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AN ASYMPTOTIC EXPANSION FOR THE GEOMETRIC MEAN OF PRIMES

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Abstract

We obtain asymptotic expansions for the geometric mean of prime numbers.

1. Main Results

Let p_n be the n-th prime and $\pi(x)$ be the prime counting function. The following limit was proved in [2] by use of the prime number theorem $p_n \sim n \ln n$.

$$\lim_{n \to \infty} \frac{\sqrt[n]{p_1 p_2 \cdots p_n}}{p_n} = \frac{1}{e}. \tag{1}$$

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Almost an immediate consequence of limit (1) is the limit

$$\lim_{x\to\infty}\frac{\left(\prod_{p\leq x}p\right)^{\frac{1}{\pi(x)}}}{x}=\frac{1}{e}, \tag{2}$$

where p denotes a generic prime. Really limit (1) and limit (2) are equivalent, since clearly (2) implies (1) if we put $x = p_n$.

In this article, we obtain asymptotic expansions for (1) and (2). We have the following theorem.

Theorem 1.1. *Let m be an arbitrary but fixed positive integer.*

The following asymptotic expansion holds.

$$\frac{\left(\prod_{p\leq x}p\right)^{\frac{1}{\pi(x)}}}{x}=\frac{1}{e}+\sum_{i=1}^{m-1}\frac{b_i}{\ln^i x}+o\left(\frac{1}{\ln^{m-1} x}\right), \tag{3}$$

where a method to determinate the coefficients b_i *is given below* (*in the proof*).

The following asymptotic expansion holds.

$$\frac{\left(\prod_{i=1}^{n}p_i\right)^{\frac{1}{n}}}{p_n}=\frac{1}{e}+\sum_{h=1}^{m-1}\frac{f_h(\ln\ln n)}{\ln^h n}+o\left(\frac{1}{\ln^{m-1} n}\right), \tag{4}$$

where the $f_h(x)$ *are polynomials. A method to determinate the polynomials* $f_h(x)$ *is given below* (*in the proof*).

The following asymptotic expansion holds.

$$\left(\prod_{i=1}^{n}p_i\right)^{\frac{1}{n}}=\frac{n\ln n}{e}+n\sum_{h=0}^{m-2}\frac{q_h(\ln\ln n)}{\ln^h n}+o\left(\frac{n}{\ln^{m-2} n}\right), \tag{5}$$

where the $q_h(x)$ *are polynomials. A method to determinate the polynomials* $q_h(x)$ *is given below* (*in the proof*).

Proof. We have the following Taylor's polynomial

$$e^x = 1 + \sum_{k=1}^{n} \frac{x^k}{k!} + o(x^n) \quad (x \to 0). \tag{6}$$

We also have the following well-known formula [4], where a is a positive constant.

$$\vartheta(x) = \sum_{p \le x} \ln p = x + O\left(\frac{x}{e^{a\sqrt{\ln x}}}\right) = x + o\left(\frac{x}{\ln^{m+1} x}\right), \tag{7}$$

and the following Panaitopol's asymptotic expansion [3]

$$\frac{1}{\pi(x)} = \frac{\ln x}{x} - \frac{1}{x} + \sum_{k=1}^{m} \frac{a_k}{x \ln^k x} + o\left(\frac{1}{x \ln^m x}\right), \tag{8}$$

where the coefficients a_k can be obtained recursively (see [3]).

We have (see (7) and (8))

$$\ln\left(\frac{\left(\prod_{p \le x} p\right)^{\frac{1}{\pi(x)}}}{x}\right) = \frac{1}{\pi(x)} \vartheta(x) - \ln x$$

$$= \left(\ln x - 1 + \sum_{i=1}^{m} \frac{a_i}{\ln^i x} + o\left(\frac{1}{\ln^m x}\right)\right)\left(1 + o\left(\frac{1}{\ln^m x}\right)\right) - \ln x$$

$$= -1 + \sum_{i=1}^{m-1} \frac{a_i}{\ln^i x} + o\left(\frac{1}{\ln^{m-1} x}\right). \tag{9}$$

Equations (9) and (6) give

$$\frac{\left(\prod_{p\leq x} p\right)^{\frac{1}{\pi(x)}}}{x} = \frac{1}{e}\exp\left(\sum_{i=1}^{m-1}\frac{a_i}{\ln^i x}\right)\left(1+o\left(\frac{1}{\ln^{m-1} x}\right)\right)$$

$$= \frac{1}{e}\left(1+\sum_{i=1}^{m-1}\frac{1}{i!}\left(\sum_{i=1}^{m-1}\frac{a_i}{\ln^i x}\right)+o\left(\frac{1}{\ln^{m-1} x}\right)\right)\left(1+o\left(\frac{1}{\ln^{m-1} x}\right)\right)$$

$$= \frac{1}{e}\left(1+\sum_{i=1}^{m-1}\frac{1}{i!}\left(\sum_{i=1}^{m-1}\frac{a_i}{\ln^i x}\right)^i\right)+o\left(\frac{1}{\ln^{m-1} x}\right)$$

$$= \frac{1}{e}+\sum_{i=1}^{m-1}\frac{b_i}{\ln^i x}+o\left(\frac{1}{\ln^{m-1} x}\right). \tag{10}$$

Therefore Equation (3) is proved.

We have the following Taylor's formula

$$\frac{1}{1+x} = 1+\sum_{k=1}^{n}(-1)^k x^k + o(x^n) \quad (x \to 0). \tag{11}$$

Cipolla [1] proved the following asymptotic expansion for $\ln p_n$

$$\ln p_n = \ln n + \ln\ln n + \sum_{i=1}^{r}\frac{g_i(\ln\ln n)}{\ln^i n} + o\left(\frac{1}{\ln^r n}\right), \tag{12}$$

where the $g_i(x)$ are polynomials of degree i and rational coefficients. Cipolla [1] gave a recursive method to obtain the polynomials $g_i(x)$.

Next, we obtain an asymptotic expansion for $\frac{1}{\ln p_n}$.

If we put

$$x = \frac{\ln \ln n}{\ln n} + \sum_{i=1}^{r} \frac{g_i(\ln \ln n)}{\ln^{i+1} n} + o\left(\frac{1}{\ln^{r+1} n}\right), \tag{13}$$

and use Equations (11), (12) and (13) then we obtain

$$\begin{aligned}
\frac{1}{\ln p_n} &= \frac{1}{\ln n} \frac{1}{1+x} \\
&= \frac{1}{\ln n}\left(1 - x + x^2 - \cdots + (-1)^{r+1} x^{r+1} + (-1)^{r+2} x^{r+2}(1 + o(1))\right) \\
&= \frac{1}{\ln n}\left(1 - x + x^2 - \cdots + (-1)^{r+1} x^{r+1} + o\left(\frac{1}{\ln^{r+1} n}\right)\right) \\
&= \frac{1}{\ln n}\left(1 + \sum_{i=1}^{r+1} (-1)^i \left(\frac{\ln \ln n}{\ln n} + \sum_{j=1}^{r} \frac{g_j(\ln \ln n)}{\ln^{j+1} n}\right)^i\right) + o\left(\frac{1}{\ln^{r+2} n}\right) \\
&= \frac{1}{\ln n} + \sum_{i=1}^{r+1} \frac{h_i(\ln \ln n)}{\ln^{i+1} n} + o\left(\frac{1}{\ln^{r+2} n}\right),
\end{aligned} \tag{14}$$

where the $h_i(x)$ are polynomials of rational coefficients. This is the asymptotic expansion for $\frac{1}{\ln p_n}$. That is,

$$\frac{1}{\ln p_n} = \frac{1}{\ln n} + \sum_{i=1}^{r+1} \frac{h_i(\ln \ln n)}{\ln^{i+1} n} + o\left(\frac{1}{\ln^{r+2} n}\right). \tag{15}$$

Substituting $x = p_n$ into (3), we obtain

$$\frac{\left(\prod_{i=1}^{n} p_i\right)^{\frac{1}{n}}}{p_n} = \frac{1}{e} + \sum_{i=1}^{m-1} \frac{b_i}{\ln^i p_n} + o\left(\frac{1}{\ln^{m-1} n}\right) \tag{16}$$

since $\ln p_n \sim \ln n$. Substituting (15) (with $r = m - 3$) into (16) we find that

$$\frac{\left(\prod_{i=1}^{n} p_i\right)^{\frac{1}{n}}}{p_n} = \frac{1}{e} + \sum_{h=1}^{m-1} b_h \left(\frac{1}{\ln n} + \sum_{i=1}^{m-2} \frac{h_i(\ln \ln n)}{\ln^{i+1} n} + o\left(\frac{1}{\ln^{m-1} n}\right)\right)^h$$

$$+ o\left(\frac{1}{\ln^{m-1} n}\right) = \frac{1}{e} + \sum_{h=1}^{m-1} b_h \left(\frac{1}{\ln n} + \sum_{i=1}^{m-2} \frac{h_i(\ln \ln n)}{\ln^{i+1} n}\right)^h$$

$$+ o\left(\frac{1}{\ln^{m-1} n}\right) = \frac{1}{e} + \sum_{h=1}^{m-1} \frac{f_h(\ln \ln n)}{\ln^h n} + o\left(\frac{1}{\ln^{m-1} n}\right), \tag{17}$$

where the $f_h(x)$ are polynomials. This is the asymptotic expansion (4) that we desired.

Cipolla [1] proved the following asymptotic expansion for p_n.

$$p_n = n \ln n + n \ln \ln n - n + n \sum_{i=1}^{r} \frac{k_i(\ln \ln n)}{\ln^i n} + o\left(\frac{n}{\ln^r n}\right), \tag{18}$$

where the $k_i(x)$ are polynomials of degree i and rational coefficients. Cipolla [1] gave a recursive method to obtain the polynomials $k_i(x)$.

If $r = 0$, then the Cipolla's formula is

$$p_n = n \ln n + n \ln \ln n - n + o(n).$$

If $r = 1$, then the Cipolla's formula is

$$p_n = n \ln n + n \ln \ln n - n + \frac{n \ln \ln n - 2n}{\ln n} + o\left(\frac{n}{\ln n}\right) \tag{19}$$

etc.

Equation (4) gives

$$\left(\prod_{i=1}^{n} p_i\right)^{\frac{1}{n}} = \left(\frac{1}{e} + \sum_{h=1}^{m-1} \frac{f_h(\ln \ln n)}{\ln^h n} + o\left(\frac{1}{\ln^{m-1} n}\right)\right) p_n. \tag{20}$$

Substituting Equation (18) (with $r = m - 2$) into Equation (20) we obtain Equation (5). The theorem is proved.

Example 1.2. We choose $m = 3$. Equation (10) becomes

$$\frac{\left(\prod_{p \le x} p\right)^{\frac{1}{\pi(x)}}}{x} = \frac{1}{e}\left(1 + \sum_{i=1}^{2} \frac{1}{i!}\left(\sum_{i=1}^{2} \frac{a_i}{\ln^i x}\right)^i\right) + o\left(\frac{1}{\ln^2 x}\right)$$

$$= \frac{1}{e}\left(1 - \frac{1}{\ln x} - \frac{5}{2}\frac{1}{\ln^2 x}\right) + o\left(\frac{1}{\ln^2 x}\right) \tag{21}$$

since $a_1 = -1$ and $a_2 = -3$ (see [3]). Equation (21) is Equation (3) for $m = 3$. Equation (14) is for $r = 0$.

$$\frac{1}{\ln p_n} = \frac{1}{\log n}\left(1 + \sum_{i=1}^{1} (-1)^i \left(\frac{\ln \ln n}{\ln n}\right)^i\right) + o\left(\frac{1}{\ln^2 n}\right)$$

$$= \frac{1}{\ln n} - \frac{\ln \ln n}{\ln^2 n} + o\left(\frac{1}{\ln^2 n}\right). \tag{22}$$

Equation (17) becomes (see Equations (21) and (22))

$$\frac{\left(\prod_{i=1}^{n} p_i\right)^{\frac{1}{n}}}{p_n} = \frac{1}{e}\left(1 - \left(\frac{1}{\ln n} - \frac{\ln \ln n}{\ln^2 n}\right) - \frac{5}{2}\left(\frac{1}{\ln n} - \frac{\ln \ln n}{\ln^2 n}\right)^2\right)$$

$$+ o\left(\frac{1}{\ln^2 n}\right) = \frac{1}{e}\left(1 - \frac{1}{\ln n} + \frac{\ln \ln n - \frac{5}{2}}{\ln^2 n}\right) + o\left(\frac{1}{\ln^2 n}\right). \tag{23}$$

This is Equation (4) for $m = 3$.

Finally Equations (23) and (19) give

$$\left(\prod_{i=1}^{n} p_i\right)^{\frac{1}{n}} = \frac{1}{e}\left(1 - \frac{1}{\ln n} + \frac{\ln\ln n - \frac{5}{2}}{\ln^2 n} + o\left(\frac{1}{\ln^2 n}\right)\right)$$

$$\times\left(n\ln n + n\ln\ln n - n + \frac{n\ln\ln n - 2n}{\ln n} + o\left(\frac{n}{\ln n}\right)\right)$$

$$= \frac{1}{e}\left(n\ln n + n\ln\ln n - 2n + \frac{n\ln\ln n - \frac{7}{2}n}{\ln n}\right) + o\left(\frac{n}{\ln n}\right).$$

This is Equation (5) for $m = 3$.

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