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GENERALIZATION OF SOME SUMS ON PRIMES AND ASYMPTOTIC EXPANSIONS

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Abstract

In this article, we study some sums on primes and obtain asymptotic expansions for sums of primes, sums of least prime factors, and sum of the geometric mean of the prime factors of an integer.

1. Generalization of Some Sums on Primes

We have

$$\operatorname{Li}(x)=\int_2^x \frac{1}{\log t}\,dt.$$

The following strong form of the prime number theorem is well-known [5].

$$\pi(x)=\operatorname{Li}(x)+O\left(\frac{x}{e^{a\sqrt{\log x}}}\right), \tag{1}$$

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where a is a positive constant. Note that

$$\frac{1}{e^{a\sqrt{\log x}}} = o\left(\frac{1}{\log^m x}\right), \tag{2}$$

for all positive number m.

We need the following well-known fundamental lemma.

Lemma 1.1. *If $f(x)$ is a function with derivative $f'(x)$ continuous on the interval $[2, \infty)$, then the following formula holds*:

$$\sum_{p \leq x} f(p) = \int_2^x \frac{f(t)}{\log t}\, dt + f(x)(\pi(x) - \mathrm{Li}(x)) - \int_2^x (\pi(t) - \mathrm{Li}(t)) f'(t)\, dt.$$

Proof (See (2.26) of [6])**.** The lemma is proved.

Now, we prove some formulas for sums on primes. Some formulas generalize the following well-known sums on primes. $\sum_{2 \leq p \leq x} \frac{1}{p}$, $\sum_{2 \leq p \leq x} \frac{\log p}{p}$ and $\vartheta(x) = \sum_{2 \leq p \leq x} \log p$.

Theorem 1.2. *If k is a nonnegative integer, then the following formula holds*:

$$\sum_{3 \leq p \leq x} \frac{(\log \log p)^k}{p} = \frac{(\log \log x)^{k+1}}{k+1} + C_k + O\left(\frac{\log x}{e^{a\sqrt{\log x}}}\right),$$

where C_k is a constant depending of k.

Proof. We have (Lemma 1.1 and (1))

$$\begin{aligned} \sum_{3 \leq p \leq x} \frac{(\log \log p)^k}{p} &= \int_3^x \frac{(\log \log t)^k}{t \log t}\, dt + O\left(\frac{(\log \log x)^k}{e^{a\sqrt{\log x}}}\right) \\ &\quad + \int_3^x \frac{O(1)}{e^{a\sqrt{\log t}}} \frac{(\log \log t)^k}{t}\, dt \\ &= \frac{(\log \log x)^{k+1}}{k+1} + C_k + O\left(\frac{\log x}{e^{a\sqrt{\log x}}}\right). \end{aligned}$$

Note that the integral in the interval $[3, \infty)$ converges since it is absolutely convergent (see (2)). On the other hand, we have

$$\left| \int_x^{\infty} \frac{O(1)}{e^{a\sqrt{\log t}}} \frac{(\log \log t)^k}{t} dt \right| \leq M \int_x^{\infty} \frac{1}{e^{a\sqrt{\log t}}} \frac{(\log \log t)^k}{t} dt,$$

where M is a positive constant and (L'Hospital's rule)

$$\lim_{x \to \infty} \frac{\displaystyle\int_x^{\infty} \frac{1}{e^{a\sqrt{\log t}}} \frac{(\log \log t)^k}{t} dt}{\dfrac{\log x}{e^{a\sqrt{\log x}}}} = 0.$$

The theorem is proved.

Theorem 1.3. *If* k *is a positive integer*, *then the following formula holds*:

$$\sum_{2 \leq p \leq x} \frac{(\log p)^k}{p} = \frac{(\log x)^k}{k} + E_k + O\left(\frac{(\log x)^{k+1}}{e^{a\sqrt{\log x}}} \right),$$

where E_k *is a constant depending of* k.

Proof. We have (Lemma 1.1 and (1))

$$\sum_{2 \leq p \leq x} \frac{(\log p)^k}{p} = \int_2^x \frac{(\log t)^{k-1}}{t} dt + O\left(\frac{(\log x)^k}{e^{a\sqrt{\log x}}} \right) + \int_2^x \frac{O(1)}{e^{a\sqrt{\log t}}} \frac{(\log t)^k}{t} dt$$

$$= \frac{(\log x)^k}{k} + E_k + O\left(\frac{(\log x)^{k+1}}{e^{a\sqrt{\log x}}} \right).$$

Note that the integral in the interval $[2, \infty)$ converges since it is absolutely convergent (see (2)). On the other hand, we have

$$\left| \int_x^{\infty} \frac{O(1)}{e^{a\sqrt{\log t}}} \frac{(\log t)^k}{t} dt \right| \leq M \int_x^{\infty} \frac{1}{e^{a\sqrt{\log t}}} \frac{(\log t)^k}{t} dt,$$

where M is a positive constant and (L'Hospital's rule)

$$\lim_{x\to\infty} \frac{\displaystyle\int_x^\infty \frac{1}{e^{a\sqrt{\log t}}} \frac{(\log t)^k}{t}\, dt}{\dfrac{(\log x)^{k+1}}{e^{a\sqrt{\log x}}}} = 0.$$

The theorem is proved.

Theorem 1.4. *If k is a positive integer, then the following asymptotic formula holds*:

$$\sum_{2\le p\le x} (\log p)^k = \sum_{i=1}^{k} a_i x(\log x)^{k-i} + O\left(\frac{x(\log x)^k}{e^{a\sqrt{\log x}}}\right), \tag{3}$$

where the a_i are integer coefficients (see below in the proof).

Proof. Lemma 1.1 and (1) give

$$\sum_{2\le p\le x} (\log p)^k = \int_2^x (\log t)^{k-1} dt + O\left(\frac{x(\log x)^k}{e^{a\sqrt{\log x}}}\right) + \int_2^x \frac{O(1)(\log t)^{k-1}}{e^{a\sqrt{\log t}}}\, dt$$

$$= \sum_{i=1}^{k} a_i x(\log x)^{k-i} + O\left(\frac{x(\log x)^k}{e^{a\sqrt{\log x}}}\right).$$

Now, we have

$$\left|\int_2^x \frac{O(1)(\log t)^{k-1}}{e^{a\sqrt{\log t}}}\, dt\right| \le M\int_2^x \frac{(\log t)^{k-1}}{e^{a\sqrt{\log t}}}\, dt,$$

and

$$\lim_{x\to\infty} \frac{\displaystyle\int_2^x \frac{(\log t)^{k-1}}{e^{a\sqrt{\log t}}}\, dt}{\dfrac{x(\log x)^k}{e^{a\sqrt{\log x}}}} = 0.$$

On the other hand, we have (integration by parts)

$$\int (\log x)^{k} dx = x(\log x)^{k} + \sum_{i=1}^{k-1} (-1)^{i} k(k-1)\cdots(k-(i-1))x(\log x)^{k-i}$$

$$+ (-1)^{k} k!\, x + c.$$

The theorem is proved.

Remark 1.5 (On the Cipolla's expansion)**.** If we put into (3) $k = 1$ then we obtain the well-known result

$$\vartheta(x) = \sum_{2 \le p \le x} \log p = x + O\left(\frac{x \log x}{e^{a\sqrt{\log x}}}\right).$$

By (2), we have the weaker result

$$\vartheta(x) = \sum_{2 \le p \le x} \log p = x + o\left(\frac{x}{\log^{m} x}\right).$$

If we put $x = p_n$ and use the prime number theorem $p_n \sim n \log n,$ then we obtain

$$\sum_{i=1}^{n} \log p_i = p_n + o\left(\frac{n}{\log^{m} n}\right), \tag{4}$$

for all nonnegative integer $m.$ The Cipolla's expansion for p_n is of the form $p_n = f_m(n) + o\left(\frac{n}{\log^{m} n}\right),$ where $f_m(n)$ is a function of n depending of the nonnegative integer $m.$ Therefore, substituting into (4) we obtain

$$\sum_{i=1}^{n} \log p_i = f_m(n) + o\left(\frac{n}{\log^{m} n}\right).$$

That is, $\sum_{i=1}^{n} \log p_i$ has the same Cipolla's asymptotic expansion as $p_n.$ This fact was proved in [2] by use of the Cipolla's theorems [1]. This is another simple proof. For example, if $m = 0$ then the first Cipolla's formula is $p_n = n \log n + n \log\log n - n + o(n)$ and consequently also we have $\sum_{i=1}^{n} \log p_i = n \log n + n \log\log n - n + o(n)$. In this case, $f_0(n) = n \log n + n \log\log n - n$.

Theorem 1.6. *If* $\alpha > 0$ *is a real number, then the following asymptotic formula holds*:

$$\sum_{2 \leq p \leq x} p^{\alpha} = \int_2^x \frac{t^{\alpha}}{\log t}\, dt + O\left(\frac{x^{\alpha+1}}{e^{a\sqrt{\log x}}}\right). \tag{5}$$

Proof. Lemma (1.1) and (1) gives

$$\sum_{2 \leq p \leq x} p^{\alpha} = \int_2^x \frac{t^{\alpha}}{\log t}\, dt + O\left(\frac{x^{\alpha+1}}{e^{a\sqrt{\log x}}}\right) + \int_2^x O(1) \frac{t^{\alpha}}{e^{a\sqrt{\log t}}}\, dt$$

$$= \int_2^x \frac{t^{\alpha}}{\log t}\, dt + O\left(\frac{x^{\alpha+1}}{e^{a\sqrt{\log x}}}\right).$$

Note that

$$\left| \int_2^x O(1) \frac{t^{\alpha}}{e^{a\sqrt{\log t}}}\, dt \right| \leq M \int_2^x \frac{t^{\alpha}}{e^{a\sqrt{\log t}}}\, dt,$$

and

$$\lim_{x \to \infty} \frac{\int_2^x \frac{t^{\alpha}}{e^{a\sqrt{\log t}}}\, dt}{\frac{x^{\alpha+1}}{(\alpha+1) e^{a\sqrt{\log x}}}} = 1.$$

The theorem is proved.

Remark 1.7. By successive integration by parts of the function $\frac{t^{\alpha}}{\log t}$ we obtain the following asymptotic expansion for all positive integer m (see (5) and (2)).

$$\sum_{2\le p\le x} p^{\alpha} = \sum_{k=1}^{m} \frac{(k-1)!\,x^{\alpha+1}}{(\alpha+1)^{k}(\log x)^{k}} + o\left(\frac{x^{\alpha+1}}{(\log x)^{m}}\right). \tag{6}$$

This asymptotic expansion was obtained by use of a different method in [4].

Remark 1.8 (Sum of least prime factors and sum of the geometric mean of the prime factors of an integer)**.** Let $a(n)$ be the least prime factor of a positive integer n. Note that if $n = p$ is a prime then $a(n) = a(p) = p$. In [3] was obtained a asymptotic formula for the sum of the powers of $a(n)$. Now, we shall prove a better asymptotic formula. If (as usual) $\Omega(n)$ is the total number of prime factors of n where multiplicities are counted then the geometric mean of the prime factors of n is $n^{\frac{1}{\Omega(n)}}$. Let $\alpha > 0$ be a real number. We have

$$\sum_{n\le x} a(n)^{\alpha} \ge \sum_{p\le x} a(p)^{\alpha} = \sum_{p\le x} p^{\alpha},$$

and

$$\sum_{n\le x} a(n)^{\alpha} \le \sum_{n\le x}\left(n^{\frac{1}{\Omega(n)}}\right)^{\alpha} = \sum_{p\le x} p^{\alpha} + \sum_{n\le x,\ n\ne p}\left(n^{\frac{1}{\Omega(n)}}\right)^{\alpha}$$

$$\le \sum_{p\le x} p^{\alpha} + \sum_{n\le x,} n^{\frac{\alpha}{2}} \le \sum_{p\le x} p^{\alpha} + x^{1+\frac{\alpha}{2}}.$$

Consequently

$$\sum_{n\leq x} a(n)^{\alpha} = \sum_{p\leq x} p^{\alpha} + O\left(x^{1+\frac{\alpha}{2}}\right),$$

and then by Equation (6) we find that

$$\sum_{n\leq x} a(n)^{\alpha} = \sum_{k=1}^{m} \frac{(k-1)!\, x^{\alpha+1}}{(\alpha+1)^{k}(\log x)^{k}} + o\left(\frac{x^{\alpha+1}}{(\log x)^{m}}\right).$$

In the same way we obtain

$$\sum_{n\leq x} n^{\frac{1}{\Omega(n)}} = \sum_{k=1}^{m} \frac{(k-1)!\, x^{2}}{(2)^{k}(\log x)^{k}} + o\left(\frac{x^{2}}{(\log x)^{m}}\right).$$

Theorem 1.9. *If k is a positive integer, then the following asymptotic formula hold*:

$$\sum_{2\leq p\leq x} p^{k} \log p = \frac{x^{k+1}}{k+1} + O\left(\frac{x^{k+1}\log x}{e^{a\sqrt{\log x}}}\right).$$

By Lemma 1.1 *and* (1), *we obtain*

$$\sum_{2\leq p\leq x} p^{k} \log p = \int_{2}^{x} t^{k} dt + O\left(\frac{x^{k+1}\log x}{e^{a\sqrt{\log x}}}\right) + \int_{2}^{x} O(1)\frac{t^{k}\log t}{e^{a\sqrt{\log t}}}\, dt$$

$$= \frac{x^{k+1}}{k+1} + O\left(\frac{x^{k+1}\log x}{e^{a\sqrt{\log x}}}\right).$$

Note that

$$\left|\int_{2}^{x} O(1)\frac{t^{k}\log t}{e^{a\sqrt{\log t}}}\, dt\right| \leq M\int_{2}^{x} \frac{t^{k}\log t}{e^{a\sqrt{\log t}}}\, dt,$$

and

$$\lim_{x\to\infty}\frac{\displaystyle\int_2^x \frac{t^k \log t}{e^{a\sqrt{\log t}}}\,dt}{\dfrac{x^{k+1}\log x}{(k+1)e^{a\sqrt{\log x}}}} = 1.$$

The theorem is proved.

Theorem 1.10. *If k is a positive integer, then the following asymptotic formula holds*:

$$\sum_{2\le p\le x}\frac{1}{(\log p)^k} = \int_2^x \frac{1}{(\log t)^{k+1}} + O\left(\frac{x}{(\log x)^k e^{a\sqrt{\log x}}}\right). \tag{7}$$

Proof. Lemma 1.1 and (1) give

$$\begin{aligned}\sum_{2\le p\le x}\frac{1}{(\log p)^k} &= \int_2^x \frac{1}{(\log t)^{k+1}} + O\left(\frac{x}{(\log x)^k e^{a\sqrt{\log x}}}\right)\\ &\quad + \int_2^x O(1)\frac{1}{e^{a\sqrt{\log t}}(\log t)^{k+1}}\,dt\\ &= \int_2^x \frac{1}{(\log t)^{k+1}} + O\left(\frac{0}{(\log x)^k e^{a\sqrt{\log x}}}\right).\end{aligned}$$

Note that

$$\left|\int_2^x O(1)\frac{1}{e^{a\sqrt{\log t}}(\log t)^{k+1}}\,dt\right| \le M\int_2^x \frac{1}{e^{a\sqrt{\log t}}(\log t)^{k+1}}\,dt,$$

and

$$\lim_{x\to\infty}\frac{\displaystyle\int_2^x \frac{1}{e^{a\sqrt{\log t}}(\log t)^{k+1}}\,dt}{\dfrac{x}{(\log x)^{k+1}e^{a\sqrt{\log x}}}} = 1.$$

The theorem is proved.

Remark 1.11. By successive integration by parts of the function $\frac{1}{(\log t)^{k+1}}$, we obtain the asymptotic expansion (see (7) and (2))

$$\sum_{2\leq p\leq x}\frac{1}{(\log p)^{k}}=\frac{1}{(\log x)^{k+1}}+\sum_{i=2}^{m}(k+1)(k+2)\cdots(k+i-1)\frac{x}{(\log x)^{k+i}}$$

$$+\,o\left(\frac{x}{(\log x)^{k+m}}\right).$$

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