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CAUCHY INTEGRAL FORMULA FOR MATRIX FUNCTIONS

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Abstract

Cauchy integral formula for functions of single complex matrix is introduced. With this in hand, some specific examples on certain matrix functions are discussed.

1. Introduction

Throughout the history of pure and applied mathematics, matrix functions have been used as powerful instruments in the solution of wide

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variety of important problems (see, e.g., [1, 2, 3, 4, 5, 6, 7, 16, 17]). In [8, 9] studied some properties for analytic functions of square complex matrices and discussed matrix real integrations, complex integration of matrix functions and Cauchy's integral formula for type functions of single complex matrix. In this paper, we derive a new Cauchy integral formula for functions of single complex matrix. Using this formula, applied examples on some complex matrix functions are given.

Let us turn now to some notations and definitions that will help us navigate through this paper. Let $\mathbb{C}^{N\times N}$ denote the complex space of complex matrices of common order N and $\mathcal{M}(\mathbb{C}^{N\times N})$ is the set of holomorphic matrix functions. The matrices I and $\mathbf{0}$ stand for the identity matrix and the null matrix in $\mathbb{C}^{N\times N}$, respectively. A matrix X is a positive stable matrix in $\mathbb{C}^{N\times N}$ if $\operatorname{Re}(\lambda) > 0$ for all $\lambda \in \sigma(X)$, where $\sigma(X)$ is the set of all eigenvalues of X.

If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z which are defined in an open set $\Omega \subset \mathbb{C}$ and X is a matrix in $\mathbb{C}^{N\times N}$ such that $\sigma(X) \subset \Omega$, then (see [1, 2, 3])

$$f(X)g(X) = g(X)f(X).$$

Hence, if Y in $\mathbb{C}^{N\times N}$ is a matrix for which $\sigma(Y) \subset \Omega$ and if $XY = YX$, then

$$f(X)g(Y) = g(Y)f(X).$$

Definition 1.1. Let $f(X)$ be a complex matrix function, then its derivative is defined by (see [8, 9, 10, 11])

$$\frac{d}{d(X)} f(X) = \lim_{h\to 0} \frac{f(X + Ih) - f(X)}{h}, \tag{1.1}$$

where X is a square matrix over the complex field and h is a scalar complex variable.

According to [1, pp. 569], the integration of matrix functions. Suppose $f(tA)$ is defined for all $t \in [a, b]$ and that we wish to compute

$$F = \int_a^b f(tA)dt,$$

where A is a matrix and the integration is on an element-by-element basis.

According to Dunford-Schwartz in [1, pp. 556] and [2, pp. 8], the matrix function via Cauchy's integral formula defined by for a matrix $A \in \mathbb{C}^{N \times N}$, then

$$f(A) = \frac{1}{2\pi i} \oint_\Gamma f(z)(zI - A)^{-1} dz,$$

where f is holomorphic function on and inside the closed contour Γ that encloses $\sigma(A)$.

The following result is useful for the coming work:

Theorem 1.1 ([10, 12]). *Let $f(X)$ be a matrix function analytic in Ω, where $X = [x_{ij}(z)]$, then*

$$f(X) = \sum_{n=0}^{\infty} A_n (z - a)^n.$$

2. Main Results

Theorem 2.1. *Let $f(X) = \sum_{n=0}^{\infty} a_n X^n$ be in $\mathcal{M}(\mathbb{C}^{N \times N})$; $X = [x_{ij}(z)]$, $x_{ij}(z)$; $z \in \mathbb{C}$ are analytic functions for all $i, j = 1, 2, 3, \ldots, N$, X and $\frac{dX}{dz}$ are commutative and $[x_{ij}(z)]$ is non-singular. Then for any point $z = a$ inside Γ, we have*

$$f(A) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(X)}{(X - A)} dX. \tag{2.1}$$

In general

$$f^{(n)}(A) = \frac{n!}{2\pi i}\oint_{\Gamma} \frac{f(X)}{(X - A)^{n+1}}\, dX;\; n = 0, 1, 2, \ldots, \tag{2.2}$$

where $A = [x_{ij}(a)]$ *and* $(X - A)$ *are non-singular matrices for all* $z \neq a$ *inside and on the closed contour* Γ *and*

$$\frac{df(X)}{dz} = \frac{df(X)}{dX} \cdot \frac{dX}{dz}.$$

Proof. Since $x_{ij}(z)$ are analytic functions for all z inside and on Γ, $\forall i,\ j = 1, 2, 3, \ldots, N,$ then from Theorem 1.1, we can write

$$x_{ij}(z) = \sum_{n=0}^{\infty} \frac{x_{ij}^{(n)}(a)}{n!}(z - a)^n, \quad \forall i,\ j = 1, 2, 3, \ldots, N.$$

$$X - A = \left[\sum_{n=1}^{\infty} \frac{x_{ij}^{(n)}(a)}{n!}(z - a)^n\right]$$

$$= (z - a)\left\{[x'_{ij}(a)] + [\sum_{n=2}^{\infty} \frac{x_{ij}^{(n)}(a)}{n!}(z - a)^{n-1}]\right\}.$$

Then

$$\frac{1}{2\pi i}\oint_{\Gamma} \frac{f(X)}{(X - A)}\, dX = f(A)[x'_{ij}(a)][x'_{ij}(a)]^{-1},$$

from which

$$\frac{1}{2\pi i}\oint_{\Gamma} \frac{f(X)}{(X - A)}\, dX = f(A). \tag{2.3}$$

As for $n = 2,$ we have

$$\frac{1}{2\pi i}\oint_{\Gamma} \frac{f(X)}{(X - A)^2}\, dX$$

$$= \left\{f'(A)[x'_{ij}(a)]^2 + f(A)[x''_{ij}(a)]\right\}[x'_{ij}(a)]^{-2}$$

$$- f(A)[x'_{ij}(a)]^{-3}[x'_{ij}(a)][x''_{ij}(a)] = f'(A).$$

Thus,

$$\frac{1}{2\pi i}\oint_{\Gamma}\frac{f(X)}{(X-A)^2}dX = f'(A). \tag{2.4}$$

Mathematical induction technique leads for any positive integer n, we can prove that

$$f^{(n)}(A) = \frac{n!}{2\pi i}\oint_{\Gamma}\frac{f(X)}{(X-A)^{n+1}}dX;\ n = 0, 1, 2, \ldots. \tag{2.5}$$

□

Remark 2.1. Differentiating both sides in (2.1) n times $(n \geq 1)$ with respect to a, we get the results.

$$f'(A)\frac{dA}{da} = f'(A)[x'_{ij}(a)] = \frac{1}{2\pi i}\oint_{\Gamma} f(X)\frac{\partial}{\partial a}(X-A)^{-1}\frac{dX}{dz}.dz$$

$$= \frac{1}{2\pi i}\oint_{\Gamma} f(X)(X-A)^{-2}[x'_{ij}(a)][x'_{ij}(z)]dz.$$

Hence,

$$f'(A) = \frac{1}{2\pi i}\oint_{\Gamma}\frac{f(X)}{(X-A)^2}dX.$$

Now, suppose that

$$f^{(n)}(A) = \frac{n!}{2\pi i}\oint_{\Gamma}\frac{f(X)}{(X-A)^{n+1}}dX. \tag{2.6}$$

Differentiating both sides of relation (2.6) with respect to a, leads to

$$f^{(n+1)}(A)[x'_{ij}(a)] = \frac{(n+1)!}{2\pi i}\oint_{\Gamma}\frac{f(X)[x'_{ij}(a)]}{(X-A)^{n+2}}dX;\ n = 0, 1, 2, \ldots,$$

i.e.,

$$f^{(n+1)}(A) = \frac{(n+1)!}{2\pi i}\oint_{\Gamma}\frac{f(X)}{(X-A)^{n+2}}dX;\ n = 0, 1, 2, \ldots.$$

Thus according to mathematical induction, we obtain

$$f^{(n)}(A) = \frac{n!}{2\pi i}\oint_{\Gamma} \frac{f(X)}{(X - A)^{n+1}}\, dX;\ n = 0, 1, 2, \dots.$$

2.1. Examples

This subsection presents some application examples to show the usability of our approach.

Example 2.1. Evaluate the following matrix functions integrals:

(1)

$$\oint_{\Gamma} \frac{e^X}{X^{n+1}}\, dX;\ n = 0, 1, 2, 3, \dots,$$

where Γ is $|z| = 1$ and

$$X = \begin{pmatrix} e^z - 1 & 3\sin z & 5\sinh z & \cosh z - 1 \\ 3\sin z & e^z - 1 & \cosh z - 1 & 5\sinh z \\ 5\sinh z & \cosh z - 1 & e^z - 1 & 3\sin z \\ \cosh z - 1 & 5\sinh z & 3\sin z & e^z - 1 \end{pmatrix}.$$

(2)

$$\oint_{\Gamma} \frac{d(X)}{(X - A)^n};\ n = 1, 2, 3, \dots,$$

where Γ is $|z| = 1$ and

$$X = \begin{pmatrix} 2e^z & e^{-z} \\ e^{-z} & 2e^z \end{pmatrix}, \quad A = [x_{ij}(0)] = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Solution: Applying relation (2.2), we obtain

(1) $$\oint_{|z|=1} \frac{e^x}{X^{n+1}} dX = \frac{2\pi i}{n!} I; \; n = 0, 1, 2, \ldots.$$

(2) $$\oint_{|z|=1} \frac{dX}{(X-A)^n} = \begin{cases} 2\pi i I; & n = 1, \\ 0; & n = 2, 3, \ldots. \end{cases}$$

Also, applying Cauchy integral formula (2.2), we get the following results concerning Legendre matrix polynomials, Hermite matrix polynomials, and Laguerre matrix polynomials, respectively, as follows (cf. [13, 14, 15]):

$$P_n(A) = \frac{1}{2^{n+1}\pi i} \oint_\Gamma \frac{(X^2 - I)^n}{(X-A)^{n+1}} dX; \; n = 0, 1, 2, \ldots, \tag{2.7}$$

$$H_n(A) = \frac{(-1)^n n! \, e^{A^2}}{2\pi i} \oint_\Gamma \frac{e^{-X^2}}{(X-A)^{n+1}} dX; \; n = 0, 1, 2, \ldots, \tag{2.8}$$

$$L_n(A) = \frac{e^A}{2\pi i} \oint_\Gamma \frac{X^n e^{-X}}{(X-A)^{n+1}} dX; \; n = 0, 1, 2, \ldots. \tag{2.9}$$

It is clear that the above examples give a direct generalization to the standard complex case of Cauchy integral formula for matrix functions. Therefore our result in this section can be exploited to establish further consequences regarding other several problems in this area.

References

[1] G. H. Golud and C. F. Van Loan, Matrix Computations, The Johns Hopkins Press Ltd., London, 1996.

[2] J. Higham, Functions of Matrices Theory and Computation, SIAM, USA, 2008.

[3] R. Brualdi and D. Cvetkvić, A Combinatorial Approach to Matrix Theory and its Applications, New York, 2009.

[4] H. Gingold, Factorization of matrix functions and their inverses via power product expansions, Linear Algebra and its Applications 430 (2009), 2835-2858.

[5] A. M. Mathai, Jacobians of Matrix Transformations and Functions of Matrix Argument, World Scientific Publishing, New York, 1997.

[6] A. M. Mathai, A Handbook of Generalized Special Functions for Statistical and Physical Sciences, Oxford University Press, Oxford, 1993.

[7] M. Abdalla, On Some Convergence Properties of Matrix Functions, MSc. Diss. South Valley University, Egypt, 2010. (Lambert Academic Publishing, Germany, ISBN 978-3-659-23592-4, (2012)).

[8] Z. Kishka, Functions of several square commutative complex matrices, S. S. Q. J. Pure Appl. Math. 66 (1992), 69-77.

[9] M. Abul-Dahab and Z. Kishka, Analytic functions of complex matrices, Trans. J. Math. Anal. Appl. 2 (2015), 105-130.

[10] Z. Kishka, M. Abul-Ez, M. Saleem and H. Abd-Elmageed, L'Hospital rule for matrix functions, J. Egyp. Math. Soc. 21 (2013), 115-118.

[11] R. F. Rinehart, The derivative of a matrix function, Proc. Amer. Math. Soc. 7 (1955), 2-5.

[12] Z. M. Kishka and M. Abul-Dahab, On power series expansions of complex matrix functions, AJOMCOR 3 (2015), 190-200.

[13] E. D. Rainville, Special Functions, New York, 1960.

[14] N. N. Lebedev, Special Functions and their Applications, Dover Publications Inc., New York, 1972.

[15] L. Y. Luke, The Special Functions and their Approximations, Vol. 2, Academic Press, New York, 1969.

[16] M. S. Metwally, On p-Kummers matrix function of complex variable under differential operators their properties, Southeast Asian Bull. Math. 35 (2011), 261-276.

[17] Z. M. Kishka, M. A. Saleem, M. T. Mohammed and M. Abul-Dahab, On the p and q-Appell matrix function, Southeast Asian Bull. Math. 36 (2012), 837-848.

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