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AVERAGES OF FRACTIONAL PARTS

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Abstract

Let us consider a strictly increasing sequence of positive integers a_n such that $A(x)$ is the distribution function of the sequence. That is, $A(x) = \sum_{a_n \leq x} 1$. We prove the asymptotic formula $\sum_{a_n \leq x} \left\{ \frac{x}{a_n} \right\} = CA(x) + o(A(x))$, where C is a constant depending of the sequence a_n. The distribution functions $A(x)$ considered are very general. The methods used are very elementary.

1. Introduction and Main Results

It is well-known the formula proved by Dirichlet in 1849.

$$\sum_{n \leq x} \left\{ \frac{x}{n} \right\} = (1 - \gamma)x + o(x), \tag{1}$$

where n denotes a positive integer and γ is Euler's constant.

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In 1898, de la Vallée Poussin [1] obtained some generalizations of the Dirichlet's formula doing some restrictions on the divisors n, equation (1) is also known as de la Vallée Poussin's formula. De la Vallée Poussin [1] consider numbers in arithmetic progression and prime numbers. Pillichshammer [9] obtained another generalization of the Dirichlet's formula also doing a restriction on the divisors n. Pillichshammer [9] consider k-th powers, where $k \geq 2$ is a positive integer. In this article, we prove that all these restrictions are particular cases of more general theorems. The proofs are simple, short and very elementary.

Let us consider a strictly increasing sequence a_n of positive integers. We shall denote a positive integer in this sequence a. Let $A(x)$ be the number of a not exceeding x, that is, $A(x)$ is the distribution function of the sequence a_n, $A(x) = \sum_{a \leq x} 1$. In this article we study the more general sum $\sum_{a \leq x} \left\{ \frac{x}{a} \right\}$. We shall prove that $\sum_{a \leq x} \left\{ \frac{x}{a} \right\} = CA(x) + o(A(x))$, where C is a constant depending of the sequence a_n. The distribution functions $A(x)$ considered are very general (see below).

We shall need the following well-known theorem (Abel summation).

Theorem 1.1. *Let* $c_n (n \geq 1)$ *be a sequence of real numbers. Let us consider the function*

$$A(x) = \sum_{n \leq x} c_n.$$

Suppose that $f(x)$ *has a continuous derivative* $f'(x)$ *on the interval* $[1, \infty]$, *then the following formula holds*:

$$\sum_{n \leq x} c_n f(n) = A(x) f(x) - \int_1^x A(t) f'(t) dt.$$

Proof. See ([2], Chapter XXII).

We also shall need the following definition.

Definition 1.2. Let us consider a positive function $f(x)$ such that $f'(x)$ is positive, strictly decreasing and $\lim_{x \to \infty} f(x) = \infty$. The function $f(x)$ is of slow increase if and only if the following limit holds:

$$\lim_{x \to \infty} \frac{xf'(x)}{f(x)} = 0.$$

Typical functions of slow increase are $\log x, \log \log x, \dfrac{\log x}{\log \log x}$, etc. The functions of slow increase are studied in [7]. We shall need the following properties of the functions of slow increase:

$$\lim_{x \to \infty} \frac{f(x)}{x^{\alpha}} = 0,$$

for all $\alpha > 0$ and

$$\lim_{x \to \infty} \frac{f(Cx)}{f(x)} = 1, \tag{2}$$

for all $C > 0$.

Note that

$$\sum_{a \le x} \left\{ \frac{x}{a} \right\} = x \sum_{a \le x} \frac{1}{a} - \sum_{a \le x} \left\lfloor \frac{x}{a} \right\rfloor. \tag{3}$$

We have the following general theorem.

Theorem 1.3. *We have the equation*

$$\sum_{\frac{x}{k} < a \le x} \left\lfloor \frac{x}{a} \right\rfloor = \left(\sum_{j=1}^{k} A\left(\frac{x}{j} \right) \right) - kA\left(\frac{x}{k} \right). \tag{4}$$

Proof. Note that if $\frac{x}{j+1} < a \leq \frac{x}{j}$, then $\left\lfloor \frac{x}{a} \right\rfloor = j$. Consequently,

$$\sum_{\frac{x}{k} < a \leq x} \left\lfloor \frac{x}{a} \right\rfloor = \sum_{j=1}^{k-1} j\left(A\left(\frac{x}{j}\right) - A\left(\frac{x}{j+1}\right)\right)$$

$$= \left(\sum_{j=1}^{k} A\left(\frac{x}{j}\right)\right) - kA\left(\frac{x}{k}\right).$$

The theorem is proved.

More precise formulas can be obtained if we have more information on $A(x)$. We have the following theorem.

Theorem 1.4. *Suppose that* $c > 0$, $0 < \alpha \leq 1$ *and* $f(x)$ *is a function of slow increase. If* $A(x) \sim cx^{\alpha}$, *then*

$$\sum_{\frac{x}{k} < a \leq x} \left\lfloor \frac{x}{a} \right\rfloor = \left(\sum_{j=1}^{k} \frac{1}{j^{\alpha}} - \frac{k}{k^{\alpha}}\right) cx^{\alpha} + o(x^{\alpha}). \tag{5}$$

If $A(x) \sim \frac{x^{\alpha}}{f(x)}$, *then*

$$\sum_{\frac{x}{k} < a \leq x} \left\lfloor \frac{x}{a} \right\rfloor = \left(\sum_{j=1}^{k} \frac{1}{j^{\alpha}} - \frac{k}{k^{\alpha}}\right) \frac{x^{\alpha}}{f(x)} + o\left(\frac{x^{\alpha}}{f(x)}\right). \tag{6}$$

Proof. Equation (5) is an immediate consequence of Equation (4). Equation (6) is an immediate consequence of Equation (4) and the limit $\lim_{x\to\infty} \frac{f\left(\frac{x}{j}\right)}{f(x)} = 1$ (see Equation (2)). The theorem is proved.

Theorem 1.5. *Suppose that* $A(x) \sim cx$, *where* $c > 0$. *If* $k \geq 2$ *is an arbitrary but fixed positive integer*, *then*

$$\sum_{\frac{x}{k} < a \leq x} \left\{\frac{x}{a}\right\} = \left(1 - \left(\sum_{i=1}^{k} \frac{1}{i} - \log k\right)\right) cx + o(x)$$

$$= \left(1 - \left(\sum_{i=1}^{k} \frac{1}{i} - \log k\right)\right) A(x) + o(A(x)). \tag{7}$$

Proof. We have

$$\sum_{a \leq x} 1 = A(x).$$

If we put $f(x) = \dfrac{1}{x}$ and apply Theorem 1.1, then we obtain

$$\sum_{a \leq x} \frac{1}{a} = A(x)\frac{1}{x} + \int_1^x \frac{A(t)}{t^2}\,dt.$$

Therefore

$$\sum_{a \leq \frac{x}{k}} \frac{1}{a} = A\left(\frac{x}{k}\right)\frac{k}{x} + \int_1^{\frac{x}{k}} \frac{A(t)}{t^2}\,dt,$$

and consequently,

$$x \sum_{\frac{x}{k} \leq a \leq x} \frac{1}{a} = \left(1 - \frac{A\left(\frac{x}{k}\right)}{A(x)} k + \left(\frac{x}{A(x)} \int_{\frac{x}{k}}^{x} \frac{A(t)}{t^2}\,dt\right)\right) A(x). \tag{8}$$

Now, we have

$$\frac{x}{A(x)}\int_{\frac{x}{k}}^{x}\frac{A(t)}{t^2}\,dt = \left(\frac{1}{c}+o(1)\right)\int_{\frac{x}{k}}^{x}\frac{ct+o(t)}{t^2}\,dt = \left(\frac{1}{c}+o(1)\right)c\int_{\frac{x}{k}}^{x}\frac{1}{t}\,dt$$

$$+\left(\frac{1}{c}+o(1)\right)\int_{\frac{x}{k}}^{x}o(1)\frac{1}{t}\,dt = \log k + o(1). \tag{9}$$

Substituting (9) into (8) and using (3) and (5) we obtain (7). The theorem is proved.

Theorem 1.6. *Suppose that* $A(x) \sim \dfrac{x}{f(x)}$, *where* $f(x)$ *is a function of slow increase. If* $k \geq 2$ *is an arbitrary but fixed positive integer, then*

$$\sum_{\frac{x}{k}<a\leq x}\left\{\frac{x}{a}\right\} = \left(1-\left(\sum_{i=1}^{k}\frac{1}{i}-\log k\right)\right)\frac{x}{f(x)} + o\left(\frac{x}{f(x)}\right)$$

$$= \left(1-\left(\sum_{i=1}^{k}\frac{1}{i}-\log k\right)\right)A(x) + o(A(x)). \tag{10}$$

Proof. As in Theorem 1.5 we have Equation (8). Now, we have

$$\frac{x}{A(x)}\int_{\frac{x}{k}}^{x}\frac{A(t)}{t^2}\,dt = (1+o(1))f(x)\int_{\frac{x}{k}}^{x}\frac{t+o(t)}{f(t)t^2}\,dt$$

$$= (1+o(1))f(x)\int_{\frac{x}{k}}^{x}\frac{1}{tf(t)}\,dt + (1+o(1))f(x)\int_{\frac{x}{k}}^{x}o(1)\frac{1}{tf(t)}\,dt$$

$$= \log k + o(1). \tag{11}$$

Since (see Equation (2))

$$\log k + o(1) = \frac{f(x)}{f(x)}\int_{\frac{x}{k}}^{x}\frac{1}{t} \leq f(x)\int_{\frac{x}{k}}^{x}\frac{1}{tf(t)}\,dt \leq \frac{f(x)}{f\left(\frac{x}{k}\right)}\int_{\frac{x}{k}}^{x}\frac{1}{t} = \log k + o(1),$$

and consequently,

$$f(x)\int_{\frac{x}{k}}^{x} \frac{1}{tf(t)}\,dt = \log k + o(1),$$

$$f(x)\int_{\frac{x}{k}}^{x} o(1)\frac{1}{tf(t)}\,dt = o(1).$$

Substituting (11) into (8) and using (3) and (6) we obtain (10). The theorem is proved.

Theorem 1.7. *Suppose that* $A(x) \sim cx^{\alpha}$, *where* $c > 0$ *and* $0 < \alpha < 1$. *If* $k \geq 2$ *is an arbitrary but fixed positive integer, then*

$$\sum_{\frac{x}{k}<a\leq x} \left\{\frac{x}{a}\right\} = \left(1 - \left(\sum_{i=1}^{k} \frac{1}{i^{\alpha}} - \int_{1}^{k} t^{-\alpha}dt\right)\right) cx^{\alpha} + o(x^{\alpha})$$

$$= \left(1 - \left(\sum_{i=1}^{k} \frac{1}{i^{\alpha}} - \int_{1}^{k} t^{-\alpha}dt\right)\right) A(x) + o(A(x)). \tag{12}$$

Proof. The proof is the same as the proof of Theorem 1.5. Note that in this case we have

$$\frac{x}{A(x)}\int_{\frac{x}{k}}^{x} \frac{ct^{\alpha}}{t^{2}}\,dt = \frac{k^{1-\alpha}}{1-\alpha} - \frac{1}{1-\alpha} + o(1) = \int_{1}^{k} \frac{1}{t^{\alpha}}\,dt + o(1).$$

The theorem is proved.

Theorem 1.8. *Suppose that* $A(x) \sim \dfrac{x^{\alpha}}{f(x)}$, *where* $f(x)$ *is a function of slow increase and* $0 < \alpha < 1$. *If* $k \geq 2$ *is an arbitrary but fixed positive integer, then*

$$\sum_{\frac{x}{k}<a\leq x} \left\{\frac{x}{a}\right\} = \left(1 - \left(\sum_{i=1}^{k} \frac{1}{i^{\alpha}} - \int_{1}^{k} t^{-\alpha}dt\right)\right) \frac{x^{\alpha}}{f(x)} + o\left(\frac{x^{\alpha}}{f(x)}\right)$$

$$= \left(1 - \left(\sum_{i=1}^{k} \frac{1}{i^{\alpha}} - \int_{1}^{k} t^{-\alpha}dt\right)\right) A(x) + o(A(x)). \tag{13}$$

Proof. The proof is the same as the proof of Theorem 1.6. Note that in this case we have

$$\frac{x}{A(x)}\int_{\frac{x}{k}}^{x}\frac{t^{\alpha}}{f(t)t^{2}}\,dt = \frac{k^{1-\alpha}}{1-\alpha}-\frac{1}{1-\alpha}+o(1) = \int_{1}^{k}\frac{1}{t^{\alpha}}\,dt+o(1).$$

The theorem is proved.

Theorem 1.9. *Suppose that either* $A(x) \sim cx^{\alpha}$ *or* $A(x) \sim \frac{x^{\alpha}}{f(x)}$, *where* $c > 0$, $0 < \alpha \leq 1$ *and* $f(x)$ *is a function of slow increase. Suppose also that*

$$\sum_{\frac{x}{k}<a\leq x}\left\{\frac{x}{a}\right\} = h(k)A(x)+o(A(x)) \quad (k \geq 2)$$

and $\lim_{k\to\infty} h(k) = l > 0$. *Then*

$$\sum_{a\leq x}\left\{\frac{x}{a}\right\} = lA(x)+o(A(x)).$$

Proof. We have

$$\sum_{a\leq x}\left\{\frac{x}{a}\right\} = \frac{\sum_{a\leq\frac{x}{k}}\left\{\frac{x}{a}\right\}}{A\left(\frac{x}{k}\right)}\frac{A\left(\frac{x}{k}\right)}{A(x)}A(x)$$

$$+\ (h(k)-l)A(x)+lA(x)+o(A(x)).$$

That is,

$$\frac{\sum_{a\leq x}\left\{\frac{x}{a}\right\}}{A(x)}-l = \frac{\sum_{a\leq\frac{x}{k}}\left\{\frac{x}{a}\right\}}{A\left(\frac{x}{k}\right)}\frac{A\left(\frac{x}{k}\right)}{A(x)}+(h(k)-l)+o(1).$$

Note that

$$0 \le \frac{\sum_{a \le \frac{x}{k}} \left\{ \frac{x}{a} \right\}}{A\left(\frac{x}{k}\right)} \le 1,$$

and

$$\frac{A\left(\frac{x}{k}\right)}{A(x)} \sim \frac{1}{k^{\alpha}}.$$

Therefore given $\epsilon > 0$ arbitrarily small there exists a k sufficiently large such that if $x \ge x_{\epsilon}$, we have

$$\left| \frac{\sum_{a \le x} \left\{ \frac{x}{a} \right\}}{A(x)} - l \right| \le \epsilon + \epsilon + \epsilon = 3\epsilon \qquad (x \ge x_{\epsilon}).$$

The theorem is proved.

The Euler's constant is defined in the form

$$\lim_{k \to \infty} \left(\sum_{j=1}^{k} \frac{1}{j} - \log k \right) = \lim_{k \to \infty} \left(\sum_{j=1}^{k} \frac{1}{j} - \int_1^k \frac{1}{t}\, dt \right) = \gamma.$$

In the following theorem we generalize this definition.

Theorem 1.10. *If* $0 < \alpha \le 1$, *we have*

$$\int_1^k \frac{1}{t^{\alpha}}\, dt - \sum_{j=2}^{k} \frac{1}{j^{\alpha}} = (1 - l_{\alpha}) + o(1),$$

where $0 < l_{\alpha} < 1$. *Therefore*

$$\sum_{j=1}^{k} \frac{1}{j^{\alpha}} - \int_1^k \frac{1}{t^{\alpha}}\, dt = l_{\alpha} + o(1).$$

In particular if $\alpha = 1$, *then* $l_1 = \gamma$.

Proof. Note that the function $g(t) = \frac{1}{t^{\alpha}}$ is strictly decreasing in the interval $[1, \infty]$ and $g(1) = 1.$ The integral $\int_1^k \frac{1}{t^{\alpha}} dt$ is the area below the function $g(t)$ in the interval $[1, k]$. The sum $\sum_{j=2}^{k} \frac{1}{j^{\alpha}}$ is the sum of the areas of $k - 1$ rectangles of base 1 and height $\frac{1}{j^{\alpha}}$ $(j = 2, 3, \ldots, k)$. Therefore $\int_1^k \frac{1}{t^{\alpha}} dt - \sum_{j=2}^{k} \frac{1}{j^{\alpha}}$ is the sum of the areas of the $k - 1$ figures "as triangles" above of the rectangles. Clearly this sum of areas of figures "as triangles" is strictly increasing and bounded by 1. Therefore, this series has sum $0 < 1 - l_{\alpha} < 1.$ The theorem is proved.

Now, we can establish and to prove our main theorem.

Theorem 1.11. *Suppose that* $A(x) \sim cx,$ *where* $c > 0,$ *then*

$$\sum_{a \leq x} \left\{ \frac{x}{a} \right\} = c(1 - \gamma)x + o(x) = (1 - \gamma)A(x) + o(A(x)).$$

Suppose that $A(x) \sim \frac{x}{f(x)}$, *then*

$$\sum_{a \leq x} \left\{ \frac{x}{a} \right\} = (1 - \gamma) \frac{x}{f(x)} + o\left(\frac{x}{f(x)} \right) = (1 - \gamma)A(x) + o(A(x)).$$

Suppose that $A(x) \sim cx^{\alpha},$ *where* $0 < \alpha < 1,$ *then*

$$\sum_{a \leq x} \left\{ \frac{x}{a} \right\} = c(1 - l_{\alpha})x^{\alpha} + o(x^{\alpha}) = (1 - l_{\alpha})A(x) + o(A(x)).$$

Suppose that $A(x) \sim \frac{x^{\alpha}}{f(x)}$, *where* $0 < \alpha < 1,$ *then*

$$\sum_{a \leq x} \left\{ \frac{x}{a} \right\} = (1 - l_{\alpha}) \frac{x^{\alpha}}{f(x)} + o\left(\frac{x^{\alpha}}{f(x)} \right) = (1 - l_{\alpha})A(x) + o(A(x)).$$

Proof. It is an immediate consequence of Theorems 1.5, 1.6, 1.7, 1.8, 1.9 and 1.10. The theorem is proved.

Remark 1.12. By use of Theorems 1.5, 1.6, 1.7, 1.8 and 1.11, we can easily obtain asymptotic formulas for the sum

$$\sum_{a \leq \frac{x}{k}} \left\{\frac{x}{a}\right\}.$$

Example 1.13. There are many sequences in number theory such that $A(x) \sim cx\,(c > 0)$. That is, sequences with positive density. The sequence a of all positive integers. The sequence a of integers in arithmetic progression. The sequence a of h-free numbers $(h \geq 2)$, where $A(x) \sim \frac{1}{\zeta(h)} x$ (see, for example, [5]). In particular, for the sequence of squarefree numbers or quadratfrei numbers we have $A(x) \sim \frac{6}{\pi^2} x$, etc.

Example 1.14. There are many sequences in number theory such that $A(x) \sim cx^{\alpha}\,(c > 0)\,(0 < \alpha < 1)$. The sequence a of k-th powers $(k \geq 2)$ where $A(x) \sim x^{\frac{1}{k}}$. The sequence a of all perfect powers where $A(x) \sim x^{\frac{1}{2}}$ (see [4]). The sequence a of h-full numbers $(h \geq 2)$ since that $A(x) \sim cx^{\frac{1}{h}}$, where the constant c depends of h (see, for example, either [3] or [6], for elementary methods), etc.

Example 1.15. There exist infinite sequences of positive integers in number theory such that $A(x) \sim \frac{x^{\alpha}}{f(x)}$, where $0 < \alpha \leq 1$ and $f(x)$ is a function of slow increase. The sequence of prime numbers, the sequence of prime powers, the sequence of numbers with exactly h prime factors in their prime factorization and infinite sequences of composite numbers with certain restrictions on their prime factorization (see [8]), etc.

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