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TRAVELLING WAVE SOLUTIONS FOR STOCHASTIC FRACTIONAL HIROTA-SATSUMA COUPLED KdV EQUATIONS WITH CONFORMABLE DERIVATIVES

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Abstract

A modified fractional sub-equation method is used for constructing exact travelling wave solutions of nonlinear stochastic fractional Hirota-Satsuma Coupled KdV Equations with Conformable Derivatives. The main idea of this method is to take full advantage of the fractional Riccati equation, which has many exact solutions. Abundant white noise functional solutions are obtained for the Wick-type stochastic fractional Hirota-Satsuma coupled KdV equations via Hermite transform and white noise analysis. Eventually, by an application example, we show how the stochastic solutions can be given as Brownian motion functional solutions.

1. Introduction

The nonlinear fractional differential equations (FDEs) are constructed by mathematical modelling of some complex physical phenomena. The study of such nonlinear physical models through wave solutions analysis corresponding to their FDEs, has a dynamic role in applied sciences. This work is devoted to investigate the stochastic fractional Hirota-Satsuma coupled equation with conformable derivatives:

$$\begin{cases} D_t^{\alpha}U = F(t) \diamond D_x^{3\alpha}U + G(t) \diamond U \diamond D_x^{\alpha}U + H(t) \diamond D_x^{\alpha}(V \diamond W), \\ D_t^{\alpha}V = -D_x^{3\alpha}V + 3U \diamond D_x^{\alpha}V, \\ D_t^{\alpha}W = -D_x^{3\alpha}W + 3U \diamond D_x^{\alpha}W, \end{cases} \tag{1.1}$$

where $(x, t) \in \mathbb{R} \times \mathbb{R}_+$ and $0 < \alpha \leq 1$. While, $F(t), G(t)$ and $H(t)$ are non-zero integrable functions from $\mathbb{R}^+$ to the Kondrative distribution space $(\mathcal{S})_{-1}$ which was defined by Holden et al. in [30] as a Banach algebra with the Wick-product "$\diamond$". Equation (1.1) is the perturbation of the variable coefficients fractional Hirota-Satsuma coupled KdV equation with conformable derivatives:

$$\begin{cases} D_t^{\alpha}u = f(t)D_x^{3\alpha}u + g(t)uD_x^{\alpha}u + h(t)D_x^{\alpha}(vw), \\ D_t^{\alpha}v = -D_x^{3\alpha}v + 3uD_x^{\alpha}v, \\ D_t^{\alpha}w = -D_x^{3\alpha}w + 3uD_x^{\alpha}w, \end{cases} \tag{1.2}$$

where $f(t)$, $g(t)$ and $h(t)$ are non-zero integrable functions on $\mathbb{R}_+$. Equation (1.2) is a general model which describes shallow water waves of small amplitude and long wavelength [28]. Moreover, if Equation (1.2) is considered in a random environment, we have a random fractional Hirota-Satsuma coupled KdV equation. In order to obtain the exact solutions of the random fractional Hirota-Satsuma coupled KdV equation, we only consider it in a white noise environment, that is, we will discuss the Wick-type stochastic fractional Hirota-Satsuma coupled KdV Equation (1.1). Many important phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry, cosmology, and material science can be better described by fractional partial differential equations FPDEs [36, 38, 39]. Consequently, considerable attention has been given to the solution of the FPDEs. There are many methods for calculating the approximate solutions for nonlinear FPDEs such as the variational iterations method [41], Adomian decomposition method [7, 8], the homotopy perturbation method [23, 24] and the Exp-function method [35, 43-45, 52]. The exact solutions for nonlinear FPDEs are still under study until now. Li and He [34] introduced complex transform for reducing FPDEs into ordinary differential equations, so that all analytical methods for advanced calculus can be easily applied to fractional calculus. It is well known that the solitons are stable against mutual collisions and behave like particles. In this sense, it is very important to study the nonlinear equations in random environment. However, variable coefficients nonlinear equations, as well as constant coefficients equations, cannot describe the realistic physical phenomena exactly. Wadati [42] first answered the interesting question, "How does external noise affect the motion of solitons ?" and studied the diffusion of soliton of the KdV equation under Gaussian noise, which satisfies a diffusion equation in transformed coordinates. Ghany et al. [13-20] studied more intensely the white noise functional solutions for some nonlinear stochastic PDEs. There are many studies have done for the definition and properties of the conformable derivative. Conformable forms of the chain rule, Gronwalls inequality, exponential functions, Taylor power series expansions, integration by parts and Laplace transform have been presented by Abdeljawad in [1]. Benkhettoua et al. [3] have been expressed the calculus of the conformable time-scale. The heat equation with conformable derivatives was investigated by Hammad and Khalil in [25]. Chung [6] used the conformable derivative and integral to study the fractional Newtonian mechanics. Moreover, the deterministic conformable partial differential equations (PDEs) became an important subject in mathematical physics. So, there are many scholars paid more attention to their approximate and analytical solutions. The existence and uniqueness theorems for linear sequential differential equations with conformable derivatives was proved by Gokdogan et al. in [22]. Eslami and Rezazadeh [9] gave a set of analytical solutions to Wu-Zhang system with conformable derivative via the first integral method. The stochastic travelling wave solutions for the fractional coupled KdV and 2D KdV equations are obtained by the modified fractional sub-equation method in [18] and [21], respectively.

Recently, many research work have done to investigate the conformable PDEs and their exact solutions via various methods. In [50], a conformable sub-equation method was proposed to construct exact solutions of the space-time resonant nonlinear Schrdinger equation. Using the generalized exponential rational function method, new periodic and hyperbolic soliton solutions were constructed to the conformable Ginzburg-Landau equation with the Kerr law nonlinearity [11]. Also, a family of exact solutions were obtained for space-time conformable generalized Hirota-Satsuma-coupled KdV equation and coupled mKdV equation using the Atangana's conformable derivative and conformable sub-equation method [47]. The analysis of the first integral method was given in [48, 49] to construct exact solutions of the nonlinear PDEs described by beta-derivative. Moreover, new optical, dark, complex and singular soliton solutions were obtained for some nonlinear PDEs with *M*-derivative [2, 12]. The investigation of exact and approximate solutions of nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena. Our aim in this work is to obtain new stochastic soliton wave solutions for the variable coefficients fractional Hirota-Satsuma coupled KdV equation and Wick-type stochastic fractional Hirota-Satsuma coupled KdV equation with conformable derivatives. Using white noise theory and Hermite transform, the Wick-type stochastic fractional Hirota-Satsuma coupled KdV equation with conformable derivatives can be transformed to a deterministic fractional Hirota-Satsuma coupled KdV equation containing conformable derivatives. Using some symbolic computation and the software program “Mathematica”, we can find soliton and periodic wave solutions for the variable coefficients fractional Hirota-Satsuma coupled KdV equation with conformable derivatives. Under pronounced conditions, we can apply the inverse Hermite transform to obtain stochastic soliton and periodic wave solutions for the Wick-type stochastic fractional Hirota-Satsuma coupled KdV equation with conformable derivatives. Finally, by an application example, we show how the stochastic solutions can be given as Brownian motion functional solutions. This paper is organized as follows: In Section 2, we recall the definitions and some properties of the conformable derivative and integral, some requisites from Gaussian white noise analysis and the main steps for solving the conformable nonlinear PDEs. In Section 3, we use the sub-eqution method, white noise theory and Hermite transform to obtain new stochastic soliton wave solutions for the Wick-type stochastic fractional Hirota-Satsuma coupled KdV equation with conformable derivatives. In Section 4, we give an example to show that the stochastic solutions can be given as Brownian motion functional solutions. Section 5 is devoted to conclusion.

2. Preliminaries

In this section, we recall the definitions and some properties of the conformable derivative and integral.

Definition 2.1 ([32, 5]). Let f be a function from $(0, \infty)$ into $\mathbb{R}$. For $\alpha \in (0, 1]$, we define the con formable derivative of f of order α as follows:

$$D_t^\alpha f(t) = \lim_{h \to 0} \frac{f(t + ht^{1-\alpha}) - f(t)}{h}, \quad t > 0. \tag{2.3}$$

Definition 2.2 ([32, 5]). Let f be an α-conformable differentiable function for $t \in (0, a), a > 0$ and $\lim_{t \to 0^+} D_t^\alpha f(t)$ exists. Then, $D_t^\alpha f(0) = \lim_{t \to 0^+} D_t^\alpha f(t)$ and the conformable integral of the function f beginning from $a \geq 0$ is given by

$$I^{\alpha, a} f(t) = \int_a^t \frac{f(\tau)}{\tau^{1-\alpha}} \, d\tau, \tag{2.4}$$

where the integral in the right hand side is the classical improper Riemann integral and $\alpha \in (0, 1]$.

The following theorems gives some sustainable properties for the conformable derivative.

Theorem 2.1 ([32, 5]). *Assume that* $\alpha \in (0, 1], f$ *and* g *are* α-*conformable differentiable functions at* $t \in (0, \infty)$ *and* f *is differentiable* (*in the usual sense*) *with respect to* t. *Then*,

(1) $D_t^\alpha (af + bg) = a \ D_t^\alpha f + b \ D_t^\alpha g$, *for all* $a, b \in \mathbb{R}$,

(2) $D_t^\alpha (t^a) = a \ t^{a-\alpha}$, *for all* $a \in \mathbb{R}$,

(3) $D_t^\alpha (fg) = f D_t^\alpha g + g D_t^\alpha f$,

(4) $D_t^{\alpha}\left(\frac{f}{g}\right) = \frac{gD_t^{\alpha}f - f\,D_t^{\alpha}g}{g^2}$,

(5) $D_t^{\alpha}(f(t)) = t^{1-\alpha}f'(t)$,

where $'$ *denotes the usual derivative with respect to* t.

Theorem 2.2 ([33]). *Assume that the function* f *is a differentiable and* α*-conformable differentiable function on* $(0, \infty)$. *Also, assume that* g *is differentiable function defined on the range of* f. *Then,*

$$D_t^{\alpha}(f \circ g)(t) = t^{1-\alpha}[g(t)]^{\alpha-1}g'(t)\,(D_t^{\alpha}f(t))_{t=g}. \tag{2.5}$$

Now, we outline the main idea of the modified fractional sub-equation method. Many authors considered nonlinear FPDE, say, in two variables

$$P(u, u_x, u_t, D_x^{\alpha}u, D_t^{\alpha}u, \dots) = 0, \quad 0 < \alpha \le 1, \tag{2.6}$$

where P is a nonlinear function with respect to the indicated variables. To determine the solution $u = u(x, t)$ explicitly, we first introduce the following transformation:

$$u = u(\xi), \quad \xi = \xi(x, t), \tag{2.7}$$

which converts Equation (2.6) into a fractional ordinary differential equation

$$Q(u, u', u'', D_{\xi}^{\alpha}u, D_{\xi}^{2\alpha}u, \dots) = 0. \tag{2.8}$$

Next we introduce a new variable $Y = Y(\xi)$ which is a solution of the fractional Riccati equation

$$D_{\xi}^{\alpha}Y = \alpha_0 + \alpha_1 Y + \alpha_2 Y^2, \quad 0 < \alpha \le 1, \tag{2.9}$$

where α_0, α_1, and α_2 are arbitrary constants. Equation (2.9) is the fractional Riccati differential equation, where α is a parameter describing the order of the fractional derivative. In the case of $\alpha = 1$, Equation (2.9) is reduced to the classical Riccati differential equation. The importance of this equation usually arises in the optimal control problems. The feed back gain of the linear quadratic optimal control depends on a solution of a Riccati differential equation which has to be found for the whole time horizon of the control process [37, 51]. Then we propose the following series expansion as a solution of Equation (2.6):

$$u(x,\, t) = u(\xi) = \sum_{k=0}^{n} a_k(x,\, t)Y^k(\xi) + \sum_{k=1}^{n} b_k(x,\, t)Y^{-k}(\xi), \tag{2.10}$$

where $a_k(k = 0, 1, \ldots, n)$, $b_k(k = 1, \ldots, n)$ are functions to be determined later and n is a positive integer which can be determined via the balancing of the highest derivative term with the nonlinear term in Equation (2.8). Inserting Equation (2.10) into Equation (2.8) and using Equation (2.9) will give an algebraic equation in powers of Y. Since all coefficients of Y^k must vanish, this will give a system of algebraic equations with respect to a_k and b_k. With the aid of Mathematica, we can determine a_k and b_k. According to the recent paper by Zhang et al. [51], we can deduce the following set of solutions of Equation (2.9):

$$\begin{cases} Y_1(\xi) = E_\alpha(\xi) - 1, & \alpha_0 = \alpha_1 = 1,\ \alpha_2 = 0, \\ Y_2(\xi) = \coth_\alpha(\xi) \pm \operatorname{csch}_\alpha(\xi), & \\ Y_3(\xi) = \tanh_\alpha(\xi) \pm i\operatorname{sech}_\alpha(\xi), & \alpha_0 = -\alpha_2 = \dfrac{1}{2},\ \alpha_1 = 0, \\ Y_4(\xi) = \dfrac{1}{2}\tan_\alpha(2\xi), & \\ Y_5(\xi) = \dfrac{1}{2}\cot_\alpha(2\xi), & \alpha_0 = \dfrac{1}{4}\alpha_2 = 1,\ \alpha_1 = 0, \end{cases} \tag{2.11}$$

with the generalized hyperbolic and trigonometric functions

$$\sinh_\alpha(x) = \frac{E_\alpha(x^\alpha) - E_\alpha(-x^\alpha)}{2},\ \cosh_\alpha(x) = \frac{E_\alpha(x^\alpha) + E_\alpha(-x^\alpha)}{2},$$

$$\tanh_\alpha(x) = \frac{\sinh_\alpha(x)}{\cosh_\alpha(x)},\ \coth_\alpha(x) = \frac{\cosh_\alpha(x)}{\sinh_\alpha(x)},$$

$$\operatorname{csch}_\alpha(x) = \frac{1}{\sinh_\alpha(x)},\ \operatorname{sech}_\alpha(x) = \frac{1}{\cosh_\alpha(x)},$$

$$\sin_\alpha(x) = \frac{E_\alpha(ix^\alpha) - E_\alpha(-ix^\alpha)}{2i},\ \cos_\alpha(x) = \frac{E_\alpha(ix^\alpha) + E_\alpha(-ix^\alpha)}{2},$$

$$\cot_\alpha(x) = \frac{\cos_\alpha(x)}{\sin_\alpha(x)},\ \tan_\alpha(x) = \frac{\sin_\alpha(x)}{\cos_\alpha(x)},$$

$$\csc_\alpha(x) = \frac{1}{\sin_\alpha(x)},\ \sec_\alpha(x) = \frac{1}{\cos_\alpha(x)},$$

defined by the Mittag-Leffler function $E_\alpha(y) = \sum_{j=0}^{\infty} \frac{y^j}{\Gamma(1 + j\alpha)}$. For more details about the generalized exponential, hyperbolic and trigonometric functions, see [39].

3. Travelling Wave Solutions for Equation (1.2)

This section is devoted to give the exact travelling wave solutions for fractional Hirota-Satsuma coupled KdV equation with conformable derivative. We apply white noise analysis, Hermite transform and modified fractional sub-equation methodto explore exact travelling wave solutions for Equation (1.2). Taking the Hermite transform of Equation (1.1), we get the deterministic system

$$\begin{cases} D_t^\alpha \widetilde{U}(x, t, z) = \widetilde{F}(t, z)\, D_x^{3\alpha} \widetilde{U}(x, t, z) + \widetilde{G}(t, z)\, \widetilde{U}(x, t, z) D_x^\alpha \widetilde{U}(x, t, z) \\ \qquad\qquad + \widetilde{H}(t, z) D_x^\alpha (\widetilde{V}(x, t, z) \widetilde{W}(x, t, z)), \\ D_t^\alpha \widetilde{V}(x, t, z) = -D_x^{3\alpha} \widetilde{V}(x, t, z) + 3\widetilde{U}(x, t, z)\, D_x^\alpha \widetilde{V}(x, t, z), \\ D_t^\alpha \widetilde{W}(x, t, z) = -D_x^{3\alpha} \widetilde{W}(x, t, z) + 3\widetilde{U}(x, t, z)\, D_x^\alpha \widetilde{W}(x, t, z), \end{cases} \tag{3.1}$$

where $z = (z_1, z_2, \ldots) \in (\mathbb{C}^{\mathbb{N}})_c$ is a vector parameter. To look for the travelling wave solution of Equation (3.1), we make the transformations

$u(x, t, z) := \widetilde{U}(x, t, z) = \phi(\xi(x, t, z))$, $v(x, t, z) := \widetilde{V}(x, t, z) = \psi(\xi(x, t, z))$, and $w(x, t, z) := \widetilde{W}(x, t, z) = \chi(\xi(x, t, z))$, with

$$\xi(x, t, z) = kx + s\int_0^t l(\tau, z)d\tau + c,$$

where k, s and c are arbitrary constants which satisfy $ks \neq 0$, $l(t, z)$ is a non-zero functions of the indicated variables to be determined. So, Equation (3.1) can be changing into the form

$$\begin{cases} (sl)^{\alpha} D_{\xi}^{\alpha}\phi = k^{3\alpha} pD_{\xi}^{3\alpha}\phi + k^{\alpha} q\phi D_{\xi}^{\alpha}\phi + k^{\alpha} rD_{\xi}^{\alpha}(\psi\chi), \\ (sl)^{\alpha} D_{\xi}^{\alpha}\psi = -k^{3\alpha} D_{\xi}^{3\alpha}\psi + 3k^{\alpha}\phi D_{\xi}^{\alpha}\psi, \\ (sl)^{\alpha} D_{\xi}^{\alpha}\chi = -k^{3\alpha} D_{\xi}^{3\alpha}\chi + 3k^{\alpha}\phi D_{\xi}^{\alpha}\chi, \end{cases} \tag{3.2}$$

where $f(t, z) := \widetilde{F}(t, z)$, $g(t, z) := \widetilde{G}(t, z)$ and $h(t, z) := \widetilde{H}(t, z)$. Balancing the highest order linear terms and nonlinear terms in Equation (3.2), gives the following ansatzes:

$$\begin{cases} u(x, t, z) = a_0(t, z) + a_1(t, z)Y(\xi) + a_2(t, z)Y^2(\xi) \\ \qquad\qquad + b_1(t, z)Y^{-1}(\xi) + b_2(t, z)Y^{-2}(\xi), \\ v(x, t, z) = c_0(t, z) + c_1(t, z)Y(\xi) + c_2(t, z)Y^2(\xi) \\ \qquad\qquad + d_1(t, z)Y^{-1}(\xi) + d_2(t, z)Y^{-2}(\xi), \\ w(x, t, z) = e_0(t, z) + e_1(t, z)Y(\xi) + e_2(t, z)Y^2(\xi) \\ \qquad\qquad + f_1(t, z)Y^{-1}(\xi) + f_2(t, z)Y^{-2}(\xi), \end{cases} \tag{3.3}$$

where $Y(\xi)$ satisfies the fractional Riccati equation (2.9). By substituting Equation (3.3) along with Equation (2.9) into Equation (3.2), collect the coefficients of $Y^k(k = -5, -4, \ldots, 5)$ and set them to be zero, we will obtain a system of algebraic equations in the unknowns $a_k, c_k, e_k (k = 0, 1, 2)$, $b_k, d_k, f_k (k = 1, 2)$ and l of the form

$$\begin{cases} -k^{2\alpha}K_i^{\psi}+3\zeta_i^{\psi}=0, & i=4,5,\\ -k^{2\alpha}L_i^{\psi}+3\eta_i^{\psi}=0, & i=4,5,\\ -k^{2\alpha}K_i^{\eta}+3\zeta_i^{\chi}=0, & i=4,5,\\ -k^{2\alpha}L_i^{\eta}+3\eta_i^{\chi}=0, & i=4,5,\\ (sl)^{\alpha}G_i^{\psi}=-k^{3\alpha}K_i^{\psi}+3k^{\alpha}\zeta_i^{\psi}, & i=0,1,2,3,\\ (sl)^{\alpha}H_i^{\psi}=-k^{3\alpha}L_i^{\psi}+3k^{\alpha}\eta_i^{\psi}, & i=1,2,3,\\ (sl)^{\alpha}G_i^{\chi}=-k^{3\alpha}K_i^{\chi}+3k^{\alpha}\zeta_i^{\chi}, & i=0,1,2,3,\\ (sl)^{\alpha}H_i^{\chi}=-k^{3\alpha}L_i^{\chi}+3k^{\alpha}\eta_i^{\chi}, & i=1,2,3,\\ k^{2\alpha}fK_i^{\phi}+g\zeta_i^{\phi}+h(\rho_i^1+\rho_i^2)=0, & i=4,5,\\ k^{2\alpha}fL_i^{\phi}+g\eta_i^{\phi}+h(\lambda_i^1+\lambda_i^2)=0, & i=4,5,\\ (sl)^{\alpha}G_i^{\phi}=k^{3\alpha}fK_i^{\phi}+k^{\alpha}g\zeta_i^{\phi}+k^{\alpha}h(\rho_i^1+\rho_i^2), & i=0,1,2,3,\\ (sl)^{\alpha}H_i^{\phi}=k^{3\alpha}fL_i^{\phi}+k^{\alpha}g\eta_i^{\phi}+k^{\alpha}h(\lambda_i^1+\lambda_i^2), & i=1,2,3, \end{cases} \tag{3.4}$$

where $\zeta_0^{\phi}=a_0G_0^{\phi}+a_1H_1^{\phi}+a_2H_2^{\phi}+b_1G_1^{\phi}+b_2G_2^{\phi}$, $\zeta_1^{\phi}=a_0G_1^{\phi}+a_1G_0^{\phi}+a_2H_1^{\phi}+b_1G_2^{\phi}+b_2G_3^{\phi}$, $\zeta_2^{\phi}=a_0G_2^{\phi}+a_1G_1^{\phi}+a_2G_0^{\phi}+b_1G_3^{\phi}$, $\zeta_3^{\phi}=a_0G_3^{\phi}+a_1G_2^{\phi}+a_2G_1^{\phi}$, $\zeta_4^{\phi}=a_1G_3^{\phi}+a_2G_2^{\phi}$, $\zeta_5^{\phi}=a_2G_3^{\phi}$, $\eta_1^{\phi}=a_0H_1^{\phi}+a_1H_2^{\phi}+a_2H_3^{\phi}+b_1G_0^{\phi}+b_2G_1^{\phi}$, $\eta_2^{\phi}=a_0H_2^{\phi}+a_1H_3^{\phi}+b_2G_0^{\phi}+b_1H_1^{\phi}$, $\eta_3^{\phi}=a_0H_3^{\phi}+b_1H_2^{\phi}+b_2H_1^{\phi}$, $\eta_4^{\phi}=b_1H_3^{\phi}+b_2H_2^{\phi}$, $\eta_5^{\phi}=b_2H_3^{\phi}$, $\zeta_0^{\psi}=c_0G_0^{\phi}+c_1H_1^{\phi}+c_2H_2^{\phi}+d_1G_1^{\phi}+d_2G_2^{\phi}$, $\zeta_1^{\psi}=c_0G_1^{\phi}+c_1G_0^{\phi}+c_2H_1^{\phi}+d_1G_2^{\phi}+d_2G_3^{\phi}$, $\zeta_2^{\psi}=c_0G_2^{\phi}+c_1G_1^{\phi}+c_2G_0^{\phi}+d_1G_3^{\phi}$, $\zeta_3^{\psi}=c_0G_3^{\phi}+c_1G_2^{\phi}+c_2G_1^{\phi}$, $\zeta_4^{\psi}=c_1G_3^{\phi}+c_2G_2^{\phi}$, $\zeta_5^{\psi}=c_2G_3^{\phi}$, $\eta_1^{\psi}=c_0H_1^{\phi}+c_1H_2^{\phi}+c_2H_3^{\phi}+d_1G_0^{\phi}+d_2G_1^{\phi}$, $\eta_2^{\psi}=c_0H_2^{\phi}+c_1H_3^{\phi}+d_2G_0^{\phi}+d_1H_1^{\phi}$, $\eta_3^{\psi}=c_0H_3^{\phi}+d_1H_2^{\phi}+d_2H_1^{\phi}$, $\eta_4^{\psi}=d_1H_3^{\phi}+d_2H_2^{\phi}$, $\eta_5^{\psi}=d_2H_3^{\phi}$, $\zeta_0^{\chi}=e_0G_0^{\phi}+e_1H_1^{\phi}+e_2H_2^{\phi}+f_1G_1^{\phi}+f_2G_2^{\phi}$, $\zeta_1^{\chi}=e_0G_1^{\phi}+e_1G_0^{\phi}+e_2H_1^{\phi}+f_1G_2^{\phi}+f_2G_3^{\phi}$, $\zeta_2^{\chi}=e_0G_2^{\phi}+e_1G_1^{\phi}+e_2$

$G_0^{\phi} + f_1 G_3^{\phi}$, $\zeta_3^{\chi} = e_0 G_3^{\phi} + e_1 G_2^{\phi} + e_2 G_1^{\phi}$, $\zeta_4^{\chi} = e_1 G_3^{\phi} + e_2 G_2^{\phi}$, $\zeta_5^{\chi} = e_2 G_3^{\phi}$, $\eta_1^{\chi} = e_0 H_1^{\phi} + e_1 H_2^{\phi} + e_2 H_3^{\phi} + f_1 G_0^{\phi} + f_2 G_1^{\phi}$, $\eta_2^{\chi} = e_0 H_2^{\phi} + e_1 H_3^{\phi} + f_2 G_0^{\phi} + f_1 H_1^{\phi}$, $\eta_3^{\chi} = e_0 H_3^{\phi} + f_1 H_2^{\phi} + f_2 H_1^{\phi}$, $\eta_4^{\chi} = f_1 H_3^{\phi} + f_2 H_2^{\phi}$, $\eta_5^{\chi} = f_2 H_3^{\phi}$, $\rho_0^1 = e_0 G_0^{\psi} + e_1 H_1^{\psi} + e_2 H_2^{\psi} + f_1 G_1^{\psi} + f_2 G_2^{\psi}$, $\rho_1^1 = e_0 G_1^{\psi} + e_1 G_0^{\psi} + e_2 H_1^{\psi} + f_1 G_2^{\psi} + f_2 G_3^{\psi}$, $\rho_2^1 = e_0 G_2^{\psi} + e_1 G_1^{\psi} + e_2 G_0^{\psi} + f_1 G_3^{\psi}$, $\rho_3^1 = e_0 G_3^{\psi} + e_1 G_2^{\psi} + e_2 G_1^{\psi}$, $\rho_4^1 = e_1 G_3^{\psi} + e_2 G_2^{\psi}$, $\rho_5^1 = e_2 G_3^{\psi}$, $\lambda_1^1 = e_0 H_1^{\psi} + e_1 H_2^{\psi} + e_2 H_3^{\psi} + f_1 G_0^{\psi} + f_2 G_1^{\psi}$, $\lambda_2^1 = e_0 H_2^{\psi} + e_1 H_3^{\psi} + f_2 G_0^{\psi} + f_1 H_1^{\psi}$, $\lambda_3^1 = e_0 H_3^{\psi} + f_1 H_2^{\psi} + f_2 H_1^{\psi}$, $\lambda_4^1 = f_1 H_3^{\psi} + f_2 H_2^{\psi}$, $\lambda_5^1 = f_2 H_3^{\psi}$, $\rho_0^2 = c_0 G_0^{\chi} + c_1 H_1^{\chi} + c_2 H_2^{\chi} + d_1 G_1^{\chi} + d_2 G_2^{\chi}$, $\rho_1^2 = c_0 G_1^{\chi} + c_1 G_0^{\chi} + c_2 H_1^{\chi} + d_1 G_2^{\chi} + d_2 G_3^{\chi}$, $\rho_2^2 = c_0 G_2^{\chi} + c_1 G_1^{\chi} + c_2 G_0^{\chi} + d_1 G_3^{\chi}$, $\rho_3^2 = c_0 G_3^{\chi} + c_1 G_2^{\chi} + c_2 G_1^{\chi}$, $\rho_4^2 = c_1 G_3^{\chi} + c_2 G_2^{\chi}$, $\rho_5^2 = c_2 G_3^{\chi}$, $\lambda_1^2 = c_0 H_1^{\chi} + c_1 H_2^{\chi} + c_2 H_3^{\chi} + d_1 G_0^{\chi} + d_2 G_1^{\chi}$, $\lambda_2^2 = c_0 H_2^{\chi} + c_1 H_3^{\chi} + d_2 G_0^{\chi} + d_1 H_1^{\chi}$, $\lambda_3^2 = c_0 H_3^{\chi} + d_1 H_2^{\chi} + d_2 H_1^{\chi}$, $\lambda_4^2 = d_1 H_3^{\chi} + d_2 H_2^{\chi}$ and $\lambda_5^2 = d_2 H_3^{\chi}$, $G_0^{\phi} = h_0 a_1 - h_2 b_1$, $G_1^{\phi} = 2h_0 a_2 + h_1 a_1$, $G_2^{\phi} = 2h_1 a_2 + h_2 a_1$, $G_3^{\phi} = 2h_2 a_2$, $G_0^{\psi} = h_0 c_1 - h_2 d_1$, $G_1^{\psi} = 2h_0 c_2 + h_1 c_1$, $G_2^{\psi} = 2h_1 c_2 + h_2 c_1$, $G_3^{\psi} = 2h_2 c_2$, $G_0^{\chi} = h_0 e_1 - h_2 f_1$, $G_1^{\chi} = 2h_0 e_2 + h_1 e_1$, $G_2^{\chi} = 2h_1 e_2 + h_2 e_1$, $G_3^{\chi} = 2h_2 e_2$, $H_1^{\phi} = -(2h_2 b_2 + h_1 b_1)$, $H_2^{\phi} = -(2h_1 b_2 + h_0 b_1)$, $H_3^{\phi} = -2h_0 b_2$, $H_1^{\psi} = -(2h_2 d_2 + h_1 d_1)$, $H_2^{\psi} = -(2h_1 d_2 + h_0 d_1)$, $H_3^{\psi} = -2h_0 d_2$ $H_1^{\chi} = -(2h_2 f_2 + h_1 f_1)$, $H_2^{\chi} = -(2h_1 f_2 + h_0 f_1)$, $H_3^{\chi} = -2h_0 f_2$, $K_0^{\phi} = h_0(2h_0 G_2^{\phi} + h_1 G_1^{\phi}) + h_2(2h_2 H_2^{\phi} + h_1 H_1^{\phi})$, $K_1^{\phi} = 2h_0(3h_0 G_3^{\phi} + 2h_1 G_2^{\phi} + h_2 G_1^{\phi}) + h_1(2h_0 G_2^{\phi} + h_1 G_1^{\phi})$, $K_2^{\phi} = 3h_0(3h_1 G_3^{\phi} + 2h_2 G_2^{\phi}) + 2h_1(3h_0 G_3^{\phi} + 2h_1 G_2^{\phi} + h_2 G_1^{\phi}) + h_2(2h_0 G_2^{\phi} + h_1 G_1^{\phi})$, $K_3^{\phi} = 2h_2(3h_0 G_3^{\phi} + 2h_1 G_2^{\phi} + h_2 G_1^{\phi}) + 3h_1(3h_1 G_3^{\phi} + 2h_2 G_2^{\phi}) + 12h_0 h_2 G_3^{\phi}$, $K_4^{\phi} = 3h_2(3h_1 G_3^{\phi} + 2h_2 G_2^{\phi}) + 12h_1 h_2 G_3^{\phi}$, $K_5^{\phi} = 12h_2^2 G_3^{\phi}$, $K_0^{\psi} = h_0(2h_0 G_2^{\psi} + h_1 G_1^{\psi}) + h_2(2h_2 H_2^{\psi} + h_1 H_1^{\psi})$, $K_1^{\psi} = 2h_0(3h_0 G_3^{\psi} + 2h_1 G_2^{\psi} + h_2 G_1^{\psi}) + h_1(2h_0 G_2^{\psi} + h_1 G_1^{\psi})$, $K_2^{\psi} = 3h_0(3h_1 G_3^{\psi} + 2h_2 G_2^{\psi}) + 2h_1(3h_0 G_3^{\psi} + 2h_1 G_2^{\psi} + h_2 G_1^{\psi}) + h_2(2h_0$

$G_2^{\psi} + h_1 G_1^{\psi})$, $K_3^{\psi} = 2h_2(3h_0 G_3^{\psi} + 2h_1 G_2^{\psi} + h_2 G_1^{\psi}) + 3h_1(3h_1 G_3^{\psi} + 2h_2 G_2^{\psi}) + 12 h_0 h_2 G_3^{\psi}$, $K_4^{\psi} = 3h_2(3h_1 G_3^{\psi} + 2h_2 G_2^{\psi}) + 12 h_1 h_2 G_3^{\psi}$, $K_5^{\psi} = 12 h_2^2 G_3^{\psi}$, $K_0^{\chi} = h_0 (2h_0 G_2^{\chi} + h_1 G_1^{\chi}) + h_2(2h_2 H_2^{\chi} + h_1 H_1^{\chi})$, $K_1^{\chi} = 2h_0(3h_0 G_3^{\chi} + 2h_1 G_2^{\chi} + h_2 G_1^{\chi}) + h_1(2h_0 G_2^{\chi} + h_1 G_1^{\chi})$, $K_2^{\chi} = 3h_0(3h_1 G_3^{\chi} + 2h_2 G_2^{\chi}) + 2h_1(3h_0 G_3^{\chi} + 2h_1 G_2^{\chi} + h_2 G_1^{\chi}) + h_2(2h_0 G_2^{\chi} + h_1 G_1^{\chi})$, $K_3^{\chi} = 2h_2(3h_0 G_3^{\chi} + 2h_1 G_2^{\chi} + h_2 G_1^{\chi}) + 3h_1(3h_1 G_3^{\chi} + 2h_2 G_2^{\chi}) + 12 h_0 h_2 G_3^{\chi}$, $K_4^{\chi} = 3h_2(3h_1 G_3^{\chi} + 2h_2 G_2^{\chi}) + 12 h_1 h_2 G_3^{\chi}$, $K_5^{\chi} = 12 h_2^2 G_3^{\chi}$, $L_1^{\phi} = 2h_2(3h_2 H_3^{\phi} + 2h_1 H_2^{\phi} + h_0 H_1^{\phi}) + h_1(2h_2 H_2^{\phi} + h_1 H_1^{\phi})$, $L_2^{\phi} = 3h_2(3h_1 H_3^{\phi} + 2h_0 H_2^{\phi}) + 2h_1(3h_2 H_3^{\phi} + 2h_1 H_2^{\phi} + h_0 H_1^{\phi}) + h_0(2h_2 H_2^{\phi} + h_1 H_1^{\phi})$, $L_3^{\phi} = 2h_0(3h_2 H_3^{\phi} + 2h_1 H_2^{\phi} + h_0 H_1^{\phi}) + 3h_1(3h_1 H_3^{\phi} + 2h_0 H_2^{\phi}) + 12 h_0 h_2 H_3^{\phi}$, $L_4^{\phi} = 3h_0(3h_1 H_3^{\phi} + 2h_0 H_2^{\phi}) + 12 h_1 h_0 H_3^{\phi}$, $L_5^{\phi} = 12 h_0^2 H_3^{\phi}$, $L_1^{\psi} = 2h_2(3h_2 H_3^{\psi} + 2h_1 H_2^{\psi} + h_0 H_1^{\psi}) + h_1(2h_2 H_2^{\psi} + h_1 H_1^{\psi})$, $L_2^{\psi} = 3h_2(3h_1 H_3^{\psi} + 2h_0 H_2^{\psi}) + 2h_1(3h_2 H_3^{\psi} + 2h_1 H_2^{\psi} + h_0 H_1^{\psi}) + h_0(2h_2 H_2^{\psi} + h_1 H_1^{\psi})$, $L_3^{\psi} = 2h_0(3h_2 H_3^{\psi} + 2h_1 H_2^{\psi} + h_0 H_1^{\psi}) + 3h_1(3h_1 H_3^{\psi} + 2h_0 H_2^{\psi}) + 12 h_0 h_2 H_3^{\psi}$, $L_4^{\psi} = 3h_0(3h_1 H_3^{\psi} + 2h_0 H_2^{\psi}) + 12 h_1 h_0 H_3^{\psi}$ $L_5^{\psi} = 12 h_0^2 H_3^{\psi}$, $L_1^{\chi} = 2h_2(3h_2 H_3^{\chi} + 2h_1 H_2^{\chi} + h_0 H_1^{\chi}) + h_1(2h_2 H_2^{\chi} + h_1 H_1^{\chi})$, $L_2^{\chi} = 3h_2(3h_1 H_3^{\chi} + 2h_0 H_2^{\chi}) + 2h_1(3h_2 H_3^{\chi} + 2h_1 H_2^{\chi} + h_0 H_1^{\chi}) + h_0(2h_2 H_2^{\chi} + h_1 H_1^{\chi})$, $L_3^{\chi} = 2h_0(3h_2 H_3^{\chi} + 2h_1 H_2^{\chi} + h_0 H_1^{\chi}) + 3h_1(3h_1 H_3^{\chi} + 2h_0 H_2^{\chi}) + 12 h_0 h_2 H_3^{\chi}$, $L_4^{\chi} = 3h_0(3h_1 H_3^{\chi} + 2h_0 H_2^{\chi}) + 12 h_1 h_0 H_3^{\chi}$, $L_5^{\chi} = 12 h_0^2 H_3^{\chi}$.

In the remaining part of this section we investigate and solve our problem for some particular cases for the fractional Riccati equation (2.9) as follows.

Case A. If we set $\alpha_0 = \alpha_1 = 1, \alpha_2 = 0$ in Equation (2.9), and use Mathematica to solve the resulting system, we will obtain the following set of solutions:

Set 1. $a_1 = a_2 = b_2 = c_2 = d_2 = e_1 = e_2 = f_2 = 0$, $a_0 = \frac{1}{g}(\frac{sl}{K})^{\alpha} - \frac{7f}{g}$ $k^{2\alpha} + \frac{3}{8gh}(1 - \frac{f}{g})$, $c_0 = \frac{3k^{\alpha}\left(1 - \frac{f}{g}\right)}{2h}$, $e_0 = -24k^{3\alpha}f$, $b_1 = 6k^{2\alpha}\frac{f}{g}$, $c_1 = -\frac{2k^{\alpha}}{h}$, $d_1 = f_1 = \frac{3k^{\alpha}}{2gh}$, $l = \pm\frac{k^3}{s}i^{2\alpha}\sqrt[\alpha]{7 + \frac{f}{g}}$.

Substituting these values in Equation (3.3) and using Equation (2.11), we obtain the following exponential decay wave solution of Equation (3.1):

$$u_1(x, t, z) = \frac{(sl)^{\alpha}}{k^{\alpha}g(t, z)} - \frac{7k^{2\alpha}f(t, z)}{g(t, z)} + \frac{3(g(t, z) - f(t, z))}{8g^2(t, z)h(t, z)} + \frac{6k^{2\alpha}f(t, z)}{g(t, z)}Y_1^{-1}[\xi_1(x, t, z)], \tag{3.5}$$

$$v_1(x, t, z) = \frac{3k^{\alpha}(g(t, z) - f(t, z))}{2g(t, z)h(t, z)} - \frac{2k^{\alpha}}{h(t, z)}Y_1[\xi_1(x, t, z)] + \frac{3k^{\alpha}f(t, z)}{2g(t, z)h(t, z)}Y_1^{-1}[\xi_1(x, t, z)], \tag{3.6}$$

$$w_1(x, t, z) = -24k^{\alpha}f(t, z) + \frac{3k^{\alpha}f(t, z)}{2g(t, z)h(t, z)}Y_1^{-1}[\xi_1(x, t, z)], \tag{3.7}$$

with

$$\xi_1(x, t, z) = kx \pm \frac{i^{2\alpha}k^3}{s}\int_0^t \sqrt[\alpha]{7 + \frac{f(\tau, z)}{g(\tau, z)}}d\tau + c_1.$$

Case B. If we set $\alpha_0 = -\alpha_2 = \frac{1}{2}$, $\alpha_1 = 0$ in Equation (2.9), and use Mathematica to solve the resulting system, we will obtain the following sets of solutions:

Set 2. $a_1 = b_1 = c_1 = c_2 = d_1 = e_1 = e_2 = f_1 = 0$, $a_0 = \dfrac{(2+3p)k^{2\alpha}}{q}$,

$c_0 = -e_0 = \dfrac{pk^{2\alpha}}{3}$, $a_2 = 2b_2 = \dfrac{-3pk^{2\alpha}}{q}$, $f_2 = d_2 = k^{2\alpha}$, $l = \left(\dfrac{(2+p)k^{3\alpha}}{s^{\alpha}}\right)^{\frac{1}{\alpha}}$.

Substituting these values in Equation (3.3) and using Equation (2.11), we obtain the following soliton wave solutions of Equation (3.1):

$$u_i(x, t, z) = \frac{(2+3p(t, z))k^{2\alpha}}{q(t, z)} - \frac{3p(t, z)k^{2\alpha}}{2q(t, z)}(2Y_i^2[\xi_2(x, t, z)] + Y_i^{-2}[\xi_2(x, t, z)]), \tag{3.8}$$

$$v_i(x, t, z) = \frac{p(t, z)k^{2\alpha}}{3} + k^{2\alpha}Y_i^{-2}[\xi_2(x, t, z)], \tag{3.9}$$

$$w_i(x, t, z) = \frac{p(t, z)k^{2\alpha}}{3} + k^{2\alpha}Y_i^{-2}[\xi_2(x, t, z)], \tag{3.10}$$

with $i = 2, 3$ and

$$\xi_2(x, t, z) = kx + k^3\int_0^t (2 + p(\tau, z))^{\frac{1}{\alpha}} d\tau + c_2.$$

Set 3. $a_1 = a_2 = b_1 = c_1 = c_2 = d_1 = e_1 = e_2 = f_1 = 0$, $a_0 = \dfrac{(2+4p)k^{2\alpha}}{q}$,

$c_0 = -e_0 = \dfrac{pk^{2\alpha}}{3}$, $b_2 = \dfrac{-3pk^{2\alpha}}{2q}$, $f_2 = d_2 = k^2$, $l = \left(\dfrac{(2+p)k^{3\alpha}}{s^{\alpha}}\right)^{\frac{1}{\alpha}}$.

Substituting these values in Equation (3.3) and using Equation (2.11), we obtain the following soliton wave solutions of Equation (3.1):

$$u_{i+2}(x, t, z) = \frac{(2 + 4p(t, z))k^{2\alpha}}{q(t, z)} - \frac{3p(t, z)k^{2\alpha}}{2q(t, z)} Y_i^{-2}[\xi_2(x, t, z)], \tag{3.11}$$

$$v_{i+2}(x, t, z) = \frac{p(t, z)k^{2\alpha}}{3} + k^2 Y_i^{-2}[\xi_2(x, t, z)], \tag{3.12}$$

$$w_{i+2}(x, t, z) = \frac{p(t, z)k^{2\alpha}}{3} + k^2 Y_i^{-2}[\xi_2(x, t, z)], \tag{3.13}$$

with $i = 2, 3$.

Case C. If we set $\alpha_0 = \frac{1}{4}\alpha_2 = 1, \alpha_1 = 0$ in Equation (2.9), and use Mathematica to solve the resulting system, we will obtain the following set of solutions:

Set 4. $a_1 = b_1 = b_2 = c_1 = d_1 = d_2 = e_1 = f_1 = f_2 = 0$, $a_0 = \frac{(32 + 31p)k^{2\alpha}}{q} - \frac{4rk^{2\alpha}}{9}$, $a_2 = -\frac{96pk^{2\alpha}}{q}$, $c_0 = -e_0 = \frac{pk^{2\alpha}}{3}$, $c_2 = e_2 = 64k^{2\alpha}$, $l = \left(\frac{(p - 32)k^{3\alpha}}{s^{\alpha}}\right)^{\frac{1}{\alpha}}$.

Substituting these values in Equation (3.3) and using Equation (2.11), we obtain the following periodic wave solutions of Equation (3.1):

$$u_{i+2}(x, t, z) = \frac{(32 + 31p(t, z))k^{2\alpha}}{q(t, z)} - \frac{4r(t, z)k^{2\alpha}}{9} - \frac{96p(t, z)k^{2\alpha}}{q(t, z)} Y_i^{2}[\xi_3(x, t, z)], \tag{3.14}$$

$$v_{i+2}(x, t, z) = \frac{p(t, z)k^{2\alpha}}{3} + 64k^{2\alpha} Y_i^{-2}[\xi_3(x, t, z)], \tag{3.15}$$

$$w_{i+2}(x, t, z) = \frac{p(t, z)k^{2\alpha}}{3} + 64k^2 Y_i^{-2}[\xi_3(x, t, z)], \tag{3.16}$$

with $i = 4, 5$ and

$$\xi_3(x, t, z) = kx + k^3 \int_0^t (p(\tau, z) - 32)^{\frac{1}{\alpha}} d\tau + c_3.$$

At the end of this section we should remark that, there exists an infinitely number of exact travelling wave solutions for Equation (1.2); these solutions come from solving the system (3.4) with regard to the fractional Riccati equation (2.9). The above mentioned cases are just to clarify how far our technique is applicable.

4. White Noise Functional Solutions of Equation (1.1)

In this section, we use the inverse Hermite transform and [30, Theorem 4.1.1] to obtain white noise functional solutions for Equation (1.2). The properties of generalized exponential, hyperbolic and trigonometric functions yield that there exists a bounded open set $G \subset \mathbb{R} \times \mathbb{R}_+$, $m < \infty$, $n > 0$ such that the solution $\{u(x, t, z), v(x, t, z), w(x, t, z)\}$ of Equation (3.1) and all its fractional derivatives which are involved in Equation (3.1) are uniformly bounded for $(x, t, z) \in G \times K_m(n)$, continuous with respect to $(x, t) \in G$ for all $z \in K_m(n)$ and analytic with respect to $z \in K_m(n)$, for all $(x, t) \in G$. From [30, Theorem 4.1.1], there exist $U(x, t), V(x, t), W(x, t) \in (\mathcal{S})_{-1}$ such that $u(x, t, z) = \widetilde{U}(x, t)(z)$, $v(x, t, z) = \widetilde{V}(x, t)(z)$, and $w(x, t, z) = \widetilde{W}(x, t)(z)$ for all $(x, t, z) \in G \times K_m(n)$ and $\{U(x, t), V(x, t), W(x, t)\}$ solves (in the strong sense in $(\mathcal{S})_{-1}$) Equation (1.2) in $(\mathcal{S})_{-1}$. Hence, for $P(t)Q(t)R(t) \neq 0$, the white noise functional solutions of Equation (1.2) can be obtained by applying the inverse Hermite transform to Equations (3.5)-(3.16) as follows:

• Stochastic exponential decay wave solution:

$$U_1(x,\ t) = \frac{(sl)^{\alpha}}{k^{\alpha}G(t)} - \frac{7k^{2\alpha}F(t)}{G(t)} + \frac{3(G(t) - F(t))}{8G^{\diamond 2}(t) \diamond H(t)} + \frac{6k^{2\alpha}F(t)}{G(t)} Y_1^{-\diamond}[\Xi_1(x,\ t)], \tag{4.1}$$

$$V_1(x,\ t) = \frac{3k^{\alpha}(G(t) - F(t))}{2G(t) \diamond H(t)} - \frac{2k^{\alpha}}{H(t)} Y_1^{\diamond}[\Xi_1(x,\ t)]$$

$$+ \frac{3k^{\alpha}F(t)}{2G(t) \diamond H(t)} Y_1^{-\diamond}[\Xi_1(x,\ t)], \tag{4.2}$$

$$W_1(x,\ t) = -\ 24k^{\alpha}F(t) + \frac{3k^{\alpha}F(t)}{2G(t) \diamond H(t)} Y_1^{-\diamond}[\Xi_1(x,\ t)], \tag{4.3}$$

with

$$\Xi_1(x,\ t) = kx \pm \frac{i^{2\alpha}k^3}{s} \int_0^t \sqrt[\alpha]{7 + \frac{F(\tau)}{G(\tau)}} d\tau + c_1.$$

Stochastic soliton wave solutions:

$$U_i(x,\ t) = \frac{k^{2\alpha}(2 + 3P(t))}{Q(t,\ z)} - \frac{3k^{2\alpha}P(t)}{2Q(t)} \diamond (2Y_i^{\diamond\, 2}[\Xi_2(x,\ t)]$$

$$+ Y_i^{\diamond\,(-2)}[\Xi_2(x,\ t)]), \tag{4.3}$$

$$V_i(x,\ t) = \frac{k^{2\alpha}P(t)}{3} + k^{2\alpha}Y_i^{\diamond\,(-2)}[\Xi_2(x,\ t)], \tag{4.4}$$

$$W_i(x,\ t) = \frac{k^{2\alpha}P(t)}{3} + k^{2\alpha}Y_i^{\diamond\,(-2)}[\Xi_2(x,\ t)], \tag{4.5}$$

$$U_{i+2}(x,\ t) = \frac{k^{2\alpha}(2 + 4P(t))}{Q(t)} - \frac{3k^{2\alpha}P(t)}{2Q(t)} \diamond Y_i^{\diamond\,(-2)}[\Xi_2(x,\ t)], \tag{4.6}$$

$$V_{i+2}(x,\ t) = \frac{k^{2\alpha}P(t)}{3} + k^{2\alpha}Y_i^{\diamond\,(-2)}[\Xi_2(x,\ t)], \tag{4.7}$$

$$W_{i+2}(x,\ t) = \frac{k^{2\alpha}P(t)}{3} + k^{2\alpha}Y_i^{\diamond\,(-2)}[\Xi_2(x,\ t)], \tag{4.8}$$

with $i = 2, 3$ and

$$\Xi_2(x, t) = kx + k^3 \int_0^t (2 + P(\tau))^{\diamond \frac{1}{\alpha}} d\tau + c_2.$$

Stochastic periodic wave solutions:

$$U_{i+2}(x, t) = \frac{k^{2\alpha}(32 + 31P(t))}{Q(t)} - \frac{4k^{2\alpha}R(t)}{9} - \frac{96k^{2\alpha}P(t)}{Q(t)} \diamond Y_i^{\diamond 2}[\Xi_3(x, t)], \tag{4.9}$$

$$V_{i+2}(x, t) = \frac{k^{2\alpha}P(t)}{3} + 64k^{2\alpha}Y_i^{\diamond (-2)}[\Xi_3(x, t)], \tag{4.10}$$

$$W_{i+2}(x, t) = \frac{k^{2\alpha}P(t)}{3} + 64k^{2\alpha}Y_i^{\diamond (-2)}[\Xi_3(x, t)], \tag{4.11}$$

with $i = 4, 5$ and

$$\Xi_3(x, t) = kx + k^3 \int_0^t (P(\tau) - 32)^{\diamond \frac{1}{\alpha}} d\tau + c_3.$$

We observe that for different forms of $F(t)$, $G(t)$ and $H(t)$, we can get different solutions of Equation (1.2) from Equations (4.1)-(3.12).

5. Summary and Discussion

Our first interest in this work is to implement new strategies that give white noise functional solutions of the variable coefficients Wick-type stochastic fractional Hirota-Satsuma coupled KdV equations. The strategies that will be pursued in this work rest mainly on Hermite transform, white noise theory and modified fractional sub-equation method, all of which are employed to find white noise functional solutions of Equation (1.1). The proposed schemes, as we believe, are entirely new and introduce new solutions in addition to the well-known traditional solutions. The ease of using these methods are showed its power to determine shock or solitary type of solutions. Obviously, the planner which we have proposed in this paper can be also applied to other nonlinear PDEs in mathematical physics such as KdV-Burgers, modified KdV-Burgers, Sawada-Kotera, Zhiber-Shabat and Benjamin-Bona-Mahony equations. Note that, if $\alpha = 1,$ Equation (1.1) is reduced to the stochastic generalized Hirota-Satsuma coupled KdV equations. Also, if $\alpha = 1, f = 0.5$ and $g = -h = 3,$ Equation (1.2) is reduced to the generalized Hirota-Satsuma coupled KdV equations. Hence, our results can be considered a generalization of the work due to Ghany [13] and Wu [46]. Moreover, there is a unitary mapping between the Gaussian white noise space and the Poisson white noise space, this connection was given by Benth and Gjerde [4]. Hence, with the help of this connection, we can derive some Poisson white noise functional solutions, if the coefficients $F(t)$, $G(t)$ and $H(t)$ are Poisson white noise functions in Equation (1.1).

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