$$

Condition (38) holds iff

$$y_K^2 \geq y_Q^2. \tag{39}$$

Arguing as before, we can show that

$$\frac{y_Q^2}{2} < G(x_E) + \int_0^{x_E} f(x)F(x)dx < \frac{y_K^2}{2} \tag{40}$$

under condition

$$\int_0^{x_E} g(x)F(x)dx = 0,$$

where x_E is the abscissa of the point E.

Let us write equality (16) in the form

$$\int_{\alpha_2}^{0}[g(x) + f(x)F(x)]dx + \int_0^{\alpha_1}[g(x) + f(x)F(x)]dx = 0,$$

whence we obtain

$$\int_0^{\alpha_2}[g(x) + f(x)F(x)]dx = \int_0^{\alpha_1}[g(x) + f(x)F(x)]dx. \tag{41}$$

Bearing in mind that

$$\alpha_1 = x_A, \quad \alpha_2 = x_E, \int_0^{\alpha_1} g(x)dx = G(x_A), \quad \int_0^{\alpha_2} g(x)dx = G(x_E),$$

we get

$$G(x_A) + \int_0^{x_A} f(x)F(x)dx = G(x_E) + \int_0^{x_E} f(x)F(x)dx. \tag{42}$$

Equation (42) and inequalities (37) and (40) yield

$$y_Q^2 < y_B^2, \quad y_C^2 < y_K^2. \tag{43}$$

Taking into account relations (43), let us construct the figure *BACKEQB* on the plane *Oxy*. This figure is bounded by the curves *BAC, KEQ* and straight line segments *QB* and *CK* (see Figure 3).

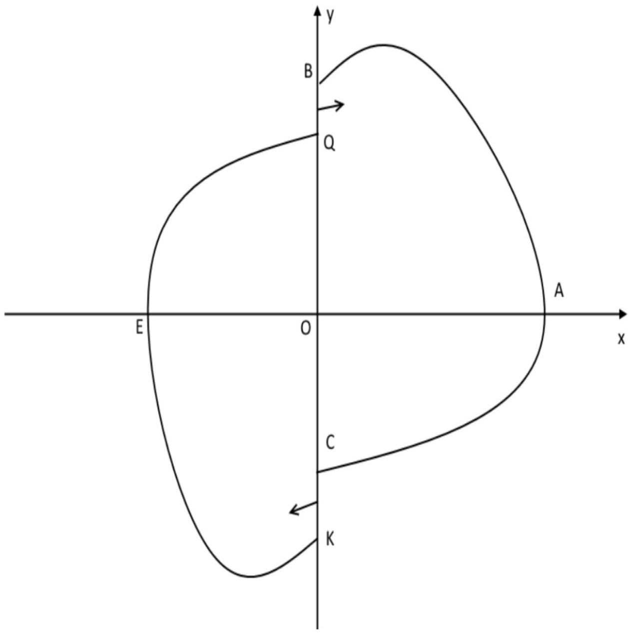

Figure 3.

Since the trajectories of system (9), starting from straight line segments *QB* and *CK*, directed inside the figure *BACKEQB*, then this figure is an invariant set of system (9). Taking into account that the figure *BACKEQB* is a bounded set, according to the Poincaré-Bendixson theorem, we can state that any trajectory of this set tends either to an asymptotically stable equilibrium position, or to a limit cycle. According to Property 1, system (9) has a unique equilibrium state $x = 0,\ y = 0$ which is unstable. Hence system (9) has a limit cycle $x_c(t),\ y_c(t),$ all points of which satisfy the inequalities $\alpha_2 < x_c(t) < \alpha_1$ (because the whole figure *BACKEQB* is located in the strip $\alpha_2 < x < \alpha_1$ of the plane *Oxy*). The proof of theorem is complete.

Remark 3.1. It is obvious that condition (16) is satisfied if system (9) is such that $f(x)$ is even function and $g(x)$ and $F(x)$ are odd functions. In this case $\alpha_2 = -\alpha_1$.

Example 3.1. Consider system (9) in which $f(x) = (x^2 - \frac{8}{15}x - \frac{6}{5}),$

$$g(x) = \begin{cases} x & \text{for } x \geq 0, \\ \frac{73}{18}x & \text{for } x < 0. \end{cases}$$

The function $f(x)$ has the following properties: $f(x) < 0$ for $x \in (-\frac{\sqrt{286}-4}{15}, \frac{\sqrt{286}+4}{15}),$ $f(x) > 0$ for $x \in (-\infty, -\frac{\sqrt{286}-4}{15}) \bigcup (\frac{\sqrt{286}+4}{15}, +\infty).$

The function $-\frac{g(x)}{f(x)}$ increases for all *x* from the domain of definition, because its derivative is positive in the intervals $(-\infty, -\frac{\sqrt{286}-4}{15}),$ $(-\frac{\sqrt{286}-4}{15}, 0),$ $(0, \frac{\sqrt{286}+4}{15}),$ $(\frac{\sqrt{286}+4}{15}, +\infty).$

We find $F(x) = \frac{x^3}{3} - \frac{4}{15}x^2 - \frac{6}{5}x$; $xF(x) = \frac{x^4}{3} - \frac{4}{15}x^3 - \frac{6}{5}x^2$; $\int_0^x sF(s)ds = \frac{x^5}{15} - \frac{1}{15}x^4 - \frac{2}{5}x^3$. $x = -2$ and $x = 3$ are the roots of the equation $\int_0^x sF(s)ds = 0$.

$$\int_{-2}^{3}[g(x) + f(x)F(x)]dx = \int_{-2}^{0}[g(x) + f(x)F(x)]dx$$

$$+ \int_{0}^{3}[g(x) + f(x)F(x)]dx = -9 + 9 = 0.$$

Thus, according to Theorem 3.1, for given $g(x)$ and $f(x)$, there exists a unique stable limit cycle of the Liénard equation, and this limit cycle is located in the strip $-2 < x < 3$.

Example 3.2. Consider system (9) in which

$$g(x) = x, \quad f(x) = \begin{cases} 3\sqrt{\frac{19}{116}}(x-1) & \text{for } x \geq 0, \\ 3\sqrt{\frac{19}{116}}(x^2-1) & \text{for } x < 0. \end{cases} \tag{44}$$

We derive

$$a_1 = 1, \quad a_2 = -1, \quad F(x) = \begin{cases} 3\sqrt{\frac{19}{116}}\left(\frac{x^2}{2} - x\right) & \text{for } x \geq 0, \\ 3\sqrt{\frac{19}{116}}\left(\frac{x^3}{3} - x\right) & \text{for } x < 0. \end{cases}$$

The function $-\frac{x}{f(x)}$ increases in intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, $(1, \infty)$ because its derivative is positive in these intervals. The function $\Phi(x) = \int_0^x sF(s)ds$ has roots $\alpha_1 = \frac{8}{3}$, $\alpha_2 = -\sqrt{5}$. Let us verify the feasibility of the equality $\int_{\alpha_2}^{\alpha_1}[x + f(x)F(x)]ds = 0$. Indeed,

$$\int_{-\sqrt{5}}^{\frac{8}{3}} [x + f(x)F(x)]dx = \int_{-\sqrt{5}}^{0} xdx - \frac{1}{2} F^2\left(-\sqrt{5}\right)$$

$$+ \int_{0}^{\frac{8}{3}} xdx + \frac{1}{2} F^2\left(\frac{8}{3}\right) = 0.$$

Moreover, $F(+\infty) = +\infty$. Therefore, if $f(x)$ and $g(x)$ are defined by equalities (44), then in the strip $-\sqrt{5} < x < \frac{8}{3}$ of the plane Oxy, according to Theorem 3.1, there exists a unique stable limit cycle of system (9), which is globally asymptotically stable.

4. The Amplitude of the Limit Cycle of the Van Der Pol Oscillator

A popular choice of the model to describe the dynamics of nonlinear physical phenomena is the van der Pol oscillator. His behaviour is governed by the differential equation (3). In economics, the van der Pol equation is equivalent to the system of differential equations of transition economy [23]. Equation (3) with the parameter value $\mu = 0$ becomes the equation of simple harmonic oscillator

$$\ddot{x} + x = 0, \tag{45}$$

which models the inflationary fluctuations in conditions of full competition [23]. The behaviour of solutions of the van der Pol equation essentially depends on a positive parameter μ. For example, if $\mu \ll 1$, then these solutions are close to sinusoidal, but if $\mu \gg 1$, then the solution becomes similar to a meander. Such behaviour of solutions, van der Pol called a relaxation oscillation [3].

It is known [24, p.300], that the differential Equation (3) has a single stable limit cycle for any $\mu > 0$; it has been shown [25] that this limit cycle is not algebraic. In general, to find the periodic solution of the van der Pol equation is not possible, therefore, the efforts of many researchers have focused on finding its approximate solution. Dorodnitsyn [26] studied Equation (3) by analytical way; to do this, he created the asymptotic method. Sudakov [27] suggested another approach for the description of a limit cycle of the van der Pol equation in the relaxation mode (in case of large μ). In paper [28], a periodic solution of Equation (3) is constructed in the form of a series converging for all values of the damping coefficient μ.

An important direction in the study of a limit cycle of the van der Pol equation is the estimation of its amplitude (the largest value of x). Zonneveld [29], solving the equation of van der Pol numerically using a computer, received approximate values of the amplitude Am for some fixed values of μ:

μ	Am	μ	Am
1	2, 00862	6	2, 01983
2	2, 01989	7	2, 01822
3	2, 02330	8	2, 01675
4	2, 02296	9	2, 01544
5	2, 02151	10	2, 01429

In paper [30], two computational methods are proposed. These methods allow to find approximately the amplitude of a periodic solution of Equation (3). The first of these methods bases on discrete mechanics. In the second of these methods, the author used the Taylor series expansion of the solution. The results are obtained as the table:

μ	Am(1)	Am(1)
0, 1	2, 005	2, 000
1, 0	2, 009	2, 009
10, 0	2, 014	2, 014

Here Am(1) and Am(2) are the amplitude values defined respectively by the first and the second method.

In contrast to papers [29] and [30], there are a lot of results allowing to evaluate the amplitude of the limit cycle of the van der Pol equation, regardless of the value of μ.

Odani [12] obtained the estimation of the amplitude of Equation (3):

$$\text{Am} < \sqrt{3 + 2\sqrt{3}} \approx 2{,}5425 \text{ for all } \mu. \tag{46}$$

Then, in paper [31], studying the Liénard equation, assuming the function $F(x)$ to be such as in Theorem 2.1, he suggested the existence of auxiliary functions $\phi, \psi : [0, b] \to [b, \infty)$, related in some way with function $F(x)$ and $g(x)$, and proved a theorem, which states that in this case, the limit cycle of the Liénard equation is located in the domain $|x| < \psi(b)$. Applying the proven theorem to the van der Pol equation, he showed that the amplitude of the limit cycle of equation (2) for any μ satisfies the constraint

$$\text{Am} < 2{,}3233. \tag{47}$$

The obtained estimation improved his earlier result (46).

Equation (4) is studied in paper [32] where $F(x)$ is assumed to be such as in Theorem 2.1. The estimate $b_1 \le \text{Am} \le u$ is obtained, where the positive root u is determined from the equation $\int_0^u F(x)dx = 0$.

For the van der Pol equation we have

$$F(x) = \frac{x^3}{3} - x, \int_0^u \left(\frac{x^3}{3} - x\right)dx = 0, \quad \frac{u^4}{12} - \frac{u^2}{2} = 0,$$

whence

$$\text{Am} \le u = \sqrt{6} \approx 2{,}4495 \text{ for all } \mu. \tag{48}$$

Now let us estimate the amplitude of the limit cycle of equation (3) applying Theorem 3.1. Equation (2) is a special case of Equation (9) where

$f(x) = \mu(x^2 - 1)$, $g(x) = x$. It is easily seen that under these $f(x)$ and $g(x)$, all conditions of Theorem 3.1 are satisfied. In this case the equation $\Phi(x) = 0$ has the form

$$\int_0^x \mu s \left(\frac{s^3}{3} - s \right) ds = 0,$$

whence we obtain $\frac{x^5}{15} - \frac{x^3}{3} = 0$. The positive and the negative roots of this equation respectively, are $x = \alpha_1 = \sqrt{5}$, $x = \alpha_2 = -\sqrt{5}$, hence

$$\text{Am} \leq \sqrt{5} \approx 2{,}2361 \text{ for all } \mu.$$

Thus, applying Theorem 3.1 to the van der Pol equation, we obtained estimations of the amplitude more exact than (46), (47) and (48).

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