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THE EXISTENCE OF CONVEXITY IN GROUP ACTIONS

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Abstract

Suppose that R is a commutative ring which has an identity. It has been shown that if B is an R-algebra such that a finite group G acts as R-linear automorphisms on B. [Here, $M_G(B)$ is the subring of $End_R(B^{|G|})$ consisting of row and column finite matrices $(a_{g,\ h})g,\ h \in G$ indexed by elements of G with entries in B.], defining a map $\Phi : B *_a G \to M_G(B)$ by $a\overline{g} \rightarrowtail \sum_{t \in G} t^{-1} \cdot (\overline{g} \cdot a) E_{gt,t}$, then, Φ is invariant. In this paper, we show further that Φ is convex.

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1. Introduction

It seems that the notion of convex function (according to Jensen) is just as fundamental as positive function or increasing function. The notion ought to find its place in elementary expositions of the theory of real functions. The word convex can refer to both functions and sets, and it has two distinct meanings. They are similar, however, in that convex functions and convex sets are extremely desirable. If the feasible region is a convex set, and if the objective function is a convex function, then it is much easier to find the optimal solution. Geometrically, a convex function lies below its secant lines. Recall that a secant line is the line segment joining two points on the function. No matter what two points we pick, the function always lies below its secant line. On the other hand, not every secant line lies above the function; some lie below it, and some lie both above and below it. Even though we can draw some secant lines which are above the function, this isn't enough. Every possible secant must lie below the function.

Convexity is a simple and natural notion which can be traced back to Archimedes (circa 250 B.C.), in connection with his famous estimates (using inscribed and circumscribed regular polygons). He noticed the important fact that the perimeter of a convex figure is smaller than the perimeter of any other convex figure, surrounding it. As a matter of facts, we experience convexity all the time and in many ways. The most prosaic example is our standing up position, which is secured as long as the vertical projection of our center of gravity lies inside the convex envelope of our feet! Also, convexity has a great impact on our everyday life through its numerous applications in industry, business, medicine, art, etc. So are the problems on optimum allocation of resources and equilibrium of non-cooperative games. The theory of convex functions is part of the general subject of convexity since a convex function is one whose epigraph is a convex set. Nonetheless it is a theory important per se, which touches almost all branches of mathematics. Probably, the first topic that necessitates the encounter with this theory is the graphical analysis. With this occasion, the second derivative test of convexity is learnt, which is a powerful tool in recognizing convexity. Then comes the problem of finding the extremal values of functions of several variables and the use of Hessian as a higher dimensional generalization of the second derivative. Passing to optimization problems in infinite dimensional spaces is the next step, but despite the technical sophistication in handling such problems, the basic ideas are pretty similar with those underlying the one variable case. The recognition of the subject of convex functions as one that deserves to be studied in its own is generally ascribed to Jensen [5, 6].

However he was not the first dealing with such functions. Among his predecessors we should recall here Hermite [7], Holder [1] and Stolz [9]. During the whole 20th Century an intense research activity was done and significant results were obtained in geometric functional analysis, mathematical economics, convex analysis, nonlinear optimization etc. A great role in the popularization of the subject of convex functions was played by the famous book of Hardy et al. [10], on inequalities. Roughly speaking, there are two basic properties of convex functions that made them so widely used in theoretical and applied mathematics: The maximum is attained at a boundary point. Any local minimum is a global one. Moreover, a strictly convex function admits at most one minimum. The modern viewpoint on convex functions entails a powerful and elegant interaction between analysis and geometry. In a memorable paper dedicated to the Brunn-Minkowski inequality, Gardner [11, 12], described this reality in beautiful phrases: [convexity] appears like an octopus, tentacles reaching far and wide, its shape and color changing as it roams one area to the next. By this, it is quite clear that research opportunities abound.

At the core of the notion of convexity is the comparison of means. By a mean (on an interval I) we understand any function $M : I \times I \to I$ which verifies the property of intermediacy,

Convex sets: A set S is said to be convex if the line segment between any two points in S lies in S, i.e., if for any $x_1, x_2 \in S$ and any α with $0 \leq \alpha \leq 1$, we have $\alpha x_1 + (1 - \alpha) x_2 \in S$.

Roughly speaking, a set is convex if every point in the set can be seen by every other point, along an unobstructed straight path between them, where unobstructed means lying in the set. Every affine set is also convex, since it contains the entire line between any two distinct points in it, and therefore also the line segment between the points. A point of the form: $\alpha_1 x_1 + \cdots + \alpha_k x_k$, where $\alpha_1 + \cdots + \alpha_k = 1$ and $\alpha_i \geq 0$, $i = 1, \cdots, k$, can be called a convex combination of the points $x_1, \cdots, x_k$. As with affine sets, it can be shown that a set is convex if and only if it contains every convex combination of its points. A convex combination of points can be thought of as a mixture or weighted average of the points, with α_i the fraction of x_i in the mixture. The convex hull of a set S, denoted $convS$, is the set of all convex combinations of points in $S : convS = \{\alpha_1 x_1 + \cdots + \alpha_k x_k | x_i \in S, \alpha_i \geq 0, i = 1, \cdots, k, \alpha_1 + \cdots + \alpha_k = 1\}$. As the name suggests, the convex hull $convS$ is always convex. It is the smallest convex set that contains S. If B is any convex set that contains S, then $convS \subseteq B$. The idea of a convex combination can be generalized to include infinite sums, integrals, and, in the most general form, probability distributions. Suppose that $\alpha_1, \alpha_2, \cdots$ satisfy $\alpha_i \geq 0$, $i = 1, 2, \cdots$,

$$\sum_{i=1}^{\infty} \alpha_i = 1,$$

and $x_1, x_2, \cdots \in S$, where $S \subseteq R^n$ is convex. Then,

$$\sum_{i=1}^{\infty} \alpha_i x_i \in S,$$

if the series converges. More generally, suppose $p : R^n \to R$ satisfies $p(x) \geq 0$ for all $x \in S$ and $\int_S p(x)dx = 1$, where $S \subseteq R^n$ is convex. Then $\int_S p(x)xdx \in S$, if the integral exists. In the most general form, suppose $S \subseteq R^n$ is convex and x is a random vector with $x \in S$ with probability one. Then $Ex \in S$. Indeed, this form includes all the others as special cases. For example, suppose the random variable x only takes on the two values x_1 and x_2, with $prob(x = x_1) = \alpha$ and $prob(x = x_2) = 1\alpha$, where $0 \leq \alpha \leq 1$. Then $Ex = \alpha x_1 + (1 - \alpha)x_2$, and we are back to a simple convex combination of two points (see [2, 3, 4]).

2. Theory of Convex Functions

Definition. A function $f : R^n \to R$ is convex if its domain is a convex set and for all $x; y$ in its domain, and all $\lambda \in [0; 1]$, we have $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$.

This means that if we take any two points x, y, then f evaluated at any convex combination of these two points should not be larger than the same convex combination of $f(x)$ and $f(y)$. Geometrically, the line segment connecting $(x; f(x))$ to $(y; f(y))$ must sit above the graph of f. If f is continuous, then to ensure convexity it is enough to check the definition with $\lambda = \frac{1}{2}$ (or any other fixed $\lambda \in (0, 1)$). Also, we say that f is concave if $-f$ is convex.

3. Characterization of Strict Convexity

Recall that a function $f : R^n \to R$ is strictly convex if $\forall x$; y; $x \neq y$; $\forall \lambda \in (0; 1)$, $f(\lambda x + (1 - \lambda y) < \lambda f(x) + (1 - \lambda) f(y)$. If f is strictly convex, then f is convex (this is obvious from the definition) but the converse is not true (e.g., $f(x) = x$; $x \in R$).

Proposition A. *Let R be a commutative ring with identity and B an R-algebra. Suppose that a finite group G acts as R-linear authomorphisms on B, where $M_G(B)$ is the subring of $End(B^{|G|})$ consisting of row and column finite matrices $(a_{g,h})g, h \in G$ indexed by elements of G with entries in B. Define a map $\Phi : B *_a G \to M_G(B)$ by*

$$a\overline{g} \rightarrowtail \sum_{t \in G} t^{-1} \cdot (\overline{g}.a) E_{gt,t}. \qquad (*)$$

Then, Φ is invariant.

To prove this proposition, we need the following short lemma.

Lemma. *Let a, b be two elements of a group G. Then,*

(i) $(gh)^{-1} = h^{-1} g^{-1}$,

(ii) $(g^{-1})^{-1} = g$.

Proof. (i) Consider $h^{-1} g^{-1} (gh) = h^{-1} (g^{-1} g) h$ (by associativity) $= h^{-1} (eh) = h^{-1} h = e$, where e is the identity element. So, $h^{-1} g^{-1}$ is the inverse of gh. (ii) follows from (i).

Proof of Proposition A. For some $a\overline{g} \in B *_a G \Rightarrow a^{-1} \overline{g} \in B *_a G$. Now, let $a^{-1} = d$. It has been proved that Φ is an algebra homomorphism (see Lomp [8]). Therefore, from (i) above,

$$\Phi\left[(a\overline{g})\left(d\overline{h}\right)\right] = \Phi(a\overline{g})\Phi(d\overline{g})\left(d\overline{h}\right)$$

$$= \left(\sum_{\nu\in G} \nu^{-1} \cdot (g^{-1} \cdot a)E_{g\nu,\nu}\right)\left(\sum_{p\in G} p^{-1} \cdot (h^{-1} \cdot d)E_{gp,p}\right)$$

$$= \sum_{\nu, p\in G} (\nu^{-1} \cdot (g^{-1} \cdot a)p^{-1} \cdot (h^{-1} \cdot d))E_{g\nu,\nu,\nu}E_{hp,p}$$

$$= \sum_{h, p\in G} ((hp)^{-1} \cdot (g^{-1} \cdot a))(p^{-1} \cdot (h^{-1} \cdot d))E_{ghp,\nu,hp}$$

$$= \sum_{p\in G} (p^{-1} \cdot ((gh)^{-1} \cdot (a(g \cdot d)))E_{hp,p}$$

$$= \sum_{p\in G} (p^{-1}(((gh)^{-1} \cdot (ad \cdot g)E_{hp,p}. \qquad (\gamma)$$

But $d = a^{-1}$

$$(\gamma) = \sum_{p\in G} (p^{-1}(((gh)^{-1} \cdot (aa^{-1}) \cdot g)E_{hp,p}$$

$$= \sum_{p\in G} (p^{-1}(((gh)^{-1} \cdot g)E_{hp,p}$$

$$= \sum_{p\in G} (p^{-1}h^{-1}(g^{-1}g)E_{hp,p}$$

$$= \sum_{p\in G} (p^{-1}(h^{-1} \cdot e)E_{hp,p}, \text{ where } e \text{ is the identity}$$

$$= \Phi\left(e\overline{h}\right) = \Phi\left(\overline{h}\right).$$

Putting $g = h \Rightarrow \Phi\left(ad\overline{h}\right) = \Phi\left(\overline{h}\right)$. □

4. Groups and Group Actions

Definition. A subgroup of a group G is a subset of G such that it itself is a group under the operation in G.

Definition. A group G acts on a set S when there is a map $G \times S :\rightarrow S$ such that the conditions of associativity and identity hold for all $s \in S$. For more and details on groups, subgroups, their characteristic properties, our readers are implored to see [13-20].

5. Existence of Convexity

Definition [Convex function]**.** A function $f : D \rightarrow R$ defined in a nonempty subset D of R^n and taking real values is known to be convex if the following conditions are satisfied:

* the domain D of the function is convex;
* for any $x,\ y \in D$ and $\lambda \in [0, 1]$, we have,

$$f(\lambda x + (1 - \lambda y) \leq \lambda f(x) + (1 - \lambda) f(y).$$

Proposition B. *From Proposition A above,* Φ *is convex.*

Proof. Let $\alpha \in [0, 1]$. Then, we have

$$\begin{aligned}\Phi(\alpha a\overline{g} + (1 - \alpha)(b\overline{g}) &= \Phi[(\alpha a + (1 - \alpha) b\overline{g})] \\ &= \Phi(\alpha a + \alpha b)\overline{g} = \Phi[(\alpha a + (1 - \alpha)(b\overline{g}] \\ &= \sum_{p \in G} p^{-1} \cdot (g^{-1} \cdot (\alpha a + (1 - \alpha) b) E_{gp,\, p} \\ &= \sum_{p \in G} p^{-1} \cdot (g^{-1} \cdot (\alpha a + g^{-1} \cdot \beta b) E_{gp,\, p},\end{aligned}$$

where $\beta = 1 - \alpha \in [0, 1]$

$$= \sum_{p \in G} [p^{-1} \cdot (g^{-1} \cdot \alpha a + p^{-1} \cdot g^{-1} \beta b] E_{gp,\, p}$$

$$= \sum_{p \in G} [p^{-1} \cdot (g^{-1} \cdot \alpha a) E_{gp,\, p} + \sum_{p \in G} [p^{-1} \cdot (g^{-1} \cdot \beta b) E_{gp,\, p}$$

$$= \alpha \sum_{p \in G} p^{-1} \cdot (g^{-1} \cdot a) E_{gp,\, p} + \beta \sum_{p \in G} p^{-1} \cdot (g^{-1} b] E_{gp,\, p}$$

$$= \alpha\Phi(a\overline{g} + \beta\Phi(b\overline{g})) \text{ (since } \Phi \text{ is an algebra homomorphism),}$$

where $\beta = 1 - \alpha,\ \alpha \in [0, 1]$.

$\therefore\ \Phi$ is convex as required. □

6. Conclusion

So far in this research, we have been able to prove the existence of convexity in group action. As such, this is part of the side products whenever any particular group of a given characteristic features or characterizations acts on any given entities.

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References

[1] O. Holder, Uber einen Mittelwertsatz, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen (1889), 38-47.

[2] Constantin P. Niculescu and Lars-Erik Persson, Convex Functions and their Applications, Springer, Berlin Heidelberg New York, Hong Kong London Milan Paris Tokyo, 2004.

[3] A. A. Ahmadi and G. Hall, aaa.princeton.edu/orf523, ORF 523, Lecture 7, Spring Princeton University, 2016.

[4] Stephen Boyd and Lieven Vandenberghe, Convex optimization, University of California, Los Angeles Cambridge University Press, 2004.

[5] J. L. W. V. Jensen, Om konvexe Funktioner og Uligheder mellem Middelvaerdier, Nyt. Tidsskrift for Mathematik 16(B) (1905), 49-69.

[6] J. L. W. V. Jensen, Sur les fonctions convexes et les inegalites entre les valeurs moyennes, Acta Mathematica 30 (1906), 175-193.

DOI: https://doi.org/10.1007/BF02418571

[7] Ch. Hermite, Sur deux limites d'une integrale definie, Mathesis 3 (1883), 82.

[8] Christian Lomp, Duality for partial group actions, International Electronic Journal of Algebra 4 (2008), 53-62

[9] O. Stolz, Grunzuge der Differential und Integralrechnung, Volume 1, Teubner, Leipzig, 1893.

[10] G. H. Hardy, J. E. Littlewood and G. Polya, Inequalities, Cambridge Mathematical Library, 2nd Edition, 1952, Reprinted 1988.

[11] R. J. Gardner, The Brunn-Minkowski inequality: A survey with proofs, Preprint, 2001.

Available at http://www.ac.wwu.edu/

[12] R. J. Gardner, The Brunn-Minkowski inequality, Bulletin of the American Mathematical Society 39(3) (2002), 355-405.

DOI: https://doi.org/10.1090/S0273-0979-02-00941-2

[13] Marius Tarnauceanu, A characterization of the quaternion group by the sum of subgroup orders, International Journal of Open Problems in Computer Science & Mathematics 14(2) (2021), 15-19.

[14] Matheus P. Lobo, Groups: A Concise Approach, OSF Preprints, 20 Apr. 2020.

DOI: https://doi.org/10.31219/osf.io/bfyhm

[15] Matheus P. Lobo, Subgroups: A Concise Approach, OSF Preprints, 22 Apr. 2020

DOI: https://doi.org/10.31219/osf.io/wcaez

[16] Matheus P. Lobo, Homomorphisms: A Concise Approach, OSF- Preprints, 26 Apr 2020.

DOI: https://doi.org/10.31219/osf.io/7pgcs

[17] Matheus P. Lobo, Group Actions: A Concise Approach, OSF Preprints, 4 May 2020.

DOI: https://doi.org/10.31219/osf.io/9dt34

[18] Matheus P. Lobo, Microarticles, OSF Preprints, 28 Oct. 2019,

DOI: https://doi.org/10.31219/osf.io/ejrct

[19] Matheus P. Lobo, Simple Guidelines for Authors: Open Journal of Mathematics and Physics, OSF Preprints, 15 Nov. 2019.

DOI: https://doi.org/10.31219/osf.io/fk836

[20] Matheus P. Lobo, Open Journal of Mathematics and Physics (OJMP), OSF, 21 Apr 2020.

DOI: https://doi.org/10.17605/OSF.IO/6HZYP

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