Transnational Journal of Mathematical Analysis and Applications
Vol. 9, Issue 1, 2021, Pages 33-48
ISSN 2347-9086
Published Online on October 27, 2021

http://jyotiacademicpress.org

NEW RESULTS ON KANNAN TYPE CONTRACTIVE SELFMAPS OF BOUNDEDLY COMPACT METRIC SPACES

MOHAMED AKKOUCHI

Department of Mathematics
Faculty of Sciences-Semlalia
University Cadi Ayyad
Av. Prince My. Abdellah, BP. 2390
Marrakesh (40.000-Marrakech)
Morocco (Maroc)
e-mail: akkm555@yahoo.fr

Abstract

Let (X, d) be a boundedly compact metric space. Let T be a Kannan type contractive selfmap of X. That is, T fulfills the following property:

$$d(Tx, Ty) < \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}, \text{ for all } x \neq y \in X. \qquad \text{(K-S)}$$

The condition (K-S) is not sufficient to ensure fixed point for T. The aim of this paper is to investigate necessary and sufficient conditions making T be a Picard operator. This work is motivated by some recent papers published by Górnicki, Garai, Senapati and Dey. Our work has

---

2020 Mathematics Subject Classification: 47H10, 54H25.
Keywords and phrases: metric space, boundedly compact set, T-orbitally compact space, Kannan type mapping, orbitally continuous mappings.
Communicated by Erdal Karapinar, Editor, RCMMS.
Received July 19, 2021; Revised September 16, 2021

relationships with other recent aricles on Kannan type contractive selfmaps. We propose here some new results and complements. Also, we provide examples to support our investigations.

1. Introduction and Preliminaries

Let (X, d) be a metric space and $T : X \to X$ be a selfmapping.

Throughout this paper, we shall use the following notations, definitions and recalls.

$\text{Fix}(T) := \{x \in X : Tx = x\}$. ($\text{Fix}(T)$ is the fixed point set of T).

For all $n \in \mathbb{N}$, we set $T^{n+1} := T \circ T^n$, $T^0 = I_X$ (the identity map of X) and $T^1 := T$.

Definition 1.1. Let (X, d) be a metric space and T be a selfmapping on X. Then the orbit of T at $x \in X$ is defined as $O_T(x) = \{x, Tx, T^2x, T^3x, \dots\}$.

In 1968, Kannan [11] established the following fixed point result.

Theorem 1.2 ([11]). *Let (X, d) be a complete metric space and T be a selfmapping of X satisfying*

$$d(Tx, Ty) \leq k\{d(x, Tx) + d(y, Ty)\},$$

for all $x, y \in X$ and $k \in [0, \frac{1}{2})$.

Then T has a unique fixed point $z \in X$, and for any $x \in X$ the sequence of iterates $(T^n x)$ converges to z.

The importance of this theorem is due to two facts. First, it does not require continuity of the selfmap T. The second is that we can characterize the completeness of the underlying space X in terms of fixed point of T. Indeed, in 1975, Subrahmanyam [15] proved that a metric space (X, d) is complete if and only if every Kannan mapping has a unique fixed point in X.

Let (X, d) be a metric space and T be a selfmapping of X satisfying the following condition:

$$d(Tx, Ty) < \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}, \text{ for all } x \neq y \in X. \tag{K-S}$$

Such a map will be called a Kannan contractive selfmap of X.

The value $\frac{1}{2}$ can not be taken in Theorem 1.2. Indeed, consider the following example: Take $X := \{-1, 1\}$ equipped with the usual distance and define the selfmapping T by setting $T(-1) = 1$ and $T(1) = -1$. Then we have

$$|T(1) - T(-1)| = 2 \leq \frac{1}{2}(|1 - T(1)| + |-1 - T(-1)|) = 2.$$

This example shows that there exists a continuous on a compact metric space (X, d) satisfying the following condition:

$$d(Tx, Ty) \leq \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}, \text{ for all } x, y \in X, \tag{K-L}$$

but without having fixed point.

We observe that if T is a selfmapping of X satisfies the condition (K-S), then it satisfies also the condition (K-L). The example just above says that the condition (K-L) is not strong enough to guarantee the existence of fixed points.

In this paper, we are interested with the condition (K-S).

It is clear that if a selfmap T of X satisfies the following condition:

$$d(Tx, Ty) < (d(x, Tx)d(y, Ty))^{\frac{1}{2}}, \text{ for all } x \neq y \in X,$$

is a Kannan contractive selfmap of X.

In 1978, Fisher [7] and Khan [13] simultaneously proved two fixed point results related to Kannan contractive type mappings. They proved that a continuous mapping on a compact metric space (X, d) has a unique fixed point if T satisfies

$$d(Tx, Ty) < \frac{1}{2}\{d(x, Ty) + d(y, Tx)\},$$

or

$$d(Tx, Ty) < (d(x, Tx)d(y, Ty))^{\frac{1}{2}},$$

for all $x, y \in X$ with $x \neq y$ respectively.

In 1980, Chen and Yeh [3] extended the above two results in a more general way.

In 2017, Górnicki [10] has made some new contributions to selfmappings of a metric space (X, d) satisfying the condition:

$$d(Tx, Ty) < k\{d(x, Tx) + d(y, Ty)\}, \text{ for all } x \neq y \in X, \tag{K-G}$$

where $k \in [0, 1]$. Also, in [10] one can find a new proof of Theorem 1.2 and a new proof to the following (known) result.

Theorem 1.3 ([10]). *Let (X, d) be a compact metric space and $T : X \to X$ be a continuous mapping satisfying*

$$d(Tx, Ty) < \frac{1}{2}\{d(x, Tx) + d(y, Ty)\},$$

for all $x, y \in X$ with $x \neq y$. Then T has a unique fixed point and for every $x \in X$, the sequence $(T^n x)$ converges to the fixed point.

We notice that a Kannan type contractive mapping does not have fixed point without additional conditions. Usually, in many fixed point results, the continuity of the map T and the compactness of X were required.

To weaken the compactness condition of the underlying spaces, we make use of the so called boundedly compact metric spaces.

Definition 1.4 ([6]). A metric space (X, d) is said to be boundedly compact if every bounded sequence in X has a convergent subsequence.

It is clear from definition that every compact metric space is boundedly compact, but a boundedly compact metric space need not be compact, for example, the set of real numbers $\mathbb{R}$ with usual metric is not compact but boundedly compact.

Suppose that the metric space (X, d) is boundedly compact. Then it is easy to see that for any subset Y of X, we have the following equivalence:

Y is compact, if and only if, Y is closed and bounded.

In particular, every (non-trivial) finite dimensional (real or complex) normed linear space is boundedly compact (but certainly not compact).

In 2017, Garai et al. (see [8] and [9]) have proved the following result.

Theorem 1.5 ([8]). *Let* (X, d) *be a boundedly compact metric space and* $T : X \to X$ *be a mapping such that*

$$d(Tx, Ty) < \frac{1}{2}\{d(x, Tx) + d(y, Ty)\},$$

for all $x, y \in X$ *with* $x \neq y$. *Then* T *will be a Picard operator.*

In the lines of proof of Theorem 1.5, the authors of [8] have used, without saying it, a kind of continuity. Precisely, we prove in Lemma 2.1 that, if Theorem 1.5 is true, then the selfmap T must be T-orbitally continuous in the following sense:

Definition 1.6 ([4])**.** Let (X, d) be a metric space and T be a selfmapping on X. A selfmapping T of X is said to be T-orbitally continuous on X, if for all $x, u \in X$ satisfying $\lim_{k\to\infty} T^{n_k}x = u$, then we have $\lim_{k\to\infty} T(T^{n_k}x) = Tu$.

We point out that the concept of orbitally continuous maps was introduced by Ćirić in [4]. Also, we recall the concept of T-orbitally completeness for metric spaces was introduced, in 1971, by Ćirić in the same paper [4] (see also [5]).

Definition 1.7. Let (X, d) be a metric space and T be a selfmapping on X. X is said to be T-orbitally complete, if for any $x \in X$, every Cauchy sequence of the orbit $O_T(x) := \{x, Tx, T^2x, \dots\}$ is convergent in X.

In 1966, the concept of asymptotically regular mappings in metric spaces was introduced by Browder and Petryshyn (see [2]).

Definition 1.8 ([2])**.** Let T be a selfmapping of a b-metric space (X, d). Then, T is said to be asymptotically regular at a point x in X, if $\lim_{n\to\infty} d(T^nx, T^nTx) = 0$, where T^nx denotes the n-th iterate of T at x.

T is said to be asymptotically regular on X if it is asymptotically regular at any point x in X.

We recall the following notions from [14] due to Rus.

Definition 1.9. Let (X, d) be a metric space and $T : X \to X$ a selfmapping. We say that T is weakly Picard operator (WPO) if the sequence $(T^n(x))_{n\in\mathbb{N}}$ converges, for all $x \in X$, and the limit (which may depend on x) is a fixed point of T.

If the operator T is WPO and $\text{Fix}(T) = \{x\}$ (for some $x \in X$), then T is said to be a Picard operator (PO).

After this introduction and preliminaries, we expose in the second section our results.

In Lemma 2.1, we prove that (in any metric space) any Picard operator is orbitally continuous.

In the first main result (see Theorem 2.2, we give necessary and sufficient conditions for a Kanann contractive selfmap of a boundedly compact metric space to be a Picard operator.

The second main result (see Theorem 2.3), deals with the same problem but in complete metric spaces.

In the third section, we provide examples to support our main results.

2. The Results

Before giving our first main result, we prove the following lemma, which is a general result concerning Picard operators in metric spaces.

Lemma 2.1. *Let* (M, d) *be a metric space and let* $T : M \to M$ *be a selfmapping. If* T *is a Picard operator on* M, *then* T *is orbitally continuous on* M.

Proof. Since T is supposed to be a Picard operator, there exists a unique fixed point $z \in M$ such that $z = \lim_{n\to+\infty} T^n x$ for all $x \in M$. Let $u \in M$ and let $(T^{n_k}x)_k$ be a subsequence of $(T^n x)_n$ such that $u = \lim_{k\to+\infty} T^{n_k}x$. Then, necessarily, we have $u = z$. Moreover, the sequence $(T(T^{n_k}x))_k = (T^{n_k+1}x)_k$ is also a subsequence of $(T^n x)_n$, therefore we have

$$\lim_{k\to+\infty} T(T^{n_k}x) = z = Tz = Tu.$$

Thus we have proved that T is orbitally continuous on M. This ends the proof. $\square$

Now, we give our first main result, in which we present two weakest (and equivalent) conditions (required) for a selfmap in order to be a Picard operator.

Precisely, we have the following theorem:

Theorem 2.2. *Let* (X, d) *be a boundedly compact metric space and* $T : X \to X$ *be a mapping such that*

$$d(Tx, Ty) < \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}, \text{ for all } x, y \in X \text{ with } x \neq y.$$

Then the following assertions are equivalent:

(A) T *is a Picard operator*;

(B) *For all* $x, u \in X$ *satisfying* $\lim_{k\to\infty} T^{n_k}x = u$, *then we have* $\lim_{k\to\infty} T(T^{n_k}x) = u$;

(C) T *is asymptotically regular on* X.

Proof. (A) $\Rightarrow$ (B) Since T is a Picard operator, there exists a unique fixed point $z \in X$ such that $z = \lim_{n\to+\infty} T^n x$ for all $x \in X$. Let $u \in X$ and let $(T^{n_k}x)_k$ be a subsequence of $(T^n x)_n$ such that $u = \lim_{k\to+\infty} T^{n_k}x$. Then we have $u = z$. Moreover, the sequence $(T(T^{n_k}x))_k = (T^{n_k+1}x)_k$ is also a subsequence of $(T^n x)_n$, therefore we have

$$\lim_{k\to+\infty} T(T^{n_k}x) = z = u.$$

Thus we have proved that T satisfies the condition (B).

(B) $\Rightarrow$ (C) Let $x_0 \in X$ be arbitrary but fixed and consider the iterated sequence (x_n), where $x_n = T^n x_0$ for each $n \in \mathbb{N}$. We set $\tau_n = d(x_n, x_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. Then we have

$$\begin{aligned}\tau_n &= d(T^n x_0, T^{n+1} x_0) \\ &= d(T(T^{n-1} x_0), T(T^n x_0)) \\ &\leq \frac{1}{2}\{d(T^{n-1} x_0, T^n x_0) + d(T^n x_0, T^{n+1} x_0)\} \\ &= \frac{1}{2}(\tau_{n-1} + \tau_n) \\ &\Rightarrow \tau_n \leq \tau_{n-1}.\end{aligned}$$

This shows that (τ_n) is a non-increasing sequence of nonnegative real numbers, so it must be a convergent sequence. Let us denote $\tau := \lim_{n\to+\infty} \tau_n$. We must show that $\tau = 0$.

From the inequalities above, we know that for all $n, m \in \mathbb{N}$, we have

$$\begin{aligned}d(x_n, x_m) &= d(T^n x_0, T^m x_0) \\ &= d(T(T^{n-1} x_0), T(T^{m-1} x_0)) \\ &\leq \frac{1}{2}\{d(T^{n-1} x_0, T^n x_0) + d(T^{m-1} x_0, T^m x_0)\} \\ &= \frac{1}{2}(\tau_{n-1} + \tau_{m-1}) \\ &\Rightarrow d(x_n, x_m) \leq \tau_0.\end{aligned}$$

Therefore, (x_n) is a bounded sequence in X. By the boundedly compactness property of X, the sequence (x_n) has a convergent subsequence, say (x_{n_k}) which converges to some $z \in X$. Since T satisfies the condition (B), we infer that $\lim_{k\to+\infty} T(T^{n_k} x) = z$.

Therefore, by the continuity of the distance function, we obtain

$$\lim_{k\to\infty} d(x_{n_k}, x_{n_k+1}) = d(\lim_{k\to\infty} x_{n_k}, \lim_{k\to\infty} x_{n_k+1}) = d(z, z) = 0.$$

Hence, the subsequence $(\tau_{n_k})_k$ converges to zero. Since the whole sequence (τ_n) converges to τ, we must have $\tau = 0$. So we have proved that T is asymptotically regular on X.

(C) $\Rightarrow$ (A) We suppose that T is asymptotically regular on X. Let $x_0 \in X$ be arbitrary but fixed and consider the iterated sequence (x_n) where $x_n = T^n x_0$ for each $n \in \mathbb{N}$. As above, we denote $\tau_n = d(x_n, x_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$.

By assumption we know that $\lim_{n\to\infty} \tau_n = 0$. Then for all integers $n, m \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_m) &= d(T^n x_0, T^m x_0) \\ &= d(T(T^{n-1}x_0), T(T^{m-1}x_0)) \\ &\le \frac{1}{2}\{d(T^{n-1}x_0, T^n x_0) + d(T^{m-1}x_0, T^m x_0)\} \\ &= \frac{1}{2}(\tau_{n-1} + \tau_{m-1}) \\ &\Rightarrow \lim_{n,m\to\infty} d(x_n, x_m) = 0. \end{aligned}$$

This says that (x_n) is a Cauchy sequence.

Since every boundedly compact space is complete, then the sequence (x_n) converges to a point (say) $z = z_{x_0}$ which may depend on x_0. Also, we have

$$d(z, Tz) \leq d(z, T^{n+1}x_0) + d(T^{n+1}x_0, Tz)$$

$$< d(z, T^{n+1}x_0) + \frac{1}{2}\{d(T^n x_0, T^{n+1}x_0) + d(z, Tz)\}$$

$$\Rightarrow \frac{1}{2} d(z, Tz) < d(z, T^{n+1}x_0) + \frac{1}{2} d(T^n x_0, T^{n+1}x_0) \to 0 \text{ as } n \to \infty.$$

This implies that $z = Tz$, i.e., z is a fixed point of T.

Next, we prove the uniqueness of z. We argue by contradiction, let z^* be another (different) fixed point of T, then

$$d(z, z^*) = d(Tz, Tz^*)$$

$$< \frac{1}{2}\{d(z, Tz) + d(z^*, Tz^*)\}$$

$$\Rightarrow d(z, z^*) < 0,$$

which leads to a contradiction. Hence, our assumption was wrong. Therefore, z must be the unique fixed point T.

Let $y_0 \in X$ be arbitrary but fixed and consider the iterated sequence (y_n), where $y_n = T^n y_0$ for each $n \in \mathbb{N}$. We denote $\sigma_n = d(y_n, y_{n+1})$ for each $n \in \mathbb{N} \cup \{0\}$. By assumption we know that $\lim_{n\to\infty} \tau_n = 0 = \lim_{n\to\infty} \sigma_n$. Then for all integer $n \in \mathbb{N}$, we have

$$d(x_n, y_n) = d(T^n x_0, T^n y_0)$$

$$= d(T(T^{n-1}x_0), T(T^{n-1}y_0))$$

$$\leq \frac{1}{2}\{d(T^{n-1}x_0, T^n x_0) + d(T^{n-1}y_0, T^n y_0)\}$$

$$= \frac{1}{2}(\tau_{n-1} + \sigma_{n-1})$$

$$\Rightarrow d(z_{x_0}, z_{y_0}) = \lim_{n\to\infty} d(x_n, y_m) = 0.$$

This says that the fixed point does not depend on the initial value x_0 for any arbitrary point, so for every $x \in X$, the iterated sequence $(T^n x)$ converges to the unique fixed point of T, i.e., T is a Picard operator. This ends the proof. □

We observe that if a selfmapping T satisfies the condition (B) (in any metric space), then it is orbitally continuous.

From the lines of proof of the Theorem 2.2, one can establish the following general result for complete metric spaces.

Theorem 2.3. *Let* (X, d) *be a complete metric space and* $T : X \to X$ *be a mapping such that*

$$d(Tx, Ty) < \frac{1}{2}\{d(x, Tx) + d(y, Ty)\}, \text{ for all } x, y \in X \text{ with } x \neq y.$$

Then the following assertions are equivalent:

(A) T *is a Picard operator*;

(C) T *is asymptotically regular on* X.

Theorem 2.3 is our second main result. It will infer that the condition (C) is the minimal condition to be added to the condition (K-S) allowing a Kannan type contractive T, on a complete metric space X, to be a Picard operator.

3. Examples

To support our first main result, we provide the following example.

Example 3.1. Let $X = [1, 2] \cup (-\infty, 0]$ and define $T : X \to X$ by

$$T(x) = \begin{cases} -1 & x = 2; \\ 0 & x \neq 2. \end{cases}$$

Now, for $x \neq 2,$ we have

$$d(Tx,\ T2) = |Tx - T2| = |0 + 1| = 1,$$

whereas

$$\frac{1}{2}\{d(x,\ Tx) + d(2,\ T2)\} = \frac{1}{2}\{|x| + |2 + 1|\} \geq \frac{3}{2} > 1.$$

So,

$$d(Tx,\ T2) < \frac{1}{2}\{d(x,\ Tx) + d(2,\ T2)\}.$$

Again, for $x,\ y \in X$ with $x,\ y \neq 2$ and $x \neq y,$ we have $d(Tx,\ Ty) = 0$ but

$$\frac{1}{2}\{d(x,\ Tx) + d(y,\ Ty)\} = \frac{1}{2}\{|x| + |y|\} > 0.$$

Therefore, we have $d(Tx,\ Ty) < \frac{1}{2}\{d(x,\ Tx) + d(y,\ Ty)\}.$

Thus, T is a Kannan contractive selfmap of the metric space X which is boundedly compact, without being compact. The selfmap T is asymptotically regular on X. Indeed, for each $x_0 \in X,$ we have $T^n(x_0) = 0$ for every integer $n \geq 2$ hence, the condition (C) of Theorem 2.2 is satisfied, therefore T is a Picard operator with $\text{Fix}(T) = \{0\}.$

To support our second main result, we provide the following example:

Example 3.2. Let us choose $X = \mathbb{N}$ and we define $d : X \times X \to \mathbb{R}$ by

$$d(x,\ y) = \begin{cases} \frac{1}{2} + \left|\frac{1}{2x} - \frac{1}{2y}\right|, & \text{if} \quad x \neq y, \\ 0, & \text{if} \quad x = y. \end{cases}$$

Clearly d is a metric on X. Note that every Cauchy sequence in X is eventually constant and hence $(X,\ d)$ is a complete metric space.

We define a function $T : X \to X$ by setting

$$Tx = 2x,$$

for all $x \in X$. It is easy to verify that T is continuous and a fixed point free map.

Now, it remains to show that T satisfies the Kannan type contractive condition. In order to do this, we choose $x, y \in X$ with $x < y$, then

$$d(Tx, Ty) = \frac{1}{2} + \left|\frac{1}{4x} - \frac{1}{4y}\right|$$

$$= \frac{1}{2} + \frac{1}{4x} - \frac{1}{4y} < \frac{1}{2} + \frac{1}{4x},$$

whereas,

$$\frac{1}{2}\{d(x, Tx) + d(y, Ty)\} = \frac{1}{2}\left\{\frac{1}{2} + \left|\frac{1}{2x} - \frac{1}{4x}\right| + \frac{1}{2} + \left|\frac{1}{2y} - \frac{1}{4y}\right|\right\}$$

$$= \frac{1}{2} + \frac{1}{4x} + \frac{1}{4y} > \frac{1}{2} + \frac{1}{4x}.$$

Therefore,

$$d(Tx, Ty) < \frac{1}{2}\{d(x, Tx) + d(y, Ty)\},$$

for all x, y in X with $x < y$.

Similarly, one can prove it for the case $x, y \in X$ with $x > y$.

Now from the above analysis, we can say that

$$d(Tx, Ty) < \frac{1}{2}\{d(x, Tx) + d(y, Ty)\},$$

for all x, y in X with $x \neq y$. Thus, T is a Kannan type contractive map but it is fixed point free map. Because, here the condition (C) is not satisfied. Indeed, for $x_0 := 1$, for every positive integer n, we have

$$d(x_n, x_{n+1}) = \frac{1}{2} + \frac{1}{2^{n+2}}.$$

which never tends to zero, when n tend to infinity.

So we may conclude that the condition (C) is the minimal condition to be added to the contractive condition (K-S) allowing a Kannan operator to be a Picard operator in a complete metric space.

We point out that Suzuki investigated in [16] the weakest contractive conditions for Edelstein's mappings to have a fixed point in complete metric spaces.

In [12], the authors introduced the Górnicki-Proinov type contractions and studied the uniqueness and existence of their fixed points in the framework of complete metric spaces. Górnicki-Proinov type contractions extends Kannan type contractive maps and many others. Their study used asymptotic regularity. It is interesting to know whether our results could be extended to the Górnicki-Proinov type contractions.

Acknowledgement

The author would like to express his deep thanks to the anonymous referee for his/her helpful comments and suggestions on the initial version of the manuscript which lead to the improvement of this paper.

References

[1] S. Banach, Sur les operations dans les ensembles abstraits et leur applications aux equations integrales, Fundamenta Mathematicae 3(1) (1920), 133-181.

DOI: https://doi.org/10.4064/FM-3-1-133-181

[2] F. E. Browder and W. V. Petryshyn, The solution by iteration of nonlinear functional equation in Banach spaces, Bulletin of the American Mathematical Society 72(3) (1966), 571-575.

[3] H. Y. Chen and C. C. Yeh, A note on fixed point in compact metric spaces, Indian Journal of Pure and Applied Mathematics 11(3) (1980), 297-298.

[4] Lj. B. Ćirić, On contraction type mappings, Mathematica Balkanica 1 (1971), 52-57.

[5] Lj. B. Ćirić, On some maps with nonunique fixed points, Publications de l'Institut Mathématique (Beograd) 13(31) (1974), 52-58.

[6] R. E. Edwards, Functional Analysis: Theory and Applications, Holt, Rinehart and Winston, 1965.

[7] B. Fisher, A fixed point theorem for compact metric spaces, Publications de l'Institut Mathématique 25 (1978), 193-194.

[8] H. Garai, T. Senapati and L. K. Dey, A study on Kannan type contractive mappings, arXiv.org [math FA] 1707.06383v1, 2017.

[9] H. Garai, L. K. Dey and T. Senapati, On Kannan-type contractive mappings, Numerical Functional Analysis and Optimization 39(13) (2018), 1466-1476.

DOI: https://doi.org/10.1080/01630563.2018.1485157

[10] J. Górnicki, Fixed point theorems for Kannan type mappings, Journal of Fixed Point Theory and Applications 19(3) (2017), 2145-2152.

DOI: https://doi.org/10.1007/s11784-017-0402-8

[11] R. Kannan, Some results on fixed points, Bulletin of the Calcutta Mathematical Society 60 (1968), 71-76.

[12] E. Karapinar, M. De La Sen and A. Fulga, A note on the Górnicki-Proinov type contraction, Journal of Function Spaces (2021); Article ID 6686644, 8 pages.

DOI: https://doi.org/10.1155/2021/6686644

[13] M. S. Khan, On fixed point theorems, Mathematica Japonicae 23 (1978), 201-204.

[14] I. A. Rus, Weakly Picard operators and application, Seminar on Fixed Point Theory Cluj-Napoca 2 (2001), 41-58.

[15] V. Subrahmanyam, Completeness and fixed-points, Monatshefte für Mathematik 80(4) (1975), 325-330.

DOI: https://doi.org/10.1007/BF01472580

[16] T. Suzuki, The weakest contractive conditions for Edelstein's mappings to have a fixed point in complete metric spaces, Journal of Fixed Point Theory and Applications 19(4) (2017), 2361-2368.

DOI: https://doi.org/10.1007/s11784-017-0430-4

■