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GENERAL THEOREMS ON SERIES INVOLVING THE ZETA FUNCTION

RAFAEL JAKIMCZUK

División Matemática
Universidad Nacional de Luján
Buenos Aires
Argentina
e-mail: jakimczu@mail.unlu.edu.ar

Abstract

We obtain general theorems on series involving the zeta function $\zeta(k)$, where $k \geq 2$ is an integer.

1. Introduction and Main Results

There are many papers on this subject. For example, we have the series

$$\sum_{k=2}^{\infty}(\zeta(k)-1) = 1;$$

$$\sum_{k=1}^{\infty}(\zeta(2k)-1) = \frac{3}{4};$$

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$$\sum_{k=1}^{\infty}(\zeta(2k+1)-1) = \frac{1}{4};$$

$$\sum_{k=2}^{\infty}\frac{\zeta(k)-1}{k} = 1-\gamma;$$

$$\sum_{k=2}^{\infty}(-1)^k\,\frac{\zeta(k)}{k} = \gamma;$$

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{2k2^{2k+1}} = \log\pi - \log 2.$$

See [1] (pages 43-44), for more series and references.

In this article we obtain more series of this type.

Theorem 1.1. *Let us consider a power series with convergence radius* $R > \frac{1}{2}$, *namely*,

$$f(x) = \sum_{i=0}^{\infty} a_{i+1}x^i = a_1 + a_2x + a_3x^2 + a_4x^3 + \cdots. \tag{1}$$

Then, there exists

$$\lim_{N\to\infty}\sum_{n=2}^{N}\left(f\left(\frac{1}{n}\right) - a_1 - \frac{a_2}{n}\right) = L, \tag{2}$$

where L *is a real number and*

$$L = \sum_{i=2}^{\infty} a_{i+1}(\zeta(i)-1) = a_3(\zeta(2)-1) + a_4(\zeta(3)-1) + a_5(\zeta(4)-1) + \cdots. \tag{3}$$

Proof. Note that as all power series the convergence of (1) is absolute for $|x| < R$. Equation (1) gives

$$f\left(\frac{1}{n}\right) - a_1 - \frac{a_2}{n} = \sum_{i=2}^{\infty} a_{i+1}\,\frac{1}{n^i}. \tag{4}$$

Therefore,

$$\sum_{n=2}^{N}\left(f\left(\frac{1}{n}\right) - a_1 - \frac{a_2}{n}\right) = \sum_{i=2}^{\infty} a_{i+1} \sum_{n=2}^{N} \frac{1}{n^i}. \tag{5}$$

Now, we shall prove that the series

$$\sum_{i=2}^{\infty} a_{i+1} \sum_{n=N+1}^{\infty} \frac{1}{n^i} \tag{6}$$

is absolutely convergent.

We have

$$\sum_{i=2}^{\infty} |a_{i+1}| \sum_{n=N+1}^{\infty} \frac{1}{n^i} \leq \sum_{i=2}^{\infty} |a_{i+1}| \int_N^{\infty} \frac{1}{x^i}\, dx = \sum_{i=2}^{\infty} |a_{i+1}| \frac{1}{(i-1)N^{i-1}}$$

$$\leq \sum_{i=2}^{\infty} |a_{i+1}| \frac{1}{N^{i-1}} = \frac{1}{N} \sum_{i=2}^{\infty} |a_{i+1}| \frac{1}{N^{i-2}} \leq \frac{1}{N} \sum_{i=2}^{\infty} |a_{i+1}| \frac{1}{2^{i-2}} = \frac{A}{N}. \tag{7}$$

Note that the power series (see (1)) $\sum_{i=2}^{\infty} a_{i+1} x^{i-2}$ also converges for $|x| < R$ and as all power series the convergence is absolute for $|x| < R$.

Consequently (see (7)), we have

$$\left|\sum_{i=2}^{\infty} a_{i+1} \sum_{n=2}^{N} \frac{1}{n^i} - \left(\sum_{i=2}^{\infty} a_{i+1}(\zeta(i) - 1)\right)\right| = \left|\sum_{i=2}^{\infty} a_{i+1} \sum_{n=N+1}^{\infty} \frac{1}{n^i}\right|$$

$$\leq \sum_{i=2}^{\infty} |a_{i+1}| \sum_{n=N+1}^{\infty} \frac{1}{n^i} \leq \frac{A}{N} < \epsilon, \quad N \geq N_{\epsilon},$$

where $\epsilon > 0$ can be arbitrarily small. Therefore, we have proved

$$\lim_{N \to \infty} \sum_{i=2}^{\infty} a_{i+1} \sum_{n=2}^{N} \frac{1}{n^i} = \sum_{i=2}^{\infty} a_{i+1}(\zeta(i) - 1) = L. \tag{8}$$

Equations (8) and (5) give (2). To finish, we have to prove that the series

$$\sum_{i=2}^{\infty} a_{i+1}(\zeta(i)-1) \tag{9}$$

converges.

We have

$$\zeta(k)-1 \leq \frac{1}{2^k} + \int_2^{\infty} \frac{1}{x^k}\, dx = \frac{1}{2^k} + \frac{1}{(k-1)2^{k-1}} = \frac{1}{2^{k-1}}\left(\frac{1}{2} + \frac{1}{k-1}\right)$$

$$\leq \frac{1}{2^{k-1}} \frac{3}{2} = 3\frac{1}{2^k}.$$

Hence

$$\sum_{i=2}^{\infty} |a_{i+1}|(\zeta(i)-1) \leq 3\sum_{i=2}^{\infty} |a_{i+1}| \frac{1}{2^i}.$$

That is, the series (9) is absolutely convergent. The theorem is proved.

Theorem 1.2. *Let us consider a power series such that either* $R = 1$ *and converges for* $x = 1$ *or* $R > 1$, *namely*,

$$f(x) = \sum_{i=0}^{\infty} a_{i+1}x^i = a_1 + a_2x + a_3x^2 + a_4x^3 + \cdots.$$

Then, there exists

$$\lim_{N\to\infty} \sum_{n=1}^{N} \left(f\left(\frac{1}{n}\right) - a_1 - \frac{a_2}{n} \right) = L_1,$$

where L_1 *is a real number and*

$$L_1 = \sum_{i=2}^{\infty} a_{i+1}\zeta(i) = a_3\zeta(2) + a_4\zeta(3) + a_5\zeta(4) + \cdots.$$

Proof. Since $R > \frac{1}{2}$ by Theorem 1.1 there exist

$$\lim_{N\to\infty}\sum_{n=2}^{N}\left(f\left(\frac{1}{n}\right) - a_1 - \frac{a_2}{n}\right) = L,$$

and consequently, there exists

$$\lim_{N\to\infty}\sum_{n=1}^{N}\left(f\left(\frac{1}{n}\right) - a_1 - \frac{a_2}{n}\right) = L + f(1) - a_1 - a_2 = L_1.$$

Consequently by Equation (3), we have

$$L_1 = L + f(1) - a_1 - a_2 = f(1) - a_1 - a_2 + \sum_{i=2}^{\infty} a_{i+1}(\zeta(i) - 1)$$

$$= f(1) - a_1 - a_2 - \sum_{i=2}^{\infty} a_{i+1} + \sum_{i=2}^{\infty} a_{i+1}\zeta(i) = \sum_{i=2}^{\infty} a_{i+1}\zeta(i).$$

The theorem is proved.

Example 1.3. (1) Let us consider the well-known power series

$$f(x) = -\log(1 - x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots, \quad -1 \le x < 1.$$

We apply Theorem 1.1 and obtain

$$L = \lim_{N\to\infty}\sum_{n=2}^{N}\left(\left(-\log\left(1 - \frac{1}{n}\right) - \frac{1}{n}\right)\right)$$

$$= \lim_{N\to\infty}\sum_{n=2}^{N}\left(\log n - \log(n-1) - \frac{1}{n}\right)$$

$$= \lim_{N\to\infty}\left(1 + \log N - \sum_{n=1}^{N}\frac{1}{n}\right) = 1 - \gamma.$$

Therefore, Theorem 1.1 gives the well-known result

$$\sum_{k=2}^{\infty} \frac{\zeta(k)-1}{k} = 1-\gamma.$$

(2) Let us consider the well-known power series

$$f(x) = \log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots, \quad -1 < x \leq 1.$$

If we apply Theorem 1.2 then we obtain the well-known result

$$\sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k} = \gamma.$$

(3) Let us consider the geometric power series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \quad |x| < 1.$$

If we apply Theorem 1.1 then we obtain

$$L = \lim_{N\to\infty} \sum_{n=2}^{N} \left(\frac{1}{1-\frac{1}{n}} - 1 - \frac{1}{n} \right) = \lim_{N\to\infty} \sum_{n=2}^{N} \left(\frac{1}{n-1} - \frac{1}{n} \right) = 1,$$

and consequently, we find the well-known result

$$\sum_{k=2}^{\infty} (\zeta(k) - 1) = 1.$$

(4) Let us consider the geometric power series

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots, \quad |x| < 1.$$

If we apply Theorem 1.1 then we obtain

$$L = \lim_{N\to\infty} \sum_{n=2}^{N} \left(\frac{1}{1+\frac{1}{n}} - 1 + \frac{1}{n} \right) = \lim_{N\to\infty} \sum_{n=2}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \frac{1}{2},$$

and consequently, we find the well-known result

$$\sum_{k=2}^{\infty}(-1)^k(\zeta(k)-1) = \frac{1}{2}.$$

Theorem 1.4. *Let* $k \geq 2$ *be an arbitrary but fixed positive integer and let us consider a power series with convergence radius* $R > \frac{1}{2}$, *namely,*

$$f(x) = \sum_{i=0}^{\infty} a_{i+1}x^i = a_1 + a_2x + a_3x^2 + a_4x^3 + \cdots. \tag{10}$$

Then, there exists

$$\lim_{N\to\infty}\sum_{n=2}^{N}\left(f\left(\frac{1}{n^k}\right) - a_1\right) = L, \tag{11}$$

where L *is a real number and*

$$L = \sum_{i=1}^{\infty} a_{i+1}(\zeta(ik)-1) = a_2(\zeta(k)-1) + a_3(\zeta(2k)-1) + a_4(\zeta(3k)-1) + \cdots. \tag{12}$$

Proof. Equation (10) gives

$$f\left(\frac{1}{n^k}\right) - a_1 = \sum_{i=1}^{\infty} a_{i+1}\frac{1}{n^{ik}}. \tag{13}$$

Therefore,

$$\sum_{n=2}^{N}\left(f\left(\frac{1}{n^k}\right) - a_1\right) = \sum_{i=1}^{\infty} a_{i+1}\sum_{n=2}^{N}\frac{1}{n^{ik}}. \tag{14}$$

Now, we shall prove that the series

$$\sum_{i=1}^{\infty} a_{i+1}\sum_{n=N+1}^{\infty}\frac{1}{n^{ik}} \tag{15}$$

is absolutely convergent.

We have

$$\sum_{i=1}^{\infty}|a_{i+1}|\sum_{n=N+1}^{\infty}\frac{1}{n^{ik}} \leq \sum_{i=1}^{\infty}|a_{i+1}|\int_{N}^{\infty}\frac{1}{x^{ik}}\,dx = \sum_{i=1}^{\infty}|a_{i+1}|\frac{1}{(ik-1)N^{ik-1}}$$

$$\leq \sum_{i=1}^{\infty}|a_{i+1}|\frac{1}{N^{ik-1}} = \frac{1}{N^{k-1}}\sum_{i=1}^{\infty}|a_{i+1}|\frac{1}{N^{(i-1)k}}$$

$$\leq \frac{1}{N^{k-1}}\sum_{i=1}^{\infty}|a_{i+1}|\frac{1}{2^{(i-1)k}} = \frac{B}{N^{k-1}}. \tag{16}$$

Consequently (see (16)), we have

$$\left|\sum_{i=1}^{\infty}a_{i+1}\sum_{n=2}^{N}\frac{1}{n^{ik}} - \left(\sum_{i=1}^{\infty}a_{i+1}(\zeta(ik)-1)\right)\right| = \left|\sum_{i=1}^{\infty}a_{i+1}\sum_{n=N+1}^{\infty}\frac{1}{n^{ik}}\right|$$

$$\leq \sum_{i=1}^{\infty}|a_{i+1}|\sum_{n=N+1}^{\infty}\frac{1}{n^{ik}} \leq \frac{B}{N^{k-1}} < \epsilon, \quad N \geq N_{\epsilon},$$

where $\epsilon > 0$ can be arbitrarily small. Therefore, we have proved

$$\lim_{N\to\infty}\sum_{i=1}^{\infty}a_{i+1}\sum_{n=2}^{N}\frac{1}{n^{ik}} = \sum_{i=1}^{\infty}a_{i+1}(\zeta(ik)-1) = L. \tag{17}$$

Equations (17) and (14) give (11). To finish, we have to prove that the series

$$\sum_{i=1}^{\infty}a_{i+1}(\zeta(ik)-1) \tag{18}$$

converges.

As in Theorem 1.1 we have

$$\zeta(ik)-1 \leq 3\,\frac{1}{2^{ik}}.$$

Hence

$$\sum_{i=1}^{\infty}|a_{i+1}|(\zeta(ik)-1) \leq 3\sum_{i=1}^{\infty}|a_{i+1}|\frac{1}{2^{ik}}.$$

That is, the series (18) is absolutely convergent. The theorem is proved.

Theorem 1.5. *Let* $k \geq 2$ *be an arbitrary but fixed positive integer. Let us consider a power series such that either* $R = 1$ *and converges for* $x = 1$ *or* $R > 1$, *namely*,

$$f(x) = \sum_{i=0}^{\infty} a_{i+1}x^{i} = a_1 + a_2x + a_3x^2 + a_4x^3 + \cdots.$$

Then, there exists

$$\lim_{N\to\infty}\sum_{i=1}^{N}\left(f\left(\frac{1}{n^k}\right) - a_1\right) = L_1,$$

where L_1 *is a real number and*

$$L_1 = \sum_{i=1}^{\infty} a_{i+1}\zeta(ik) = a_2\zeta(k) + a_3\zeta(2k) + a_4\zeta(3k) + \cdots.$$

Proof. The proof is the same as the proof of Theorem 1.2. The theorem is proved.

Example 1.6. (1) Let us consider the geometric power series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \quad |x| < 1.$$

If we apply Theorem 1.4 then we obtain $(k \geq 2)$

$$L = \lim_{N\to\infty}\sum_{n=2}^{N}\left(\frac{1}{1-\frac{1}{n^k}} - 1\right) = \sum_{n=2}^{\infty}\frac{1}{n^k - 1}.$$

Therefore, we find that

$$\sum_{n=2}^{\infty} \frac{1}{n^k - 1} = (\zeta(k) - 1) + (\zeta(2k) - 1) + (\zeta(3k) - 1) + \cdots .$$

In particular, if $k = 2$ then we obtain

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1} = \frac{1}{2} \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) = \frac{3}{4},$$

and consequently, we find the well-known result

$$\frac{3}{4} = (\zeta(2) - 1) + (\zeta(4) - 1) + (\zeta(6) - 1) + \cdots .$$

(2) Let us consider the geometric power series

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots, \quad |x| < 1.$$

If we apply Theorem 1.4 then we obtain $(k \geq 2)$

$$L = \lim_{N \to \infty} \sum_{n=2}^{N} \left(\frac{1}{1 + \frac{1}{n^k}} - 1 \right) = -\sum_{n=2}^{\infty} \frac{1}{n^k + 1} .$$

Therefore, we find that

$$\sum_{n=2}^{\infty} \frac{1}{n^k + 1} = (\zeta(k) - 1) - (\zeta(2k) - 1) + (\zeta(3k) - 1) - \cdots .$$

In particular, if $k = 2$ it is well-known that (see, for example, [2] (page 433))

$$\sum_{n=2}^{\infty} \frac{1}{n^2 + 1} = \frac{\pi \coth \pi}{2} - 1,$$

and consequently, we find the result

$$\frac{\pi \coth \pi}{2} - 1 = (\zeta(2) - 1) - (\zeta(4) - 1) + (\zeta(6) - 1) - \cdots,$$

where $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$.

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References

[1] S. R. Finch, Mathematical Constants, Cambridge University Press, 2003.

[2] A. D. Wunsch, Variable Compleja con Aplicaciones, Segunda Edicion, Addison-Wesley Iberoamericana, 1997.

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