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THE WARPED SCHOUTEN-VAN KAMPEN AND VRANCEANU CONNECTIONS

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Abstract

In this paper, we develop Schouten-Van Kampen and Vranceanu connections for the warped product Finsler manifolds by applying the warped Levi-Civita connection to tangent bundle. Also, necessary and sufficient conditions are obtained for ${}^{*}\mathbf{G}$ (warped Sasaki-Matsumoto metric) to be totally geodesic and bundle like.

1. Introduction

The *Schouten-Van Kampen* connection and the *Vranceanu* connections have been introduced for a study of non-holonomic manifolds (cf. [7, 8]). In this paper, we present some properties of these connections in the warped product Finsler manifolds. Let M_1 and M_2 be two real

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smooth manifolds of dimensions n_1 and n_2, respectively, and $M = M_1 \times M_2$. Then a coordinate system on $M = M_1 \times M_2$ is denoted by (x^i, u^α), where (x^i) and (u^α) are coordinate systems in M_1 and M_2, respectively.

We know that $TM = TM_1 \oplus TM_2$ and the coordinate system (x^i, u^α) on M, defines a coordinate system $(x^i, u^\alpha; y^i, v^\alpha)$ on $TM \simeq TM_1 \oplus TM_2$. We consider another coordinate system $(\tilde{x}^i, \tilde{u}^\alpha)$ on $M = M_1 \times M_2$, then the local coordinates (x, u, y, v) and $(\tilde{x}, \tilde{u}, \tilde{y}, \tilde{v})$ on $TM = TM_1 \oplus TM_2$ are related by (cf. [3])

$$\begin{cases} \tilde{x}^i = \tilde{x}^i(x^1, \ldots, x^{n_1}), \\ \tilde{u}^\alpha = \tilde{u}^\alpha(u^1, \ldots, u^{n_2}), \\ \tilde{y}^i = \dfrac{\partial \tilde{x}^i}{\partial x^j} y^j, \\ \tilde{v}^\alpha = \dfrac{\partial \tilde{u}^\alpha}{\partial u^\beta} v^\beta. \end{cases} \tag{1}$$

If $\mathfrak{u} = (x, u, y, v) \in TM = TM_1 \oplus TM_2$ we denote by $T_\mathfrak{u}(TM)$ the tangent space at $\mathfrak{u}$ to TM. This is a $2(n_1 + n_2)$-dimensional vector space and natural basis induced by a local chart $(x^i, u^\alpha; y^i, v^\alpha)$ at $\mathfrak{u}$ is $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial v^\alpha}\}$. After a change of coordinates (1) on $TM = TM_1 \oplus TM_2$, the natural basis change as follows (cf. [1])

$$\begin{cases} \dfrac{\partial}{\partial x^i} = \dfrac{\partial \tilde{x}^j}{\partial x^i}\dfrac{\partial}{\partial \tilde{x}^j} + \dfrac{\partial \tilde{y}^j}{\partial x^i}\dfrac{\partial}{\partial \tilde{y}^j}, \\ \dfrac{\partial}{\partial u^\alpha} = \dfrac{\partial \tilde{u}^\beta}{\partial u^\alpha}\dfrac{\partial}{\partial \tilde{u}^\beta} + \dfrac{\partial \tilde{v}^\beta}{\partial u^\alpha}\dfrac{\partial}{\partial \tilde{v}^\beta}, \\ \dfrac{\partial}{\partial y^i} = \dfrac{\partial \tilde{x}^j}{\partial x^i}\dfrac{\partial}{\partial \tilde{y}^j}, \\ \dfrac{\partial}{\partial v^\alpha} = \dfrac{\partial \tilde{u}^\beta}{\partial u^\alpha}\dfrac{\partial}{\partial \tilde{v}^\beta}, \\ \operatorname{rank}\begin{bmatrix} \dfrac{\partial \tilde{x}^i}{\partial x^j} & 0 \\ 0 & \dfrac{\partial \tilde{u}^\alpha}{\partial u^\beta} \end{bmatrix} = n_1 + n_2. \end{cases} \tag{2}$$

Let (TM_1, π_1, M_1) and (TM_2, π_2, M_2) be tangent bundles of M_1 and M_2, respectively, and (TM, π, M) tangent bundle of $M = M_1 \times M_2$, where $\pi = (\pi_1, \pi_2)$. Then for each $\mathfrak{u}_1 \in TM_1$ and $\mathfrak{u}_2 \in TM_2$ the linear maps $\pi_{1_*, \mathfrak{u}_1} : T_{\mathfrak{u}_1}TM_1 \to T_{\pi_1(\mathfrak{u}_1)}M_1$ and $\pi_{2_*, \mathfrak{u}_2} : T_{\mathfrak{u}_2}TM_2 \to T_{\pi_2(\mathfrak{u}_2)}M_2$ induced by the canonical submersions π_1 and π_2 are epimorphisms of linear spaces, respectively. Hence the kernels determines regular and integrable distributions $\mathcal{V}_1 : \mathfrak{u}_1 \in TM_1 \mapsto \mathcal{V}_{\mathfrak{u}_1}TM_1 := \operatorname{Ker}(\pi_{1_*, \mathfrak{u}_1}) \subset T_{\mathfrak{u}_1}TM_1$ and $\mathcal{V}_2 : \mathfrak{u}_2 \in TM_2 \mapsto \mathcal{V}_{\mathfrak{u}_2}TM_2 := \operatorname{Ker}(\pi_{2_*, \mathfrak{u}_2}) \subset T_{\mathfrak{u}_2}TM_2$, respectively, which are called the *vertical distributions*. If $\mathfrak{u} = (\mathfrak{u}_1, \mathfrak{u}_2)$ then $\pi_{*, \mathfrak{u}} = (\pi_{1_*, \mathfrak{u}_1}, \pi_{2_*, \mathfrak{u}_2})$ and so

$$\operatorname{Ker}(\pi_{*, \mathfrak{u}}) = \operatorname{Ker}(\pi_{1_*, \mathfrak{u}_1}) \oplus \operatorname{Ker}(\pi_{2_*, \mathfrak{u}_2}), \tag{3}$$

that is,

$$\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2, \tag{4}$$

where $\mathcal{V}, \mathcal{V}_1$ and $\mathcal{V}_2$ are vertical sub-bundle of M, M_1 and M_2, respectively. For every $\mathfrak{u} = (\mathfrak{u}_1, \mathfrak{u}_2) \in TM = TM_1 \oplus TM_2$, set

$\{\frac{\partial}{\partial y^i}|_{u_1}, \frac{\partial}{\partial v^\alpha}|_{u_2}\}$ is a basis of $\mathcal{V}_u TM$, where $\{\frac{\partial}{\partial x^i}|_{u_1}, \frac{\partial}{\partial y^i}|_{u_1}, \frac{\partial}{\partial u^i}|_{u_1}, \frac{\partial}{\partial v^\alpha}|_{u_2}\}$ is the natural basis of T_uTM induced by a local chart. According to (2), the natural frame fields $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial v^\alpha}\}$ and $\{\frac{\partial}{\partial \tilde{x}^j}, \frac{\partial}{\partial \tilde{y}^j}, \frac{\partial}{\partial \tilde{u}^\beta}, \frac{\partial}{\partial \tilde{v}^\alpha}\}$ are related by (cf. [1])

$$\begin{cases} \frac{\partial}{\partial x^i} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial}{\partial \tilde{x}^j} + \frac{\partial \tilde{y}^j}{\partial x^i}\frac{\partial}{\partial \tilde{y}^j}, \\ \frac{\partial}{\partial u^\alpha} = \frac{\partial \tilde{u}^\beta}{\partial u^\alpha}\frac{\partial}{\partial \tilde{u}^\beta} + \frac{\partial \tilde{v}^\beta}{\partial u^\alpha}\frac{\partial}{\partial \tilde{v}^\beta}, \\ \frac{\partial}{\partial y^i} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\partial}{\partial \tilde{y}^j}, \\ \frac{\partial}{\partial v^\alpha} = \frac{\partial \tilde{u}^\beta}{\partial u^\alpha}\frac{\partial}{\partial \tilde{v}^\beta}. \end{cases} \tag{5}$$

Proposition 1.1. *There exists a complementary distribution* $\mathcal{H}$ *to* $\mathcal{V} \cong \mathcal{V}_1 \oplus \mathcal{V}_2$ *in TTM if and only if on the domain of each local chart on* $M = M_1 \times M_2$, *there exists* $4n_1n_2$ *smooth functions* $\mathbf{N}_j^i, \mathbf{N}_\beta^i, \mathbf{N}_j^\alpha$ *and* $\mathbf{N}_\beta^\alpha$ *satisfies*

$$\mathbf{N}_i^j \frac{\partial \tilde{x}^h}{\partial x^j} = \tilde{\mathbf{N}}_j^h \frac{\partial \tilde{x}^j}{\partial x^i} + \frac{\partial \tilde{y}^h}{\partial x^i}, \tag{6}$$

$$\mathbf{N}_i^\beta \frac{\partial \tilde{u}^h}{\partial u^\beta} = \tilde{\mathbf{N}}_j^\lambda \frac{\partial \tilde{x}^j}{\partial x^i}, \tag{7}$$

$$\mathbf{N}_\alpha^j \frac{\partial \tilde{x}^h}{\partial x^j} = \tilde{\mathbf{N}}_\beta^h \frac{\partial \tilde{u}^\beta}{\partial u^\alpha}, \tag{8}$$

$$\mathbf{N}_\alpha^\beta \frac{\partial \tilde{u}^\lambda}{\partial u^\beta} = \tilde{\mathbf{N}}_\beta^\lambda \frac{\partial \tilde{u}^\beta}{\partial u^\alpha} + \frac{\partial \tilde{v}^\lambda}{\partial u^\alpha}. \tag{9}$$

The distribution $\mathcal{H}$ is called the ***warped distribution or warped non-linear connection*** *on $M = M_1 \times M_2$.*

Proof. First suppose that $\mathcal{H}$ is a complementary distribution to $\mathcal{V} \cong V_1 \oplus \mathcal{V}_2$ in TM, and take a local frame fields $\{E_i, \dot{E}_\alpha \frac{\partial}{\partial y^i}, \frac{\partial}{\partial v^\alpha}\}$ on M such that $\{E_i, \dot{E}_\alpha\} \in \Gamma(\mathcal{H})$ and $\{\frac{\partial}{\partial y^i}, \frac{\partial}{\partial v^\alpha}\} \in \Gamma(\mathcal{V})$. So

$$\begin{cases} \dfrac{\partial}{\partial x^i} = \mathrm{A}_i^k E_k + \mathrm{C}_i^\gamma \dot{E}_\gamma + \mathbf{N}_i^k \dfrac{\partial}{\partial y^k} + \mathbf{N}_i^\gamma \dfrac{\partial}{\partial v^\gamma}, \\ \dfrac{\partial}{\partial u^\alpha} = \mathrm{B}_\alpha^h E_h + \mathrm{D}_\alpha^\mu \dot{E}_\mu + \mathbf{N}_\alpha^h \dfrac{\partial}{\partial y^h} + \mathbf{N}_\alpha^\mu \dfrac{\partial}{\partial v^\mu}, \end{cases} \tag{10}$$

where $\mathrm{A}_i^j, \mathrm{B}_\alpha^j, \mathrm{C}_i^\beta, \mathrm{D}_\alpha^\beta, \mathbf{N}_i^j, \mathbf{N}_i^\beta, \mathbf{N}_\alpha^j$, and $\mathbf{N}_\alpha^\beta$ are smooth functions on a coordinate neighborhood in $M_1 \times M_2$. Hence the matrix of transition from $\{E_i, \dot{E}_\alpha, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial v^\alpha}\}$ to $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial v^\alpha}\}$ is

$$\Lambda = \begin{pmatrix} \mathrm{A}_i^k & \mathrm{B}_\alpha^k & 0 & 0 \\ \mathrm{C}_i^\gamma & \mathrm{D}_\alpha^\gamma & 0 & 0 \\ \mathbf{N}_i^k & \mathbf{N}_\alpha^k & \delta_i^k & 0 \\ \mathbf{N}_i^\gamma & \mathbf{N}_\alpha^\gamma & 0 & \delta_\alpha^\beta \end{pmatrix}. \tag{11}$$

Since the matrix Λ is non-singular, then matrix

$$\begin{pmatrix} \mathrm{A}_i^k & \mathrm{B}_\alpha^k \\ \mathrm{C}_i^\gamma & \mathrm{D}_\alpha^\gamma \end{pmatrix}, \tag{12}$$

also is non-singular. As a consequently it follows that $\mathcal{H}$ is also locally spanned by

$$\begin{cases} \dfrac{\delta^*}{\delta^* \mathbf{x}^i} = \mathrm{A}_i^k E_k + \mathrm{C}_i^\gamma \dot{E}_\gamma, \\ \dfrac{\delta^*}{\delta^* \mathbf{u}^\alpha} = \mathrm{B}_\alpha^k E_k + \mathrm{D}_\alpha^\gamma \dot{E}_\gamma. \end{cases} \tag{13}$$

Thus by using (10) and (13), we have

$$\begin{cases} \dfrac{\delta^*}{\delta^*\mathbf{x}^i} = \dfrac{\partial}{\partial x^i} - \mathbf{N}_i^k \dfrac{\partial}{\partial y^k} - \mathbf{N}_i^\beta \dfrac{\partial}{\partial v^\beta}, \\ \dfrac{\delta^*}{\delta^*\mathbf{u}^\alpha} = \dfrac{\partial}{\partial u^\alpha} - \mathbf{N}_\alpha^k \dfrac{\partial}{\partial y^k} - \mathbf{N}_\alpha^\beta \dfrac{\partial}{\partial v^\beta}. \end{cases} \tag{14}$$

Moreover by using (5) and (14) for two coordinate systems $(x^i, y^i, u^\alpha, v^\alpha)$ and $(\tilde{x}^j, \tilde{y}^j, \tilde{u}^\beta, \tilde{v}^\beta)$ with overlapping domains, we obtain

$$\begin{aligned} \frac{\delta^*}{\delta^*\mathbf{x}^i} &= \frac{\partial \tilde{x}^k}{\partial x^i}\frac{\partial}{\partial \tilde{x}^k} + \frac{\partial \tilde{y}^k}{\partial x^i}\frac{\partial}{\partial \tilde{y}^k} - \mathbf{N}_i^j \frac{\partial \tilde{x}^k}{\partial x^j}\frac{\partial}{\partial \tilde{y}^k} - \mathbf{N}_i^\beta \frac{\partial \tilde{u}^\gamma}{\partial u^\beta}\frac{\partial}{\partial \tilde{v}^\gamma} \\ &= \frac{\partial \tilde{x}^k}{\partial x^i}\left(\frac{\delta^*}{\delta^*\tilde{\mathbf{x}}^k} + \widetilde{\mathbf{N}}_k^h \frac{\partial}{\partial \tilde{y}^h} + \widetilde{\mathbf{N}}_k^\mu \frac{\partial}{\partial \tilde{v}^\mu}\right) + \frac{\partial \tilde{y}^k}{\partial x^i}\frac{\partial}{\partial \tilde{y}^k} \\ &\quad - \mathbf{N}_i^j \frac{\partial \tilde{x}^k}{\partial x^j}\frac{\partial}{\partial \tilde{y}^k} - \mathbf{N}_i^\beta \frac{\partial \tilde{u}^\gamma}{\partial u^\beta}\frac{\partial}{\partial \tilde{v}^\gamma} \\ &= \frac{\partial \tilde{x}^k}{\partial x^i}\frac{\delta^*}{\delta^*\mathbf{x}^k} + \left(\widetilde{\mathbf{N}}_h^k \frac{\partial \tilde{x}^h}{\partial x^i} - \mathbf{N}_i^j \frac{\partial \tilde{x}^k}{\partial x^j} + \frac{\partial \tilde{y}^k}{\partial x^i}\right)\frac{\partial}{\partial \tilde{y}^k} \\ &\quad + \left(\widetilde{\mathbf{N}}_k^\gamma \frac{\partial \tilde{x}^k}{\partial x^i} - \mathbf{N}_i^\beta \frac{\partial \tilde{u}^\gamma}{\partial u^\beta}\right)\frac{\partial}{\partial \tilde{v}^\gamma}, \end{aligned} \tag{15}$$

$$\begin{aligned} \frac{\delta^*}{\delta^*\mathbf{u}^\alpha} &= \frac{\partial \tilde{u}^\mu}{\partial u^\alpha}\frac{\partial}{\partial \tilde{u}^\mu} + \frac{\partial \tilde{v}^\mu}{\partial u^\alpha}\frac{\partial}{\partial \tilde{y}^k} - \mathbf{N}_\alpha^j \frac{\partial \tilde{x}^k}{\partial x^j}\frac{\partial}{\partial \tilde{y}^k} - \mathbf{N}_\alpha^\beta \frac{\partial \tilde{u}^\gamma}{\partial u^\beta}\frac{\partial}{\partial \tilde{v}^\gamma} \\ &= \frac{\partial \tilde{u}^\mu}{\partial u^\alpha}\left(\frac{\delta^*}{\delta^*\tilde{\mathbf{u}}^\mu} + \widetilde{\mathbf{N}}_\mu^h \frac{\partial}{\partial \tilde{y}^h} + \widetilde{\mathbf{N}}_\mu^\tau \frac{\partial}{\partial \tilde{v}^\tau}\right) + \frac{\partial \tilde{v}^\mu}{\partial u^\alpha}\frac{\partial}{\partial \tilde{v}^\mu} \\ &\quad - \mathbf{N}_\alpha^j \frac{\partial \tilde{x}^k}{\partial x^j}\frac{\partial}{\partial \tilde{y}^k} - \mathbf{N}_\alpha^\beta \frac{\partial \tilde{u}^\gamma}{\partial u^\beta}\frac{\partial}{\partial \tilde{v}^\gamma} \end{aligned} \tag{16}$$

$$= \frac{\partial \tilde{u}^{\mu}}{\partial u^{\alpha}} \frac{\delta^*}{\delta^* \mathbf{u}^{\mu}} + \left(\widetilde{\mathbf{N}}_{\tau}^{\mu} \frac{\partial \tilde{u}^{\tau}}{\partial u^{\alpha}} - \mathbf{N}_{\alpha}^{\lambda} \frac{\partial \tilde{u}^{\mu}}{\partial u^{\lambda}} + \frac{\partial \tilde{v}^{\mu}}{\partial u^{\alpha}} \right) \frac{\partial}{\partial \tilde{v}^{\mu}}$$

$$+ \left(\widetilde{\mathbf{N}}_{\gamma}^{k} \frac{\partial \tilde{u}^{\gamma}}{\partial u^{\alpha}} - \mathbf{N}_{\alpha}^{h} \frac{\partial \tilde{x}^{k}}{\partial x^{h}} \right) \frac{\partial}{\partial \tilde{y}^{k}}. \tag{17}$$

Hence we obtain (6), (7), (8), and (9) for the functions $\mathbf{N}_i^j$, $\mathbf{N}_i^{\beta}$, $\mathbf{N}_{\alpha}^j$, and $\mathbf{N}_{\alpha}^{\beta}$ from (13) and the $\{\frac{\delta^*}{\delta^* \mathbf{x}^i}, \frac{\delta^*}{\delta^* \mathbf{u}^{\alpha}}\}$ given by (14), (15), and (16) satisfies

$$\begin{cases} \dfrac{\delta^*}{\delta^* \mathbf{x}^i} = \dfrac{\partial \tilde{x}^j}{\partial x^i} \dfrac{\delta^*}{\delta^* \tilde{\mathbf{x}}^j}, \\ \dfrac{\delta^*}{\delta^* \mathbf{u}^{\alpha}} = \dfrac{\partial \tilde{u}^{\beta}}{\partial u^{\alpha}} \dfrac{\delta^*}{\delta^* \tilde{\mathbf{u}}^{\beta}}. \end{cases} \tag{18}$$

Conversely, similarly proof. □

Corollary 1.2. *Let* $\mathbb{F}_1^{n_1} = (M_1, F_1)$ *and* $\mathbb{F}_2^{n_2} = (M_2, F_2)$ *be two Finsler manifolds, and* $f : M_1 \to \mathbf{R}_+$ *be a smooth function. Then* $\mathbb{F}^{n_1+n_2} + (M_1 \times M_2, F)$ *is a warped product Finsler manifold and denoted by* $\mathbb{F} = (M_1 \times_f M_2, F)$, *where*

$$F^2(x, u, y, v) = F_1^2(x, y) + (f \circ \pi_1)^2(x, y) F_2^2(u, v), \tag{19}$$

for any $(x, y) \in TM_1$ *and* $(u, v) \in TM_2$.

Proof. For details and proof, see [1, 5]. □

Let $\mathcal{H} = \text{span}\{\frac{\delta^*}{\delta^* \mathbf{x}^i}, \frac{\delta^*}{\delta^* \mathbf{u}^{\alpha}}\}$ be a warped non-linear connection on $M = M_1 \times M_2$, then the following new operators can be defined (cf [6]):

$$\frac{\partial^*}{\partial^* \mathbf{y}^i} := \frac{\partial}{\partial y^i} + \mathbf{N}_i^\beta \frac{\partial}{\partial v^\beta}, \tag{20}$$

$$\frac{\partial^*}{\partial^* \mathbf{v}^\alpha} := \mathbf{N}_\alpha^j \frac{\partial}{\partial y^j} + \frac{\partial}{\partial v^\alpha}, \tag{21}$$

and

$$^*\mathcal{V}TM := \operatorname{span}\{\frac{\partial^*}{\partial^* \mathbf{y}^i}, \frac{\partial^*}{\partial^* \mathbf{v}^\alpha}\}, \tag{22}$$

can be put forward. It follows that $^*\mathcal{V}TM^\circ \cong \mathcal{V}TM^\circ$. Thus the tangent bundle of TM° admits the composition

$$TTM^\circ = \mathcal{H}TM^\circ \oplus {}^*\mathcal{V}TM^\circ, \tag{23}$$

where $\mathcal{H}TM^\circ = \operatorname{span}\{\frac{\delta^*}{\delta^* \mathbf{x}^i}, \frac{\delta^*}{\delta^* \mathbf{u}^\alpha}\}$.

Corollary 1.3 (cf [6])**.** *By using* (20), (21), (6), (7), (8), (9) *for two coordinate systems* $(x^i, y^i, u^\alpha, v^\alpha)$ *and* $(\tilde{x}^j, \tilde{y}^j, \tilde{u}^\beta, \tilde{v}^\beta)$, *we obtain*

$$\frac{\partial^*}{\partial^* \mathbf{y}^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial^*}{\partial^* \mathbf{y}^j}, \tag{24}$$

$$\frac{\partial^*}{\partial^* \mathbf{v}^\alpha} = \frac{\partial \tilde{u}^\beta}{\partial u^\alpha} \frac{\partial^*}{\partial^* \mathbf{v}^\beta}. \tag{25}$$

Using decomposition (23), the warped vertical morphism $^*v : T(TM^\circ) \to {}^*\mathcal{V}TM^\circ$ is defined as follows (cf [6])

$$^*v := \frac{\partial}{\partial y^i} \otimes \delta^* \mathbf{y}^i + \frac{\partial}{\partial v^\alpha} \otimes \delta^* \mathbf{v}^\alpha, \tag{26}$$

where

$$\delta^* \mathbf{y}^i := \mathrm{d}y^i + \mathbf{N}_j^i \mathrm{d}x^j + \mathbf{N}_\beta^i \mathrm{d}u^\beta, \tag{27}$$

$$\delta^{*}\mathbf{v}^{\alpha} := dv^{\alpha} + \mathbf{N}_{j}^{\alpha}dx^{j} + \mathbf{N}_{\beta}^{\alpha}du^{\beta}. \tag{28}$$

Therefore

$$\begin{cases} {}^{*}v(\dfrac{\partial}{\partial x^{j}}) = \mathbf{N}_{j}^{i}\dfrac{\partial}{\partial y^{i}} + \mathbf{N}_{j}^{\alpha}\dfrac{\partial}{\partial v^{\alpha}}, \\ {}^{*}v(\dfrac{\partial}{\partial u^{\beta}}) = \mathbf{N}_{\beta}^{i}\dfrac{\partial}{\partial y^{i}} + \mathbf{N}_{\beta}^{\alpha}\dfrac{\partial}{\partial v^{\alpha}}, \end{cases} \tag{29}$$

$$\begin{cases} {}^{*}v(\dfrac{\partial}{\partial y^{j}}) = \dfrac{\partial^{*}}{\partial^{*}\mathbf{y}^{j}}, \\ {}^{*}v(\dfrac{\partial}{\partial v^{\beta}}) = \dfrac{\partial^{*}}{\partial^{*}\mathbf{v}^{\beta}}, \end{cases} \tag{30}$$

$$\text{(i)}\ {}^{*}v(\frac{\delta^{*}}{\delta^{*}\mathbf{x}^{j}}) = 0, \quad \text{(ii)}\ {}^{*}v(\frac{\delta^{*}}{\delta^{*}\mathbf{u}^{\beta}}) = 0. \tag{31}$$

Thus using (30) and (31) the following can be inferred that ${}^{*}v^{2} = {}^{*}v$ and $\ker({}^{*}v) = \mathcal{H}(TM^{0})$. This mapping is called the *warped vertical projective*. Similarly the warped horizontal projective ${}^{*}h : T(TM^{0}) \to \mathcal{H}(TM^{0})$ is defined as follows:

$${}^{*}h := id - {}^{*}v, \tag{32}$$

or

$${}^{*}h = \frac{\delta^{*}}{\delta^{*}\mathbf{x}^{i}} \otimes dx^{i} + \frac{\delta^{*}}{\delta^{*}\mathbf{u}^{\alpha}} \otimes du^{\alpha}. \tag{33}$$

Thus

$$\begin{cases} {}^{*}h(\dfrac{\partial}{\partial x^{j}}) = \dfrac{\delta^{*}}{\delta^{*}\mathbf{x}^{j}}, \\ {}^{*}h(\dfrac{\partial}{\partial u^{\beta}}) = \dfrac{\delta^{*}}{\delta^{*}\mathbf{u}^{\beta}}, \\ {}^{*}h(\dfrac{\partial^{*}}{\partial^{*}\mathrm{y}^{j}}) = 0, \\ {}^{*}h(\dfrac{\partial^{*}}{\partial^{*}\mathrm{v}^{\beta}}) = 0. \end{cases} \tag{34}$$

Using (33) and (34), ${}^*h^2 = {}^*h$ and $\ker({}^*h) = \mathcal{V}^*(TM^0)$ can be inferred (cf. [1]).

2. Some Geometry Objects of the Warped Product Finsler Manifolds

Let $\mathbb{F}_1^{n_1} = (M_1, F_1)$ and $\mathbb{F}_2^{n_2} = (M_2, F_2)$ be two Finsler manifolds. The functions

$$\begin{cases} \text{(i)} \quad \rho_{ij}(x, y) = \dfrac{1}{2} \dfrac{\partial^2 F_1^2(x, y)}{\partial y^i \partial y^j}, \\ \text{(ii)} \quad \sigma_{\alpha\beta}(u, v) = \dfrac{1}{2} \dfrac{\partial^2 F_2^2(u, v)}{\partial v^\alpha \partial v^\beta}, \end{cases} \tag{35}$$

define a Finsler tensor field of type (0, 2) on TM_1° and TM_2°, respectively. Let $\mathbb{F}^{n_1+n_2} = \left(M_1 \times_f M_2, F\right)$, $\mathbf{x} \in M$ and $\mathbf{y} \in T_\mathbf{x}M$, where $\mathbf{x} = (x, u)$, $\mathbf{y} = (y, v)$, $M = M_1 \times M_2$ and $T_\mathbf{x}M = T_xM_1 \oplus T_uM_2$. Then using (19) and (35) it can be inferred that (see [5])

$$(g_{ab}) = \left(\frac{1}{2} \frac{\partial^2 F^2}{\partial \mathbf{y}^a \partial \mathbf{y}^b}\right) = \begin{bmatrix} \rho_{ij} & 0 \\ 0 & f^2(x)\sigma_{\alpha\beta} \end{bmatrix}, \tag{36}$$

where $\mathbf{y}^a = (y^i, v^\alpha)$ and $\mathbf{y}^b = (y^j, v^\beta)$ and $i, j, \ldots \in \{1, \ldots, n_1\}$, $\alpha, \beta, \ldots \in \{1, \ldots, n_2\}$ and $a, b, \ldots \in \{1, \ldots, n_1 + n_2\}$. Suppose that

$$\hat{G}^i(x, y) := \frac{1}{4} \rho^{ih}(x, y) \left\{ \frac{\partial^2 F_1^2}{\partial y^h \partial x^j} y^j - \frac{\partial F_1^2}{\partial x^h} \right\} (x, y), \tag{37}$$

$$\check{G}^\alpha(u, v) := \frac{1}{4} \rho^{\alpha\gamma}(u, v) \left\{ \frac{\partial^2 F_2^2}{\partial v^\gamma \partial u^\beta} v^\beta - \frac{\partial F_2^2}{\partial u^\gamma} \right\} (u, v), \tag{38}$$

$$\mathbf{G}^a(\mathbf{x}, \mathbf{y}) := \frac{1}{4} g^{ab}(\mathbf{x}, \mathbf{y}) \left\{ \frac{\partial^2 F^2}{\partial \mathbf{y}^b \partial \mathbf{x}^c} \mathbf{y}^c - \frac{\partial F^2}{\partial \mathbf{x}^b} \right\} (\mathbf{x}, \mathbf{y}). \tag{39}$$

Then using (19), (35), (37), (38) and (39), it can be deduced by straightforward calculation as follows

$$(\mathbf{G}^{a}) = (\mathcal{G}^{i}, \mathfrak{G}^{\alpha}), \tag{40}$$

where

$$\begin{cases} \mathcal{G}^{i}(x, u, y, v) = \hat{G}^{i}(x, y) - \dfrac{1}{4}\rho^{ih}(x, y) F_2^2(u, v)\dfrac{\partial f^2(x)}{\partial x^h}, \\ \mathfrak{G}^{\alpha}(x, u, y, v) = \check{G}^{\alpha}(u, v) + \dfrac{1}{4}\dfrac{1}{f^2(x)}\sigma^{\alpha\beta}(u, v)\dfrac{\partial F_2^2(u, v)}{\partial v^{\beta}}\dfrac{\partial f^2(x)}{\partial x^h} y^h. \end{cases} \tag{41}$$

The following is considered

$$(\mathbf{N}_b^a(x, u, y, v)) = \begin{bmatrix} \mathbf{N}_j^i(x, u, y, v) & \mathbf{N}_j^{\alpha}(x, u, y, v) \\ \mathbf{N}_{\beta}^i(x, u, y, v) & \mathbf{N}_{\beta}^{\alpha}(x, u, y, v) \end{bmatrix}, \tag{42}$$

where $\mathbf{N}_j^i = \dfrac{\partial \mathcal{G}^i}{\partial y^j}$, $\mathbf{N}_{\beta}^i = \dfrac{\partial \mathcal{G}^i}{\partial v^{\beta}}$, $\mathbf{N}_j^{\alpha} = \dfrac{\partial \mathfrak{G}^{\alpha}}{\partial y^j}$, and $\mathbf{N}_{\beta}^{\alpha} = \dfrac{\partial \mathfrak{G}^{\alpha}}{\partial v^{\beta}}$. Then

$$\begin{cases} \mathbf{N}_j^i(x, u, y, v) = \hat{G}_j^i(x, y) - \dfrac{1}{4} F_2^2(u, v)\dfrac{\partial f^2(x)}{\partial x^h}\dfrac{\partial \rho^{ih}(x, y)}{\partial y^j}, \\ \mathbf{N}_{\beta}^i(x, u, y, v) = -\dfrac{1}{4}\rho^{ih}(x, y)\dfrac{\partial f^2(x)}{\partial x^h}\dfrac{\partial F_2^2(u, v)}{\partial v^{\beta}}, \\ \mathbf{N}_j^{\alpha}(x, u, y, v) = \dfrac{1}{4f^2(x)}\sigma^{\alpha\gamma}(u, v)\dfrac{\partial F_2^2(u, v)}{\partial v^{\gamma}}\dfrac{\partial f^2(x)}{\partial x^j}, \\ \mathbf{N}_{\beta}^{\alpha}(x, u, y, v) = \check{G}_{\beta}^{\alpha}(u, v) + \dfrac{1}{2f^2(x)}\dfrac{\partial f^2(x)}{\partial x^j} y^j \delta_{\beta}^{\alpha}, \end{cases} \tag{43}$$

where $\hat{G}_j^i = \dfrac{\partial \hat{G}^i}{\partial y^j}$ and $\check{G}_{\beta}^{\alpha} = \dfrac{\partial \check{G}^{\alpha}}{\partial v^{\beta}}$.

In the continuation of this section, some geometry objects of warped product Finsler manifold type on TM^0 can be defined as follows (cf. [1])

$$\begin{cases} \text{(a)} \left[\dfrac{\delta^*}{\partial^* \mathbf{x}^i}, \dfrac{\delta^*}{\delta^* \mathbf{x}^j}\right] = {}^*\mathbf{R}^k_{ij}\dfrac{\partial}{\partial y^k} + {}^*\mathbf{R}^\gamma_{ij}\dfrac{\partial}{\partial v^\gamma}, \\ \text{(b)} \left[\dfrac{\delta^*}{\partial^* \mathbf{x}^i}, \dfrac{\delta^*}{\delta^* \mathbf{u}^\beta}\right] = {}^*\mathbf{R}^k_{i\beta}\dfrac{\partial}{\partial y^k} + {}^*\mathbf{R}^\gamma_{i\beta}\dfrac{\partial}{\partial v^\gamma}, \\ \text{(c)} \left[\dfrac{\delta^*}{\partial^* \mathbf{u}^\alpha}, \dfrac{\delta^*}{\delta^* \mathbf{x}^j}\right] = {}^*\mathbf{R}^k_{\alpha j}\dfrac{\partial}{\partial y^k} + {}^*\mathbf{R}^\gamma_{\alpha j}\dfrac{\partial}{\partial v^\gamma}, \\ \text{(d)} \left[\dfrac{\delta^*}{\partial^* \mathbf{u}^\alpha}, \dfrac{\delta^*}{\delta^* \mathbf{u}^\beta}\right] = {}^*\mathbf{R}^k_{\alpha\beta}\dfrac{\partial}{\partial y^k} + {}^*\mathbf{R}^\gamma_{\alpha\beta}\dfrac{\partial}{\partial v^\gamma}, \end{cases} \tag{44}$$

and

$$\begin{cases} \text{(a)} \left[\dfrac{\delta^*}{\partial^* \mathbf{x}^i}, \dfrac{\partial}{\partial y^j}\right] = \mathbf{N}^k_{i\,j}\dfrac{\partial}{\partial y^k} + \mathbf{N}^\gamma_{i\,j}\dfrac{\partial}{\partial v^\gamma}, \\ \text{(b)} \left[\dfrac{\delta^*}{\delta^* \mathbf{x}^i}, \dfrac{\partial}{\partial v^\beta}\right] = \mathbf{N}^k_{i\,\beta}\dfrac{\partial}{\partial y^k} + \mathbf{N}^\gamma_{i\,\beta}\dfrac{\partial}{\partial v^\gamma}, \\ \text{(c)} \left[\dfrac{\delta^*}{\delta^* \mathbf{u}^\alpha}, \dfrac{\partial}{\partial y^j}\right] = \mathbf{N}^k_{\alpha\,j}\dfrac{\partial}{\partial y^k} + \mathbf{N}^\gamma_{\alpha\,j}\dfrac{\partial}{\partial v^\gamma}, \\ \text{(d)} \left[\dfrac{\delta^*}{\delta^* \mathbf{u}^\alpha}, \dfrac{\partial}{\partial v^\beta}\right] = \mathbf{N}^k_{\alpha\,\beta}\dfrac{\partial}{\partial y^k} + \mathbf{N}^\gamma_{\alpha\,\beta}\dfrac{\partial}{\partial v^\gamma}, \end{cases} \tag{45}$$

where (cf [2])

$$\text{(i)}\ {}^*\mathbf{R}^k_{ij} := \frac{\delta^*\mathbf{N}^k_i}{\delta^*\mathbf{x}^j} - \frac{\delta^*\mathbf{N}^k_j}{\delta^*\mathbf{x}^i}, \qquad \text{(ii)}\ {}^*\mathbf{R}^k_{i\beta} := \frac{\delta^*\mathbf{N}^k_i}{\delta^*\mathbf{u}^\beta} - \frac{\delta^*\mathbf{N}^k_\beta}{\delta^*\mathbf{x}^i}; \tag{46}$$

$$\text{(i)}\ {}^*\mathbf{R}^k_{\alpha j} := \frac{\delta^*\mathbf{N}^k_\alpha}{\delta^*\mathbf{x}^j} - \frac{\delta^*\mathbf{N}^k_j}{\delta^*\mathbf{u}^\alpha}, \qquad \text{(ii)}\ {}^*\mathbf{R}^k_{\alpha\beta} := \frac{\delta^*\mathbf{N}^k_\alpha}{\delta^*\mathbf{u}^\beta} - \frac{\delta^*\mathbf{N}^k_\beta}{\delta^*\mathbf{u}^\alpha}; \tag{47}$$

$$\text{(i)}\ {}^*\mathbf{R}^\gamma_{ij} := \frac{\delta^*\mathbf{N}^\gamma_i}{\delta^*\mathbf{x}^j} - \frac{\delta^*\mathbf{N}^\gamma_j}{\delta^*\mathbf{x}^i}, \qquad \text{(ii)}\ {}^*\mathbf{R}^\gamma_{i\beta} := \frac{\delta^*\mathbf{N}^\gamma_i}{\delta^*\mathbf{u}^\beta} - \frac{\delta^*\mathbf{N}^\gamma_\beta}{\delta^*\mathbf{x}^i}; \tag{48}$$

$$\text{(i)}\ {}^*\mathbf{R}^\gamma_{\alpha j} := \frac{\delta^*\mathbf{N}^\gamma_\alpha}{\delta^*\mathbf{x}^j} - \frac{\delta^*\mathbf{N}^\gamma_j}{\delta^*\mathbf{u}^\alpha}, \qquad \text{(ii)}\ {}^*\mathbf{R}^\gamma_{\alpha\beta} := \frac{\delta^*\mathbf{N}^\gamma_\alpha}{\delta^*\mathbf{u}^\beta} - \frac{\delta^*\mathbf{N}^\gamma_\beta}{\delta^*\mathbf{u}^\alpha}; \tag{49}$$

and

$$\mathbf{N}^k_{i\,j} = \frac{\partial \mathbf{N}^k_i}{\partial y^j} = \mathbf{N}^k_{j\,i}, \tag{50}$$

$$\mathbf{N}^k_{i\,\beta} = \frac{\partial \mathbf{N}^k_\beta}{\partial y^i} = -\frac{1}{4}\frac{\partial F_2^2}{\partial v^\beta}\frac{\partial f^2}{\partial x^h}\frac{\partial \rho^{kh}}{\partial y^i} = \frac{\partial \mathbf{N}^k_i}{\partial v^\beta} = \mathbf{N}^k_{\beta\,i}, \tag{51}$$

$$\mathbf{N}^k_{\alpha\,\beta} = \frac{\partial \mathbf{N}^k_\alpha}{\partial v^\beta} = \mathbf{N}^k_{\beta\,\alpha}, \tag{52}$$

$$\mathbf{N}^\gamma_{i\,j} = \frac{\partial \mathbf{N}^\gamma_i}{\partial y^j} = \mathbf{N}^\gamma_{j\,i}, \tag{53}$$

$$\mathbf{N}^\gamma_{i\,\beta} = \frac{\partial \mathbf{N}^\gamma_\beta}{\partial y^i} = \frac{1}{2f^2}\frac{\partial f^2}{\partial x^i}\delta^\gamma_\beta = \frac{\partial \mathbf{N}^\gamma_i}{\partial v^\beta} = \mathbf{N}^\gamma_{\beta\,i}, \tag{54}$$

$$\mathbf{N}^\gamma_{\alpha\,\beta} = \frac{\partial \mathbf{N}^\gamma_\alpha}{\partial v^\beta} = \mathbf{N}^\gamma_{\beta\,\alpha}. \tag{55}$$

It is obvious, the warped horizontal distribution $\mathcal{H}TM$ is integrable if and only if the functions ${}^*\mathbf{R}^i_{j\,k}, \dots, {}^*\mathbf{R}^i_{\beta\,\gamma}$ and ${}^*\mathbf{R}^\alpha_{j\,k}, \dots, {}^*\mathbf{R}^\alpha_{\beta\,\gamma}$ are vanish identically on $M = (M_1 \times_f M_2)$ for all $i, j, k \in \{1, \cdots, n_1\}$ and $\alpha, \beta, \gamma \in \{1, \cdots, n_2\}$ (see [4] p. 32).

Let $FC^* := ((\mathbf{N}^a_b), (\mathbf{F}^a_{b\,c}), (\mathbf{C}^a_{b\,c}))$ be the Cartan connection on $M = M_1 \times_f M_2$, where (cf ([1]))

$$(\mathbf{F}^a_{b\,c}) = (\mathcal{F}^k_{i\,j}, \mathcal{F}^k_{\alpha\,j}, \mathcal{F}^k_{i\,\beta}, \mathcal{F}^k_{\alpha\,\beta}, \mathfrak{F}^\gamma_{i\,j}, \mathfrak{F}^\gamma_{\alpha\,j}, \mathfrak{F}^\gamma_{i\,\beta}, \mathfrak{F}^\gamma_{\alpha\,\beta}), \tag{56}$$

$$(\mathbf{C}^a_{b\,c}) = (\mathcal{C}^i_{j\,k}, \mathcal{C}^i_{\beta\,k}, \mathcal{C}^i_{j\,\gamma}, \mathcal{C}^i_{\beta\,\gamma}, \mathfrak{C}^\alpha_{j\,k}, \mathfrak{C}^\alpha_{\beta\,k}, \mathfrak{C}^\alpha_{j\,\gamma}, \mathfrak{C}^\alpha_{\beta\,\gamma}), \tag{57}$$

and

$$\mathcal{F}^{i}_{j\ k} := \frac{1}{2}\rho^{ih}\left(\frac{\delta^{*}\rho_{hj}}{\delta^{*}\mathbf{x}^{k}} + \frac{\delta^{*}\rho_{kh}}{\delta^{*}\mathbf{x}^{j}} - \frac{\delta^{*}\rho_{jk}}{\delta^{*}\mathbf{x}^{h}}\right), \tag{58}$$

$$\mathcal{F}^{i}_{\beta\ k} := \frac{1}{2}\rho^{ih}\left(\frac{\delta^{*}\rho_{hk}}{\delta^{*}\mathbf{u}^{\beta}}\right) =: \mathcal{F}^{i}_{k\ \beta}, \tag{59}$$

$$\mathcal{F}^{i}_{\beta\ \gamma} := -\frac{1}{2}\rho^{ih}\left(\frac{\delta^{*}\sigma_{\beta\gamma}}{\delta^{*}\mathbf{x}^{h}}\right), \tag{60}$$

$$\mathfrak{F}^{\alpha}_{j\ k} := -\frac{1}{2f^{2}}\sigma^{\alpha\lambda}\left(\frac{\delta^{*}\rho_{jk}}{\delta^{*}\mathbf{u}^{\lambda}}\right), \tag{61}$$

$$\mathfrak{F}^{\alpha}_{\beta\ k} := \frac{1}{2f^{2}}\sigma^{\alpha\lambda}\left(\frac{\delta^{*}f^{2}\sigma_{\lambda\beta}}{\delta^{*}\mathbf{x}^{k}}\right) =: \mathfrak{F}^{\alpha}_{k\ \beta}, \tag{62}$$

$$\mathfrak{F}^{\alpha}_{\beta\ \gamma} := \frac{1}{2}\sigma^{\alpha\lambda}\left(\frac{\delta^{*}\sigma_{\lambda\beta}}{\delta^{*}\mathbf{u}^{\gamma}} + \frac{\delta^{*}\sigma_{\gamma\lambda}}{\delta^{*}\mathbf{u}^{\beta}} - \frac{\delta^{*}\sigma_{\beta\gamma}}{\delta^{*}\mathbf{u}^{\lambda}}\right), \tag{63}$$

$$\begin{cases} \text{(a) } \mathcal{C}^{i}_{j\ k} := \frac{1}{2}\rho^{ih}\frac{\partial^{*}\rho_{jk}}{\partial^{*}\mathrm{y}^{h}}, \\ \text{(b) } \mathcal{C}^{i}_{j\ \beta} := \frac{1}{2}\rho^{ih}\frac{\partial^{*}\rho_{h\,j}}{\partial^{*}\mathrm{v}^{\beta}} =: \mathcal{C}^{i}_{\beta\ j}, \\ \text{(c) } \mathcal{C}^{i}_{\beta\ \gamma} := \frac{1}{2}\rho^{ih}\frac{\partial^{*}f^{2}\sigma_{\beta\gamma}}{\partial^{*}\mathrm{y}^{h}}, \\ \text{(d) } \mathfrak{C}^{\alpha}_{j\ k} := \frac{1}{2f^{2}}\sigma^{\alpha\gamma}\frac{\partial^{*}\rho_{jk}}{\partial^{*}\mathrm{v}^{\gamma}}, \\ \text{(e) } \mathfrak{C}^{\alpha}_{\beta\ k} := \frac{1}{2}\sigma^{\alpha\gamma}\frac{\partial^{*}\sigma_{\gamma\beta}}{\partial^{*}\mathrm{y}^{k}} =: \mathfrak{C}^{\alpha}_{k\ \beta}, \\ \text{(f) } \mathfrak{C}^{\alpha}_{\beta\ \gamma} := \frac{1}{2}\sigma^{\alpha\lambda}\frac{\partial^{*}\sigma_{\beta\gamma}}{\partial^{*}\mathrm{v}^{\lambda}}. \end{cases} \tag{64}$$

From the previous observation, we have the following corollaries (for detailed see [1, 6]).

Corollary 2.1. *By direct calculations using* (58), (59), *and* (63), *it is deduced that*

$$\begin{cases} (a)\ \mathcal{F}^{i}_{j\ k} = \hat{F}^{i}_{j\ k}, \\ (b)\ \mathcal{F}^{i}_{j\ \alpha} = -\frac{1}{2}\rho^{ih}G^{l}_{\alpha}\frac{\partial\rho_{hj}}{\partial y^{l}}, \\ (c)\ \mathcal{F}^{i}_{\alpha\ \beta} = -\frac{1}{2}\rho^{ih}\left(\frac{\partial f^{2}}{\partial x^{h}}\sigma_{\alpha\beta} - f^{2}\mathfrak{G}^{\mu}_{h}\frac{\partial\sigma_{\alpha\beta}}{\partial v^{\mu}}\right), \\ (d)\ \mathfrak{F}^{\gamma}_{j\ k} = -\frac{1}{2f^{2}}\sigma^{\gamma\lambda}\mathcal{G}^{l}_{\lambda}\frac{\partial\rho_{jk}}{\partial y^{l}}, \\ (e)\ \mathfrak{F}^{\gamma}_{i\ \beta} = \frac{1}{2f^{2}}\sigma^{\gamma\lambda}\left(\frac{\partial f^{2}}{\partial x^{i}}\sigma_{\beta\lambda} - f^{2}\mathfrak{G}^{\mu}_{i}\frac{\partial\sigma_{\beta\lambda}}{\partial v^{\mu}}\right), \\ (f)\ \mathfrak{F}^{\gamma}_{\alpha\ \beta} = \check{F}^{\gamma}_{\alpha\ \beta}, \end{cases} \tag{65}$$

where $\hat{F}^{i}_{j\ k} := \frac{1}{2}\rho^{ih}\left(\frac{\delta\rho_{hj}}{\delta x^{k}} + \frac{\delta\rho_{kh}}{\delta x^{j}} - \frac{\delta\rho_{jk}}{\delta x^{h}}\right)$ *and* $\check{F}^{\alpha}_{\beta\ \gamma} := \frac{1}{2}\sigma^{\alpha\lambda}\left(\frac{\delta\sigma_{\lambda\beta}}{\delta u^{\gamma}} + \frac{\delta\sigma_{\gamma\lambda}}{\delta u^{\beta}} - \frac{\delta\sigma_{\beta\gamma}}{\delta u^{\lambda}}\right).$

Corollary 2.2. *Using Corollary* 2.1, *we obtain*

$$\begin{cases} (a)\ \mathcal{F}^{i}_{j\ k}y^{k} = \hat{G}^{i}_{j}, \\ (b)\ \mathcal{F}^{i}_{j\ \alpha}v^{\alpha} = \mathcal{G}^{i}_{j} - \hat{G}^{i}_{j}, \\ (c)\ \mathcal{F}^{i}_{\alpha\ k}y^{k} = 0, \\ (d)\ \mathcal{F}^{i}_{\alpha\ \beta}v^{\beta} = \mathcal{G}^{i}_{\alpha}, \\ (e)\ \mathfrak{F}^{\gamma}_{j\ k}y^{k} = 0, \\ (f)\ \mathfrak{F}^{\gamma}_{i\ \beta}v^{\beta} = \mathfrak{G}^{\gamma}_{i}, \\ (g)\ \mathfrak{F}^{\gamma}_{\beta\ k}y^{k} = \mathfrak{G}^{\gamma}_{\beta} - \check{G}^{\gamma}_{\beta}, \\ (h)\ \mathfrak{F}^{\gamma}_{\alpha\ \beta}v^{\beta} = \check{G}^{\gamma}_{\alpha}\ . \end{cases} \tag{66}$$

Corollary 2.3. *By direct calculations using* (64)*, we have*

$$\begin{cases} \text{(a) } \mathcal{C}^{i}_{j\ k} = \hat{C}^{i}_{j\ k}, \\ \text{(b) } \mathcal{C}^{i}_{j\ \beta} = \mathcal{G}^{l}_{\beta}\hat{C}^{i}_{j\ l} =: \mathcal{C}^{i}_{\beta\ j}, \\ \text{(c) } \mathcal{C}^{i}_{\beta\ \gamma} = \dfrac{1}{4}\rho^{ih}\dfrac{\partial F_2^2}{\partial v^{\mu}}\dfrac{\partial f^2}{\partial x^h}\check{C}^{\mu}_{\beta\ \gamma}, \\ \text{(d) } \mathfrak{C}^{\alpha}_{j\ k} = -\dfrac{1}{2f^2}\dfrac{\partial f^2}{\partial x^i}\hat{C}^{i}_{j\ k}v^{\alpha}, \\ \text{(e) } \mathfrak{C}^{\alpha}_{\beta\ k} = \mathfrak{G}^{\mu}_{k}\check{C}^{\ \alpha}_{\beta\ \mu} = \mathfrak{C}^{\alpha}_{k\ \beta}, \\ \text{(f) } \mathfrak{C}^{\alpha}_{\beta\ \gamma} = \check{C}^{\ \alpha}_{\beta\ \gamma}, \end{cases} \tag{67}$$

where $\hat{C}^{i}_{j\ k} = \dfrac{1}{2}\rho^{ih}\dfrac{\partial\rho_{kj}}{\partial y^h}$ *and* $\check{C}^{\alpha}_{\beta\ \gamma} = \dfrac{1}{2}\sigma^{\alpha\mu}\dfrac{\partial\sigma_{\beta\gamma}}{\partial v^{\mu}}$.

Proposition 2.4. *Let* $(M_1 \times_f M_2, F)$ *be a warped product Finsler manifold and the* $f : M_1 \to \mathbf{R}_+$ *is not constant, then according to Corollary* 2.3*,* $(M_1 \times_f M_2, F)$ *is a Riemannian manifold, if and only if* (M_1, F_1) *and* (M_2, F_2) *are Riemannian manifolds.*

Proof. According to (67) if $f : M_1 \to \mathbf{R}_+$ is not constant, then $\hat{C}^{i}_{j\ k} = 0$ if and only if $\mathfrak{C}^{\alpha}_{j\ k} = 0$. Then by using (67) the proof is complete.

□

3. The Warped Sasaki-Matsumoto Lift

Let $\mathbb{F}^{n_1+n_2} = (M = M_1 \times_f M_2, F)$ and $g_{ab} = \rho_{ij} + f^2\sigma_{\alpha\beta}$ be the warped metric on $M = M_1 \times_f M_2$. Then, the Sasaki-Matsumoto lift of the g_{ab} can be introduced as follows (see [1]):

$$\begin{aligned} {}^{*}\mathbf{G} := {} & \rho_{ij}dx^i \otimes dx^j + \frac{1}{a}(\rho_{ij} - f^2\sigma_{\alpha\beta}\mathbf{N}^{\alpha}_{i}\mathbf{N}^{\beta}_{j})\delta^{*}\mathbf{y}^i \otimes \delta^{*}\mathbf{y}^j \\ & + f^2\sigma_{\alpha\beta}du^{\alpha} \otimes du^{\beta} + \frac{1}{a}(-\mathbf{N}^{i}_{\alpha}\mathbf{N}^{j}_{\beta}\rho_{ij} + f^2\sigma_{\alpha\beta})\delta^{*}\mathbf{v}^{\alpha} \otimes \delta^{*}\mathbf{v}^{\beta}, \end{aligned} \tag{68}$$

where

$$a := 1 - \mathbf{N}_{\gamma}^{h}\mathbf{N}_{\mu}^{k}\mathbf{N}_{h}^{\gamma}\mathbf{N}_{k}^{\mu}. \tag{69}$$

Corollary 3.1. *Let* $\mathbb{F} = (M_1 \times_f M_2, F)$ *be a warped product Finsler manifold and let* $f : M_1 \to \mathbf{R}_+$ *is not constant. Then the warped horizontal* $\mathcal{H}TM$ *is orthogonal to the warped vertical* ${}^*\mathcal{V}TM$ *with respect to the warped Sasaki-Matsumoto metric* ${}^*\mathbf{G}$.

Proof. From (14), (20), (21) and (68), we have

$$^*\mathbf{G} = \left(\frac{\delta^*}{\delta^*\mathbf{x}^i}, \frac{\partial^*}{\partial^*\mathbf{y}^j}\right) = 0, \quad ^*\mathbf{G} = \left(\frac{\delta^*}{\delta^*\mathbf{x}^i}, \frac{\partial^*}{\partial^*\mathbf{v}^\beta}\right) = 0,$$

$$^*\mathbf{G} = \left(\frac{\delta^*}{\delta^*\mathbf{u}^\alpha}, \frac{\partial^*}{\partial^*\mathbf{y}^j}\right) = 0, \quad ^*\mathbf{G} = \left(\frac{\delta^*}{\delta^*\mathbf{u}^\alpha}, \frac{\partial^*}{\partial^*\mathbf{v}^\beta}\right) = 0.$$

□

Let $(TM^\circ, {}^*\mathbf{G})$ and let ${}^*\widetilde{\nabla}$ be the Levi-Civita connection on $(TM^\circ, {}^*\mathbf{G})$, that is, ${}^*\widetilde{\nabla}$ is given by

$$2{}^*\mathbf{G}({}^*\widetilde{\nabla}_X Y, Z) = X({}^*\mathbf{G}(Y, Z)) + Y({}^*\mathbf{G}(Z, X)) - Z({}^*\mathbf{G}(X, Y))$$
$$+ {}^*\mathbf{G}([X, Y], Z) - {}^*\mathbf{G}([Y, Z], X) + {}^*\mathbf{G}([Z, X], Y), \tag{70}$$

for any $X, Y \in \Gamma(TM^\circ)$.

Proposition 3.2 ([1])**.** *Let* $\mathbb{F} = (M_1 \times_f M_2, F)$ *be warped product Finsler manifold. Then the Levi-Civita connection* ${}^*\widetilde{\nabla}$ *on* $(TM^\circ, {}^*\mathbf{G})$ *is locally expressed as follows*:

$${}^*\widetilde{\nabla}_{\frac{\delta^*}{\delta^*\mathbf{x}^j}} \frac{\delta^*}{\delta^*\mathbf{x}^i} = -\left(\mathcal{C}_{i\ j}^{k} + \frac{1}{2}{}^*\mathbf{R}_{ij}^{k}\right)\frac{\partial^*}{\partial^*\mathrm{y}^k} + \mathcal{F}_{i\ j}^{k}\frac{\delta^*}{\delta^*\mathbf{x}^k}$$
$$- \left(\mathfrak{C}_{i\ j}^{\gamma} + \frac{1}{2}{}^*\mathbf{R}_{ij}^{\gamma}\right)\frac{\partial^*}{\partial^*\mathrm{v}^\gamma} + \mathfrak{F}_{i\ j}^{\gamma}\frac{\delta^*}{\delta^*\mathbf{u}^\gamma}, \tag{71}$$

$$
\begin{aligned}
{}^{*}\widetilde{\nabla}_{\frac{\delta^*}{\delta^*\mathbf{u}^\beta}} \frac{\delta^*}{\delta^*\mathbf{x}^i} = &-(\mathcal{C}^k_{i\ \beta} + \frac{1}{2}\,{}^{*}\mathbf{R}^k_{i\beta})\frac{\partial^*}{\partial^*\mathrm{y}^k} + \mathcal{F}^k_{i\ \beta}\frac{\delta^*}{\delta^*\mathbf{x}^k} \\
&-(\mathfrak{C}^\gamma_{i\ \beta} + \frac{1}{2}\,{}^{*}\mathbf{R}^\gamma_{i\beta})\frac{\partial^*}{\partial^*\mathrm{v}^\gamma} + \mathfrak{F}^\gamma_{i\ \beta}\frac{\delta^*}{\delta^*\mathbf{u}^\gamma}, \qquad (72)
\end{aligned}
$$

$$
\begin{aligned}
{}^{*}\widetilde{\nabla}_{\frac{\delta^*}{\delta^*\mathbf{x}^j}} \frac{\delta^*}{\delta^*\mathbf{u}^\alpha} = &-(\mathcal{C}^k_{\alpha\ j} + \frac{1}{2}\,{}^{*}\mathbf{R}^k_{\alpha j})\frac{\partial^*}{\partial^*\mathrm{y}^k} + \mathcal{F}^k_{\alpha\ j}\frac{\delta^*}{\delta^*\mathbf{x}^k} \\
&-(\mathfrak{C}^\gamma_{\alpha\ j} + \frac{1}{2}\,{}^{*}\mathbf{R}^\gamma_{\alpha j})\frac{\partial^*}{\partial^*\mathrm{v}^\gamma} + \mathfrak{F}^\gamma_{\alpha\ j}\frac{\delta^*}{\delta^*\mathbf{u}^\gamma}, \qquad (73)
\end{aligned}
$$

$$
\begin{aligned}
{}^{*}\widetilde{\nabla}_{\frac{\delta^*}{\delta^*\mathbf{u}^\beta}} \frac{\delta^*}{\delta^*\mathbf{u}^\alpha} = &-(\mathcal{C}^k_{\alpha\ \beta} + \frac{1}{2}\,{}^{*}\mathbf{R}^k_{\alpha\beta})\frac{\partial^*}{\partial^*\mathrm{y}^k} + \mathcal{F}^k_{\alpha\ \beta}\frac{\delta^*}{\delta^*\mathbf{x}^k} \\
&-(\mathfrak{C}^\gamma_{\alpha\ \beta} + \frac{1}{2}\,{}^{*}\mathbf{R}^\gamma_{\alpha\beta})\frac{\partial^*}{\partial^*\mathrm{v}^\gamma} + \mathfrak{F}^\gamma_{\alpha\ \beta}\frac{\delta^*}{\delta^*\mathbf{u}^\gamma}, \qquad (74)
\end{aligned}
$$

$$
{}^{*}\widetilde{\nabla}_{\frac{\partial^*}{\partial^*\mathrm{y}^j}} \frac{\partial^*}{\partial^*\mathrm{y}^i} = (\mathcal{C}^k_{i\ j}\frac{\partial^*}{\partial^*\mathrm{y}^k} - \frac{1}{2}\rho_{ij|_* h}\rho^{hk}\frac{\delta^*}{\delta^*\mathbf{x}^k} + \mathfrak{C}^\gamma_{i\ j}\frac{\partial^*}{\partial^*\mathrm{v}^\gamma}, \qquad (75)
$$

$$
{}^{*}\widetilde{\nabla}_{\frac{\partial^*}{\partial^*\mathrm{v}^\beta}} \frac{\partial^*}{\partial^*\mathrm{y}^i} = \mathcal{C}^k_{i\ \beta}\frac{\partial^*}{\partial^*\mathrm{y}^k} + \mathfrak{C}^\gamma_{i\ \beta}\frac{\partial^*}{\partial^*\mathrm{v}^\gamma} = {}^{*}\widetilde{\nabla}_{\frac{\partial^*}{\partial^*\mathrm{y}^i}} \frac{\partial^*}{\partial^*\mathrm{v}^\beta}, \qquad (76)
$$

$$
\begin{aligned}
{}^{*}\widetilde{\nabla}_{\frac{\partial^*}{\partial^*\mathrm{v}^\beta}} \frac{\partial^*}{\partial^*\mathrm{v}^\alpha} = &\ \mathcal{C}^k_{\alpha\ \beta}\frac{\partial^*}{\partial^*\mathrm{y}^k} - \frac{1}{2}\sigma_{\alpha\beta|_* h}\rho^{hk}\frac{\delta^*}{\delta^*\mathbf{x}^k} \\
&+ \mathfrak{C}^\gamma_{\alpha\ \beta}\frac{\partial^*}{\partial^*\mathrm{v}^\gamma} - \frac{f^2}{2}\sigma_{\alpha\beta|_*\lambda}\sigma^{\lambda\gamma}\frac{\delta^*}{\delta^*\mathbf{u}^\gamma}, \qquad (77)
\end{aligned}
$$

$$
\begin{aligned}
{}^{*}\widetilde{\nabla}_{\frac{\delta^*}{\delta^*\mathbf{x}^j}} \frac{\partial^*}{\partial^* \mathrm{y}^i} &= \mathcal{F}^{k}_{i\ j} \frac{\partial^*}{\partial^* \mathrm{y}^k} + \left(\mathcal{C}^{k}_{i\ j} + \frac{1}{2}\rho_{ih}\rho^{lk}\,{}^{*}\mathbf{R}^{h}_{lj}\right)\frac{\delta^*}{\delta^*\mathbf{x}^k} \\
&\quad + \mathfrak{F}^{\gamma}_{i\ j} \frac{\partial^*}{\partial^* \mathrm{v}^\gamma} + \left(\mathfrak{C}^{\gamma}_{i\ j} + \frac{f^2}{2}\rho_{ih}\,{}^{*}\mathbf{R}^{h}_{\lambda j}\sigma^{\lambda\gamma}\right)\frac{\delta^*}{\delta^*\mathbf{u}^\gamma} \\
&= {}^{*}\widetilde{\nabla}_{\frac{\partial^*}{\partial^*\mathrm{y}^i}} \frac{\delta^*}{\delta^*\mathbf{x}^j} + \mathcal{G}^{k}_{i\ j}\frac{\partial^*}{\partial^* \mathrm{y}^k} + \mathfrak{G}^{\gamma}_{i\ j}\frac{\partial^*}{\partial^* \mathrm{v}^\gamma}, \qquad (78)
\end{aligned}
$$

$$
\begin{aligned}
{}^{*}\widetilde{\nabla}_{\frac{\delta^*}{\delta^*\mathbf{u}^\beta}} \frac{\partial^*}{\partial^* \mathrm{y}^i} &= \mathcal{F}^{k}_{i\ \beta} \frac{\partial^*}{\partial^* \mathrm{y}^k} + \left(\mathcal{C}^{k}_{i\ \beta} + \frac{1}{2}\rho_{ih}\rho^{lk}\,{}^{*}\mathbf{R}^{h}_{l\beta}\right)\frac{\delta^*}{\delta^*\mathbf{x}^k} \\
&\quad + \mathfrak{F}^{\gamma}_{i\ \beta} \frac{\partial^*}{\partial^* \mathrm{v}^\gamma} + \left(\mathfrak{C}^{\gamma}_{i\ \beta} + \frac{f^2}{2}\rho_{ih}\sigma^{\lambda\gamma}\ {}^{*}\mathbf{R}^{h}_{\lambda\beta}\right)\frac{\delta^*}{\delta^*\mathbf{u}^\gamma} \\
&= {}^{*}\widetilde{\nabla}_{\frac{\partial^*}{\partial^*\mathrm{y}^i}} \frac{\delta^*}{\delta^*\mathbf{u}^\beta} + \mathcal{G}^{k}_{i\ \beta}\frac{\partial^*}{\partial^* \mathrm{y}^k} + \mathfrak{G}^{\gamma}_{i\ \beta}\frac{\partial^*}{\partial^* \mathrm{v}^\gamma}, \qquad (79)
\end{aligned}
$$

$$
\begin{aligned}
{}^{*}\widetilde{\nabla}_{\frac{\delta^*}{\delta^*\mathbf{x}^j}} \frac{\partial^*}{\partial^* \mathrm{v}^\alpha} &= \mathcal{F}^{k}_{\alpha\ j} \frac{\partial^*}{\partial^* \mathrm{y}^k} + \left(\mathcal{C}^{k}_{\alpha\ j} + \frac{f^2}{2}\sigma_{\alpha\lambda}\rho^{lk}\,{}^{*}\mathbf{R}^{\lambda}_{lj}\right)\frac{\delta^*}{\delta^*\mathbf{x}^k} \\
&\quad + \mathfrak{F}^{\gamma}_{\alpha\ j} \frac{\partial^*}{\partial^* \mathrm{v}^\gamma} + \left(\mathfrak{C}^{\gamma}_{\alpha\ j} + \frac{1}{2}\sigma_{\alpha\lambda}\sigma^{\beta\gamma}\,{}^{*}\mathbf{R}^{\lambda}_{\beta j}\right)\frac{\delta^*}{\delta^*\mathbf{u}^\gamma} \\
&= {}^{*}\widetilde{\nabla}_{\frac{\partial^*}{\partial^*\mathrm{v}^\alpha}} \frac{\delta^*}{\delta^*\mathbf{x}^j} + \mathcal{G}^{k}_{\alpha\ j}\frac{\partial^*}{\partial^* \mathrm{y}^k} + \mathfrak{G}^{\gamma}_{\alpha\ j}\frac{\partial^*}{\partial^* \mathrm{v}^\gamma}, \qquad (80)
\end{aligned}
$$

$$
\begin{aligned}
{}^{*}\widetilde{\nabla}_{\frac{\delta^*}{\delta^*\mathbf{u}^\beta}} \frac{\partial^*}{\partial^* \mathrm{v}^\alpha} &= \mathcal{F}^{k}_{\alpha\ \beta} \frac{\partial^*}{\partial^* \mathrm{y}^k} + \left(\mathcal{C}^{k}_{\alpha\ \beta} + \frac{f^2}{2}\sigma_{\alpha\lambda}\rho^{lk}\,{}^{*}\mathbf{R}^{\lambda}_{l\beta}\right)\frac{\delta^*}{\delta^*\mathbf{x}^k} \\
&\quad + \mathfrak{F}^{\gamma}_{\alpha\ \beta} \frac{\partial^*}{\partial^* \mathrm{v}^\gamma} + \left(\mathfrak{C}^{\gamma}_{\alpha\ \beta} + \frac{1}{2}\sigma_{\alpha\lambda}\sigma^{\mu\gamma}\,{}^{*}\mathbf{R}^{\lambda}_{\mu\beta}\right)\frac{\delta^*}{\delta^*\mathbf{u}^\gamma} \\
&= {}^{*}\widetilde{\nabla}_{\frac{\partial^*}{\partial^*\mathrm{v}^\alpha}} \frac{\delta^*}{\delta^*\mathbf{u}^\beta} + \mathcal{G}^{k}_{\alpha\ \beta}\frac{\partial^*}{\partial^* \mathrm{y}^k} + \mathfrak{G}^{\gamma}_{\alpha\ \beta}\frac{\partial^*}{\partial^* \mathrm{v}^\gamma}, \qquad (81)
\end{aligned}
$$

where

$$\rho_{ij|_* h} = \frac{\delta^* \rho_{ij}}{\delta^* \mathbf{x}^h} - \rho_{kj} \mathcal{F}^k_{i\ h} - \rho_{ik} \mathcal{F}^k_{j\ h},$$

$$\sigma_{\alpha\beta|_* h} = \frac{\delta^* \sigma_{\alpha\beta}}{\delta^* \mathbf{x}^h} - \sigma_{\gamma\beta} \mathfrak{F}^\gamma_{\alpha\ h} - \sigma_{\alpha\gamma} \mathfrak{F}^\gamma_{\beta\ h},$$

$$\sigma_{\alpha\beta|_* \lambda} = \frac{\delta^* \sigma_{\alpha\beta}}{\delta^* \mathbf{u}^\lambda} - \sigma_{\gamma\beta} \mathfrak{F}^\gamma_{\alpha\ \lambda} - \sigma_{\alpha\gamma} \mathfrak{F}^\gamma_{\beta\ \lambda}.$$

4. The Schouten-Van Kampen and the Vranceanu Connections on Warped Product Finsler Manifold

In this section, we consider two linear connections on $M = M_1 \times_f M_2$ with respect to which both distributions $\mathcal{H}TM$ and ${}^*\mathcal{V}TM$ are parallel. They were introduced in the first half of the last century by Schouten and Van Kampen [7] and Vranceanu [8] for studying the Riemannian geometry of a non-holonomic spaces. In the modern terminology, a non-holonomic space is a manifold endowed with a non-integrable distribution.

Definition 4.1. Let $M = M_1 \times_f M_2$, and $(TM^\circ, {}^*\mathbf{G})$ be the Riemannian manifold, where ${}^*\mathbf{G}$ is the warped Sasaki-Matsumoto metric on TM° by given (68). Also, ${}^*\widetilde{\nabla}$ be the Levi-Civita connection on $(TM^\circ, {}^*\mathbf{G})$ and ${}^*v$ and ${}^*h$ be the warped vertical projective and the warped horizontal projective, respectively. So the **warped Schouten-Van Kampen** ∇° and the **warped Vranceanu connections** ∇ are given by as follows respectively,

$$\nabla^\circ_X Y = {}^*v({}^*\widetilde{\nabla}_X\ {}^*vY) + {}^*h({}^*\widetilde{\nabla}_X\ {}^*hY), \tag{82}$$

$$\nabla_X Y = {}^*v({}^*\widetilde{\nabla}_{{}^*vX}\ {}^*vY) + {}^*h({}^*\widetilde{\nabla}_{{}^*hX}\ {}^*hY) + {}^*v[{}^*hX, {}^*vY] + {}^*h[{}^*vX, {}^*hY], \tag{83}$$

for any $X, Y \in \Gamma(TM_1 \oplus TM_2)$.

Now, by using Proposition 3.2 and (82) and (83), we can prove the following:

Proposition 4.2. *The warped Schouten-Van Kampen connection* ∇° *on* $(TM^{\circ},\ {}^{*}\mathbf{G})$ *is locally expressed as follows*:

$$\nabla^{\circ}_{\frac{\delta^*}{\delta^*\mathbf{x}^i}} \frac{\delta^*}{\delta^*\mathbf{x}^j} = \mathcal{F}_{i\ j}^{k} \frac{\delta^*}{\delta^*\mathbf{x}^k} + \mathfrak{F}_{i\ j}^{\gamma} \frac{\delta^*}{\delta^*\mathbf{u}^\gamma}, \tag{84}$$

$$\nabla^{\circ}_{\frac{\delta^*}{\delta^*\mathbf{u}^\alpha}} \frac{\delta^*}{\delta^*\mathbf{x}^j} = \mathcal{F}_{\alpha\ j}^{k} \frac{\delta^*}{\delta^*\mathbf{x}^k} + \mathfrak{F}_{\alpha\ j}^{\gamma} \frac{\delta^*}{\delta^*\mathbf{u}^\gamma}, \tag{85}$$

$$\nabla^{\circ}_{\frac{\delta^*}{\delta^*\mathbf{x}^i}} \frac{\delta^*}{\delta^*\mathbf{u}^\beta} = \mathcal{F}_{i\ \beta}^{k} \frac{\delta^*}{\delta^*\mathbf{x}^k} + \mathfrak{F}_{i\ \beta}^{\gamma} \frac{\delta^*}{\delta^*\mathbf{u}^\gamma}, \tag{86}$$

$$\nabla^{\circ}_{\frac{\delta^*}{\delta^*\mathbf{u}^\alpha}} \frac{\delta^*}{\delta^*\mathbf{u}^\beta} = \mathcal{F}_{\alpha\ \beta}^{k} \frac{\delta^*}{\delta^*\mathbf{x}^k} + \mathfrak{F}_{\alpha\ \beta}^{\gamma} \frac{\delta^*}{\delta^*\mathbf{u}^\gamma}, \tag{87}$$

$$\nabla^{\circ}_{\frac{\partial^*}{\partial^*\mathbf{y}^i}} \frac{\partial^*}{\partial^*\mathbf{y}^j} = \mathcal{C}_{i\ j}^{k} \frac{\partial^*}{\partial^*\mathbf{y}^k} + \mathfrak{C}_{i\ j}^{\gamma} \frac{\partial^*}{\partial^*\mathbf{v}^\gamma}, \tag{88}$$

$$\nabla^{\circ}_{\frac{\partial^*}{\partial^*\mathbf{v}^\alpha}} \frac{\partial^*}{\partial^*\mathbf{y}^j} = \mathcal{C}_{\alpha\ j}^{k} \frac{\partial^*}{\partial^*\mathbf{y}^k} + \mathfrak{C}_{\alpha\ j}^{\gamma} \frac{\partial^*}{\partial^*\mathbf{v}^\gamma}, \tag{89}$$

$$\nabla^{\circ}_{\frac{\partial^*}{\partial^*\mathbf{y}^i}} \frac{\partial^*}{\partial^*\mathbf{v}^\beta} = \mathcal{C}_{i\ \beta}^{k} \frac{\partial^*}{\partial^*\mathbf{y}^k} + \mathfrak{C}_{i\ \beta}^{\gamma} \frac{\partial^*}{\partial^*\mathbf{v}^\gamma}, \tag{90}$$

$$\nabla^{\circ}_{\frac{\partial^*}{\partial^*\mathbf{v}^\alpha}} \frac{\partial^*}{\partial^*\mathbf{v}^\beta} = \mathcal{C}_{\alpha\ \beta}^{k} \frac{\partial^*}{\partial^*\mathbf{y}^k} + \mathfrak{C}_{\alpha\ \beta}^{\gamma} \frac{\partial^*}{\partial^*\mathbf{v}^\gamma}, \tag{91}$$

$$\nabla^{\circ}_{\frac{\partial^*}{\partial^*\mathbf{y}^i}} \frac{\delta^*}{\delta^*\mathbf{x}^j} = \left(\mathcal{C}^k_{i\ j} + \frac{1}{2}\rho_{ih}{}^*\mathbf{R}^h_{lj}\rho^{lk}\right)\frac{\delta^*}{\delta^*\mathbf{x}^k}$$

$$+ \left(\mathfrak{C}^{\gamma}_{i\ j} + \frac{1}{2}\frac{1}{f^2}\rho_{ih}{}^*\mathbf{R}^h_{\lambda j}\sigma^{\lambda\gamma}\right)\frac{\delta^*}{\delta^*\mathbf{u}^{\gamma}}, \tag{92}$$

$$\nabla^{\circ}_{\frac{\partial^*}{\partial^*\mathbf{v}^{\alpha}}} \frac{\delta^*}{\delta^*\mathbf{x}^j} = \left(\mathcal{C}^k_{\alpha\ j} + \frac{1}{2}f^2\sigma_{\alpha\lambda}{}^*\mathbf{R}^{\lambda}_{lj}\rho^{lk}\right)\frac{\delta^*}{\delta^*\mathbf{x}^k}$$

$$+ \left(\mathfrak{C}^{\gamma}_{\alpha\ j} + \frac{1}{2}\sigma_{\alpha\lambda}{}^*\mathbf{R}^h_{\mu j}\sigma^{\mu\gamma}\right)\frac{\delta^*}{\delta^*\mathbf{u}^{\gamma}}, \tag{93}$$

$$\nabla^{\circ}_{\frac{\partial^*}{\partial^*\mathbf{y}^i}} \frac{\delta^*}{\delta^*\mathbf{u}^{\beta}} = \left(\mathcal{C}^k_{i\ \beta} + \frac{1}{2}\rho_{ih}{}^*\mathbf{R}^h_{l\alpha}\rho^{lk}\right)\frac{\delta^*}{\delta^*\mathbf{x}^k}$$

$$+ \left(\mathfrak{C}^{\gamma}_{i\ \beta} + \frac{1}{2}\frac{1}{f^2}\rho_{ih}{}^*\mathbf{R}^h_{\lambda\beta}\sigma^{\lambda\gamma}\right)\frac{\delta^*}{\delta^*\mathbf{u}^{\gamma}}, \tag{94}$$

$$\nabla^{\circ}_{\frac{\partial^*}{\partial^*\mathbf{v}^{\alpha}}} \frac{\delta^*}{\delta^*\mathbf{u}^{\beta}} = \left(\mathcal{C}^k_{\alpha\ \beta} + \frac{1}{2}f^2\sigma_{\alpha\lambda}{}^*\mathbf{R}^{\lambda}_{l\beta}\rho^{lk}\right)\frac{\delta^*}{\delta^*\mathbf{x}^k}$$

$$+ \left(\mathfrak{C}^{\gamma}_{\alpha\ \beta} + \frac{1}{2}\sigma_{\alpha\lambda}{}^*\mathbf{R}^{\lambda}_{\mu\beta}\sigma^{\mu\gamma}\right)\frac{\delta^*}{\delta^*\mathbf{u}^{\gamma}}, \tag{95}$$

$$\nabla^{\circ}_{\frac{\delta^*}{\delta^*\mathbf{x}^i}} \frac{\partial^*}{\partial^*\mathbf{y}^j} = \mathcal{F}^k_{i\ j}\frac{\partial^*}{\partial^*\mathbf{y}^k} + \mathfrak{F}^{\gamma}_{i\ j}\frac{\partial^*}{\partial^*\mathbf{v}^{\gamma}}, \tag{96}$$

$$\nabla^{\circ}_{\frac{\delta^*}{\delta^*\mathbf{u}^{\alpha}}} \frac{\partial^*}{\partial^*\mathbf{y}^j} = \mathcal{F}^k_{\alpha\ j}\frac{\partial^*}{\partial^*\mathbf{y}^k} + \mathfrak{F}^{\gamma}_{\alpha\ j}\frac{\partial^*}{\partial^*\mathbf{v}^{\gamma}}, \tag{97}$$

$$\nabla^{\circ}_{\frac{\delta^*}{\delta^*\mathbf{x}^i}} \frac{\partial^*}{\partial^*\mathbf{v}^{\beta}} = \mathcal{F}^k_{i\ \beta}\frac{\partial^*}{\partial^*\mathbf{y}^k} + \mathfrak{F}^{\gamma}_{i\ \beta}\frac{\partial^*}{\partial^*\mathbf{v}^{\gamma}}, \tag{98}$$

$$\nabla^{\circ}_{\frac{\delta^*}{\delta^*\mathbf{u}^{\alpha}}} \frac{\partial^*}{\partial^*\mathbf{v}^{\beta}} = \mathcal{F}^k_{\alpha\ \beta}\frac{\partial^*}{\partial^*\mathbf{y}^k} + \mathfrak{F}^{\gamma}_{\alpha\ \beta}\frac{\partial^*}{\partial^*\mathbf{v}^{\gamma}}, \tag{99}$$

Proposition 4.3. *The warped Vranceanu connection* ∇ *on* $(TM^{\circ},\ {}^{*}\mathbf{G})$ *is locally expressed as follows*:

$$\nabla_{\frac{\partial^*}{\partial^* \mathbf{y}^i}} \frac{\partial^*}{\partial^* \mathbf{y}^j} = \mathcal{C}_{i\ j}^{k} \frac{\partial^*}{\partial^* \mathbf{y}^k} + \mathfrak{C}_{i\ j}^{\gamma} \frac{\partial^*}{\partial^* \mathbf{v}^\gamma}, \tag{100}$$

$$\nabla_{\frac{\partial^*}{\partial^* \mathbf{v}^\alpha}} \frac{\partial^*}{\partial^* \mathbf{y}^j} = \mathcal{C}_{\alpha\ j}^{k} \frac{\partial^*}{\partial^* \mathbf{y}^k} + \mathfrak{C}_{\alpha\ j}^{\gamma} \frac{\partial^*}{\partial^* \mathbf{v}^\gamma}, \tag{101}$$

$$\nabla_{\frac{\partial^*}{\partial^* \mathbf{y}^i}} \frac{\partial^*}{\partial^* \mathbf{v}^\beta} = \mathcal{C}_{i\ \beta}^{k} \frac{\partial^*}{\partial^* \mathbf{y}^k} + \mathfrak{C}_{i\ \beta}^{\gamma} \frac{\partial^*}{\partial^* \mathbf{v}^\gamma}, \tag{102}$$

$$\nabla_{\frac{\partial^*}{\partial^* \mathbf{v}^\alpha}} \frac{\partial^*}{\partial^* \mathbf{v}^\beta} = \mathcal{C}_{\alpha\ \beta}^{k} \frac{\partial^*}{\partial^* \mathbf{y}^k} + \mathfrak{C}_{\alpha\ \beta}^{\gamma} \frac{\partial^*}{\partial^* \mathbf{v}^\gamma}, \tag{103}$$

$$\nabla_{\frac{\delta^*}{\delta^* \mathbf{x}^i}} \frac{\delta^*}{\delta^* \mathbf{x}^j} = \mathcal{F}_{i\ j}^{k} \frac{\delta^*}{\delta^* \mathbf{x}^k} + \mathfrak{F}_{i\ j}^{\gamma} \frac{\delta^*}{\delta^* \mathbf{u}^\gamma}, \tag{104}$$

$$\nabla_{\frac{\delta^*}{\delta^* \mathbf{u}^\alpha}} \frac{\delta^*}{\delta^* \mathbf{x}^j} = \mathcal{F}_{\alpha\ j}^{k} \frac{\delta^*}{\delta^* \mathbf{x}^k} + \mathfrak{F}_{\alpha\ j}^{\gamma} \frac{\delta^*}{\delta^* \mathbf{u}^\gamma}, \tag{105}$$

$$\nabla_{\frac{\delta^*}{\delta^* \mathbf{x}^i}} \frac{\delta^*}{\delta^* \mathbf{u}^\beta} = \mathcal{F}_{i\ \beta}^{k} \frac{\delta^*}{\delta^* \mathbf{x}^k} + \mathfrak{F}_{i\ \beta}^{\gamma} \frac{\delta^*}{\delta^* \mathbf{u}^\gamma}, \tag{106}$$

$$\nabla_{\frac{\delta^*}{\delta^* \mathbf{u}^\alpha}} \frac{\delta^*}{\delta^* \mathbf{u}^\beta} = \mathcal{F}_{\alpha\ \beta}^{k} \frac{\delta^*}{\delta^* \mathbf{x}^k} + \mathfrak{F}_{\alpha\ \beta}^{\gamma} \frac{\delta^*}{\delta^* \mathbf{u}^\gamma}, \tag{107}$$

$$\nabla_{\frac{\delta^*}{\delta^* \mathbf{x}^i}} \frac{\partial^*}{\partial^* \mathbf{y}^j} = \mathbf{N}_{i\ j}^{k} \frac{\partial^*}{\partial^* \mathbf{y}^k} + \mathbf{N}_{i\ j}^{\gamma} \frac{\partial^*}{\partial^* \mathbf{v}^\gamma}, \tag{108}$$

$$\nabla_{\frac{\delta^*}{\delta^* \mathbf{u}^\alpha}} \frac{\partial^*}{\partial^* \mathbf{y}^j} = \mathbf{N}^k_{\alpha\ j} \frac{\partial^*}{\partial^* \mathbf{y}^k} + \mathbf{N}^\gamma_{\alpha\ j} \frac{\partial^*}{\partial^* \mathbf{v}^\gamma}, \tag{109}$$

$$\nabla_{\frac{\delta^*}{\delta^* \mathbf{x}^i}} \frac{\partial^*}{\partial^* \mathbf{v}^\beta} = \mathbf{N}^k_{i\ \beta} \frac{\partial^*}{\partial^* \mathbf{y}^k} + \mathbf{N}^\gamma_{i\ \beta} \frac{\partial^*}{\partial^* \mathbf{v}^\gamma}, \tag{110}$$

$$\nabla_{\frac{\delta^*}{\delta^* \mathbf{u}^\alpha}} \frac{\partial^*}{\partial^* \mathbf{v}^\beta} = \mathbf{N}^k_{\alpha\ \beta} \frac{\partial^*}{\partial^* \mathbf{y}^k} + \mathbf{N}^\gamma_{\alpha\ \beta} \frac{\partial^*}{\partial^* \mathbf{v}^\gamma}, \tag{111}$$

and

$$\nabla_{\frac{\partial^*}{\partial^* \mathbf{y}^i}} \frac{\delta^*}{\delta^* \mathbf{x}^j} = \nabla_{\frac{\partial^*}{\partial^* \mathbf{v}^\beta}} \frac{\delta^*}{\delta^* \mathbf{x}^j} = \nabla_{\frac{\partial^*}{\partial^* \mathbf{y}^i}} \frac{\delta^*}{\delta^* \mathbf{u}^\alpha} = \nabla_{\frac{\partial^*}{\partial^* \mathbf{v}^\alpha}} \frac{\delta^*}{\delta^* \mathbf{u}^\beta} = 0. \tag{112}$$

Corollary 4.4. (i) *By using Propositions* 4.2 *and* 4.3, *we can define the horizontal and vertical covariant derivatives of* $\mathbf{T} = \left(\mathbf{T}^{ij\alpha\beta}_{kh\lambda\mu}\right)$ *induced by the Schouten-Van Kampen connection are defined by*

$$\mathbf{T}^{ij\alpha\beta}_{kh\lambda\mu|^\circ t} = \frac{\delta^* \mathbf{T}^{ij\alpha\beta}_{kh\lambda\mu}}{\delta^* \mathbf{x}^t} + \mathbf{T}^{lj\alpha\beta}_{kh\lambda\mu} \mathcal{F}^{\,i}_{l\ t} + \mathbf{T}^{il\alpha\beta}_{kh\lambda\mu} \mathcal{F}^{\,j}_{l\ t} + \mathbf{T}^{lj\tau\beta}_{kh\lambda\mu} \mathfrak{F}^{\alpha}_{\tau\ t} + \mathbf{T}^{lj\alpha\tau}_{kh\lambda\mu} \mathfrak{F}^{\beta}_{\tau\ t}$$

$$- \mathbf{T}^{lj\alpha\beta}_{lh\lambda\mu} \mathcal{F}^{\,l}_{k\ t} - \mathbf{T}^{ij\alpha\beta}_{kl\lambda\mu} \mathcal{F}^{\,l}_{h\ t} - \mathbf{T}^{lj\tau\beta}_{kh\tau\mu} \mathfrak{F}^{\tau}_{\lambda\ t} - \mathbf{T}^{lj\tau\beta}_{kh\lambda\tau} \mathfrak{F}^{\tau}_{\mu\ t}, \tag{113}$$

$$\mathbf{T}^{ij\alpha\beta}_{kh\lambda\mu|^\circ \gamma} = \frac{\delta^* \mathbf{T}^{ij\alpha\beta}_{kh\lambda\mu}}{\delta^* \mathbf{u}^\gamma} + \mathbf{T}^{lj\alpha\beta}_{kh\lambda\mu} \mathcal{F}^{\,i}_{l\ \gamma} + \mathbf{T}^{il\alpha\beta}_{kh\lambda\mu} \mathcal{F}^{\,j}_{l\ \gamma} + \mathbf{T}^{lj\tau\beta}_{kh\lambda\mu} \mathfrak{F}^{\alpha}_{\tau\ \gamma} + \mathbf{T}^{lj\alpha\tau}_{kh\lambda\mu} \mathfrak{F}^{\beta}_{\tau\ \gamma}$$

$$- \mathbf{T}^{lj\alpha\beta}_{lh\lambda\mu} \mathcal{F}^{\,l}_{k\ \gamma} - \mathbf{T}^{ij\alpha\beta}_{kl\lambda\mu} \mathcal{F}^{\,l}_{h\ \gamma} - \mathbf{T}^{lj\tau\beta}_{kh\tau\mu} \mathfrak{F}^{\tau}_{\lambda\ \gamma} - \mathbf{T}^{lj\tau\beta}_{kh\lambda\tau} \mathfrak{F}^{\tau}_{\mu\ \gamma}, \tag{114}$$

and

$$\mathbf{T}^{ij\alpha\beta}_{kh\lambda\mu\|^\circ t} = \frac{\partial^* \mathbf{T}^{ij\alpha\beta}_{kh\lambda\mu}}{\partial^* \mathbf{y}^t} + \mathbf{T}^{lj\alpha\beta}_{kh\lambda\mu} \mathcal{C}^{\,i}_{l\ t} + \mathbf{T}^{il\alpha\beta}_{kh\lambda\mu} \mathcal{C}^{\,j}_{l\ t} + \mathbf{T}^{lj\tau\beta}_{kh\lambda\mu} \mathbf{L}^{\alpha}_{\tau\ t} + \mathbf{T}^{lj\alpha\tau}_{kh\lambda\mu} \mathbf{L}^{\beta}_{\tau\ t}$$

$$- \mathbf{T}^{lj\alpha\beta}_{lh\lambda\mu}\mathcal{C}^{l}_{k\ t} - \mathbf{T}^{ij\alpha\beta}_{kl\lambda\mu}\mathcal{C}^{l}_{h\ t} - \mathbf{T}^{lj\tau\beta}_{kh\tau\mu}\mathbf{L}^{\tau}_{\lambda\ t} - \mathbf{T}^{lj\tau\beta}_{kh\lambda\tau}\mathbf{L}^{\tau}_{\mu\ t}, \tag{115}$$

$$\mathbf{T}^{ij\alpha\beta}_{kh\lambda\mu\|^{\circ}\gamma} = \frac{\partial^{*}\mathbf{T}^{ij\alpha\beta}_{kh\lambda\mu}}{\partial^{*}\mathbf{v}^{\gamma}} + \mathbf{T}^{lj\alpha\beta}_{kh\lambda\mu}\mathcal{C}^{i}_{l\ \gamma} + \mathbf{T}^{il\alpha\beta}_{kh\lambda\mu}\mathcal{C}^{j}_{l\ \gamma} + \mathbf{T}^{lj\tau\beta}_{kh\lambda\mu}\mathbf{L}^{\alpha}_{\tau\ \gamma} + \mathbf{T}^{lj\alpha\tau}_{kh\lambda\mu}\mathbf{L}^{\beta}_{\tau\ \gamma}$$

$$- \mathbf{T}^{lj\alpha\beta}_{lh\lambda\mu}\mathcal{C}^{l}_{k\ \gamma} - \mathbf{T}^{ij\alpha\beta}_{kl\lambda\mu}\mathcal{C}^{l}_{h\ \gamma} - \mathbf{T}^{lj\tau\beta}_{kh\tau\mu}\mathbf{L}^{\tau}_{\lambda\ t} - \mathbf{T}^{lj\tau\beta}_{kh\lambda\tau}\mathbf{L}^{\tau}_{\mu\ \gamma}, \tag{116}$$

where $\mathbf{L}^{k}_{i\ j} = \mathcal{C}^{k}_{i\ j} + \frac{1}{2}\rho_{ih}{}^{*}\mathbf{R}^{h}_{lj}\rho^{lk}, \ldots, \mathbf{L}^{\gamma}_{\alpha\ \beta} = \mathfrak{C}^{k}_{\alpha\ \beta} + \frac{1}{2}\sigma_{\alpha\lambda}{}^{*}\mathbf{R}^{\lambda}_{\mu\beta}\sigma^{\mu\gamma}.$

(ii) *Similarly, the horizontal and vertical covariant derivatives induced by the Vranceanu connection are given by*

$$\mathbf{T}^{ij\alpha\beta}_{kh\lambda\mu|t} = \frac{\delta^{*}\mathbf{T}^{ij\alpha\beta}_{kh\lambda\mu}}{\delta^{*}\mathbf{x}^{t}} + \mathbf{T}^{lj\alpha\beta}_{kh\lambda\mu}\mathbf{N}^{i}_{l\ t} + \mathbf{T}^{il\alpha\beta}_{kh\lambda\mu}\mathbf{N}^{j}_{l\ t} + \mathbf{T}^{lj\tau\beta}_{kh\lambda\mu}\mathfrak{F}^{\alpha}_{\tau\ t} + \mathbf{T}^{lj\alpha\tau}_{kh\lambda\mu}\mathfrak{F}^{\beta}_{\tau\ t}$$

$$- \mathbf{T}^{lj\alpha\beta}_{lh\lambda\mu}\mathbf{N}^{l}_{k\ t} - \mathbf{T}^{ij\alpha\beta}_{kl\lambda\mu}\mathbf{N}^{l}_{h\ t} - \mathbf{T}^{lj\tau\beta}_{kh\tau\mu}\mathfrak{F}^{\tau}_{\lambda\ t} - \mathbf{T}^{lj\tau\beta}_{kh\lambda\tau}\mathfrak{F}^{\tau}_{\mu\ t}, \tag{117}$$

$$\mathbf{T}^{ij\alpha\beta}_{kh\lambda\mu|\gamma} = \frac{\delta^{*}\mathbf{T}^{ij\alpha\beta}_{kh\lambda\mu}}{\delta^{*}\mathbf{u}^{\gamma}} + \mathbf{T}^{lj\alpha\beta}_{kh\lambda\mu}\mathbf{N}^{i}_{l\ \gamma} + \mathbf{T}^{il\alpha\beta}_{kh\lambda\mu}\mathbf{N}^{j}_{l\ \gamma} + \mathbf{T}^{lj\tau\beta}_{kh\lambda\mu}\mathfrak{F}^{\alpha}_{\tau\ \gamma} + \mathbf{T}^{lj\alpha\tau}_{kh\lambda\mu}\mathfrak{F}^{\beta}_{\tau\ \gamma}$$

$$- \mathbf{T}^{lj\alpha\beta}_{lh\lambda\mu}\mathbf{N}^{l}_{k\ \gamma} - \mathbf{T}^{ij\alpha\beta}_{kl\lambda\mu}\mathbf{N}^{l}_{h\ \gamma} - \mathbf{T}^{lj\tau\beta}_{kh\tau\mu}\mathfrak{F}^{\tau}_{\lambda\ \gamma} - \mathbf{T}^{lj\tau\beta}_{kh\lambda\tau}\mathfrak{F}^{\tau}_{\mu\ \gamma}, \tag{118}$$

and

$$\mathbf{T}^{ij\alpha\beta}_{kh\lambda\mu\|t} = \frac{\partial^{*}\mathbf{T}^{ij\alpha\beta}_{kh\lambda\mu}}{\partial^{*}\mathbf{y}^{t}} + \mathbf{T}^{lj\alpha\beta}_{kh\lambda\mu}\mathcal{C}^{i}_{l\ t} + \mathbf{T}^{il\alpha\beta}_{kh\lambda\mu}\mathcal{C}^{j}_{l\ t} + \mathbf{T}^{lj\tau\beta}_{kh\lambda\mu}\mathfrak{C}^{\alpha}_{\tau\ t} + \mathbf{T}^{lj\alpha\tau}_{kh\lambda\mu}\mathfrak{C}^{\beta}_{\tau\ t}, \tag{119}$$

$$\mathbf{T}^{ij\alpha\beta}_{kh\lambda\mu\|\gamma} = \frac{\partial^{*}\mathbf{T}^{ij\alpha\beta}_{kh\lambda\mu}}{\partial^{*}\mathbf{v}^{\gamma}} + \mathbf{T}^{lj\alpha\beta}_{kh\lambda\mu}\mathcal{C}^{i}_{l\ \gamma} + \mathbf{T}^{il\alpha\beta}_{kh\lambda\mu}\mathcal{C}^{j}_{l\ \gamma} + \mathbf{T}^{lj\tau\beta}_{kh\lambda\mu}\mathfrak{C}^{\alpha}_{\tau\ \gamma} + \mathbf{T}^{lj\alpha\tau}_{kh\lambda\mu}\mathfrak{C}^{\beta}_{\tau\ \gamma}. \tag{120}$$

According to Proposition 3.2, we put

$$\begin{cases} \mathbf{G}_{i\ j}^{k} = -\left(\mathcal{C}_{i\ j}^{k} + \frac{1}{2}\ {}^{*}\mathbf{R}_{i\ j}^{k}\right), \\ \vdots \\ \mathbf{G}_{\alpha\ \beta}^{k} = -\left(\mathcal{C}_{\alpha\ \beta}^{k} + \frac{1}{2}\ {}^{*}\mathbf{R}_{\alpha\ \beta}^{k}\right), \\ \vdots \\ \mathbf{G}_{i\ j}^{\gamma} = -\left(\mathfrak{C}_{i\ j}^{\gamma} + \frac{1}{2}\ {}^{*}\mathbf{R}_{i\ j}^{\gamma}\right), \\ \vdots \\ \mathbf{G}_{\alpha\ \beta}^{\gamma} = -\left(\mathfrak{C}_{\alpha\ \beta}^{\gamma} + \frac{1}{2}\ {}^{*}\mathbf{R}_{\alpha\ \beta}^{\gamma}\right). \end{cases} \tag{121}$$

Now let

$$\begin{cases} \mathbf{g}_{ij} := {}^{*}\mathbf{G}\left(\frac{\delta^{*}}{\delta^{*}\mathbf{x}^{i}}, \frac{\delta^{*}}{\delta^{*}\mathbf{x}^{j}}\right), \\ \mathbf{g}_{\alpha\beta} := {}^{*}\mathbf{G}\left(\frac{\delta^{*}}{\delta^{*}\mathbf{u}^{\alpha}}, \frac{\delta^{*}}{\delta^{*}\mathbf{u}^{\beta}}\right), \\ \mathbf{g}_{i\beta} := {}^{*}\mathbf{G}\left(\frac{\delta^{*}}{\delta^{*}\mathbf{x}^{i}}, \frac{\delta^{*}}{\delta^{*}\mathbf{u}^{\beta}}\right), \end{cases} \tag{122}$$

and

$$\begin{cases} \widetilde{\mathbf{g}}_{ij} := {}^{*}\mathbf{G}\left(\frac{\partial^{*}}{\partial^{*}\mathbf{y}^{i}}, \frac{\partial^{*}}{\partial^{*}\mathbf{y}^{j}}\right), \\ \widetilde{\mathbf{g}}_{\alpha\beta} := {}^{*}\mathbf{G}\left(\frac{\partial^{*}}{\partial^{*}\mathbf{v}^{\alpha}}, \frac{\partial^{*}}{\partial^{*}\mathbf{v}^{\beta}}\right), \\ \widetilde{\mathbf{g}}_{i\beta} := {}^{*}\mathbf{G}\left(\frac{\partial^{*}}{\partial^{*}\mathbf{y}^{i}}, \frac{\partial^{*}}{\partial^{*}\mathbf{v}^{\beta}}\right). \end{cases} \tag{123}$$

Lemma 4.5. (i) *The vertical covariant derivatives* $\mathbf{g}_{ij}$, $\mathbf{g}_{\alpha\beta}$, $\mathbf{g}_{i\beta}$, $\widetilde{\mathbf{g}}_{ij}$, $\widetilde{\mathbf{g}}_{\alpha\beta}$, *and* $\widetilde{\mathbf{g}}_{i\beta}$ *with respect to the Schouten-Van Kampen connection are given by*

$$\text{(a) } \mathbf{g}_{ij\|\circ k} = 0, \quad \text{(b) } \mathbf{g}_{ij\|\circ\gamma} = 0, \tag{124}$$

$$\text{(a) } \mathbf{g}_{\alpha\beta\|\circ k} = 0, \quad \text{(b) } \mathbf{g}_{\alpha\beta\|\circ\gamma} = 0, \tag{125}$$

$$\text{(a) } \mathbf{g}_{i\beta\|\circ k} = 0, \quad \text{(b) } \mathbf{g}_{i\beta\|\circ\gamma} = 0, \tag{126}$$

$$\text{(a) } \widetilde{\mathbf{g}}_{ij\|\circ k} = 0, \quad \text{(b) } \widetilde{\mathbf{g}}_{ij\|\circ\gamma} = 0, \tag{127}$$

$$\text{(a) } \widetilde{\mathbf{g}}_{\alpha\beta\|\circ k} = 0, \quad \text{(b) } \widetilde{\mathbf{g}}_{\alpha\beta\|\circ\gamma} = 0, \tag{128}$$

$$\text{(a) } \widetilde{\mathbf{g}}_{i\beta\|\circ k} = 0, \quad \text{(b) } \widetilde{\mathbf{g}}_{i\beta\|\circ\gamma} = 0. \tag{129}$$

Similarly, with respect to the Vranceanu connection are given by

$$\begin{cases} \text{(a) } \mathbf{g}_{ij\|*k} = \frac{1}{2}\left\{\widetilde{\mathbf{g}}_{kh}(\mathbf{G}_{i\ j}^{h} + \mathbf{G}_{j\ i}^{h}) + \widetilde{\mathbf{g}}_{k\lambda}(\mathbf{G}_{i\ j}^{\lambda} + \mathbf{G}_{j\ i}^{\lambda})\right\} \\ \text{(b) } \mathbf{g}_{ij\|*\gamma} = \frac{1}{2}\left\{\widetilde{\mathbf{g}}_{\gamma h}(\mathbf{G}_{i\ j}^{h} + \mathbf{G}_{j\ i}^{h}) + \widetilde{\mathbf{g}}_{\gamma\lambda}(\mathbf{G}_{i\ j}^{\lambda} + \mathbf{G}_{j\ i}^{\lambda})\right\} \end{cases}, \tag{130}$$

$$\begin{cases} \text{(a) } \mathbf{g}_{\alpha\beta\|*k} = \frac{1}{2}\left\{\widetilde{\mathbf{g}}_{kh}(\mathbf{G}_{\alpha\ \beta}^{h} + \mathbf{G}_{\beta\ \alpha}^{h}) + \widetilde{\mathbf{g}}_{k\lambda}(\mathbf{G}_{\alpha\ \beta}^{\lambda} + \mathbf{G}_{\beta\ \alpha}^{\lambda})\right\} \\ \text{(b) } \mathbf{g}_{\alpha\beta\|*\gamma} = \frac{1}{2}\left\{\widetilde{\mathbf{g}}_{\gamma h}(\mathbf{G}_{\alpha\ \beta}^{h} + \mathbf{G}_{\beta\ \alpha}^{h}) + \widetilde{\mathbf{g}}_{\gamma\lambda}(\mathbf{G}_{\alpha\ \beta}^{\lambda} + \mathbf{G}_{\beta\ \alpha}^{\lambda})\right\} \end{cases}, \tag{131}$$

$$\begin{cases} \text{(a) } \mathbf{g}_{i\beta\|*k} = \frac{1}{2}\left\{\widetilde{\mathbf{g}}_{kh}(\mathbf{G}_{i\ \beta}^{h} + \mathbf{G}_{\beta\ i}^{h}) + \widetilde{\mathbf{g}}_{k\lambda}(\mathbf{G}_{i\ \beta}^{\lambda} + \mathbf{G}_{\beta\ i}^{\lambda})\right\} \\ \text{(b) } \mathbf{g}_{i\beta\|*\gamma} = \frac{1}{2}\left\{\widetilde{\mathbf{g}}_{\gamma h}(\mathbf{G}_{i\ \beta}^{h} + \mathbf{G}_{\beta\ i}^{h}) + \widetilde{\mathbf{g}}_{\gamma\lambda}(\mathbf{G}_{i\ \beta}^{\lambda} + \mathbf{G}_{\beta\ i}^{\lambda})\right\} \end{cases}, \tag{132}$$

and

$$\text{(a) } \widetilde{\mathbf{g}}_{ij\|*k} = 0, \quad \text{(b) } \widetilde{\mathbf{g}}_{ij\|*\gamma} = 0, \tag{133}$$

$$\text{(a) } \widetilde{\mathbf{g}}_{\alpha\beta\|*k} = 0, \quad \text{(b) } \widetilde{\mathbf{g}}_{\alpha\beta\|*\gamma} = 0, \tag{134}$$

$$\text{(a) } \widetilde{\mathbf{g}}_{i\beta\|*k} = 0, \quad \text{(b) } \widetilde{\mathbf{g}}_{i\beta\|*\gamma} = 0. \tag{135}$$

(ii) *The horizontal covariant derivatives of* $\mathbf{g}_{ij}$, $\mathbf{g}_{\alpha\beta}$, $\mathbf{g}_{i\beta}$, $\widetilde{\mathbf{g}}_{ij}$, $\widetilde{\mathbf{g}}_{\alpha\beta}$ *and* $\widetilde{\mathbf{g}}_{i\beta}$ *with respect to the Schouten-Van Kampen connection are given by*

$$\text{(a) } \mathbf{g}_{ij|\circ k} = 0, \quad \text{(b) } \mathbf{g}_{ij|\circ\gamma} = 0, \tag{136}$$

$$\text{(a) } \mathbf{g}_{\alpha\beta|\circ k} = 0, \quad \text{(b) } \mathbf{g}_{\alpha\beta|\circ\gamma} = 0, \tag{137}$$

$$\text{(a) } \mathbf{g}_{i\beta|^{\circ}k} = 0, \quad \text{(b) } \mathbf{g}_{i\beta|^{\circ}\gamma} = 0, \tag{138}$$

$$\text{(a) } \widetilde{\mathbf{g}}_{ij|^{\circ}k} = 0, \quad \text{(b) } \widetilde{\mathbf{g}}_{ij|^{\circ}\gamma} = 0, \tag{139}$$

$$\text{(a) } \widetilde{\mathbf{g}}_{\alpha\beta|^{\circ}k} = 0, \quad \text{(b) } \widetilde{\mathbf{g}}_{\alpha\beta|^{\circ}\gamma} = 0, \tag{140}$$

$$\text{(a) } \widetilde{\mathbf{g}}_{i\beta|^{\circ}k} = 0, \quad \text{(b) } \widetilde{\mathbf{g}}_{i\beta|^{\circ}\gamma} = 0, \tag{141}$$

and similarly with respect to the Vranceanu connection we have

$$\text{(a) } \mathbf{g}_{ij|*k} = 0, \quad \text{(b) } \mathbf{g}_{ij|*\gamma} = 0, \tag{142}$$

$$\text{(a) } \mathbf{g}_{\alpha\beta|*k} = 0, \quad \text{(b) } \mathbf{g}_{\alpha\beta|*\gamma} = 0, \tag{143}$$

$$\text{(a) } \mathbf{g}_{i\beta|*k} = 0, \quad \text{(b) } \mathbf{g}_{i\beta|*\gamma} = 0, \tag{144}$$

$$\text{(a) } \widetilde{\mathbf{g}}_{ij|*k} = 2\mathbf{P}_{i\ j}^{h}\mathbf{g}_{hk}, \quad \text{(b) } \widetilde{\mathbf{g}}_{ij|*\gamma} = 2\mathbf{P}_{i\ j}^{\lambda}\mathbf{g}_{\lambda\gamma}, \tag{145}$$

$$\text{(a) } \widetilde{\mathbf{g}}_{\alpha\beta|*k} = 2\mathbf{P}_{\alpha\ \beta}^{h}\mathbf{g}_{hk}, \quad \text{(b) } \widetilde{\mathbf{g}}_{\alpha\beta|*\gamma} = 2\mathbf{P}_{\alpha\ \beta}^{\lambda}\mathbf{g}_{\lambda\gamma}, \tag{146}$$

$$\text{(a) } \widetilde{\mathbf{g}}_{i\beta|*k} = 2\mathbf{P}_{i\ \beta}^{h}\mathbf{g}_{hk}, \quad \text{(b) } \widetilde{\mathbf{g}}_{i\beta|*\gamma} = 2\mathbf{P}_{i\ \beta}^{\lambda}\mathbf{g}_{\lambda\gamma}, \tag{147}$$

where $\mathbf{P}_{i\ j}^{k} := \mathcal{C}_{i\ j}^{k} + \rho_{ih}{}^{*}\mathbf{R}_{l\ j}^{h}\rho^{lk}, \ldots, \mathbf{P}_{\alpha\ \beta}^{\gamma} := \mathfrak{C}_{\alpha\ \beta}^{\gamma} + \sigma_{\alpha\mu}{}^{*}\mathbf{R}_{\tau\ \beta}^{\mu}\rho^{\tau\gamma}$.

Proof. First (127), (128), (129), (133), (134) and (135) follow from $X({}^{*}\mathbf{G}(Y, Z)) = {}^{*}\mathbf{G}(\widetilde{\nabla}_X Y, Z) + {}^{*}\mathbf{G}(Y, \widetilde{\nabla}_X Z)$ on taking $\{X = \frac{\partial^*}{\partial^* \mathbf{y}^k}, Y = \frac{\partial^*}{\partial^* \mathbf{y}^i}, Z = \frac{\partial^*}{\partial^* \mathbf{y}^j}\}, \{X = \frac{\partial^*}{\partial^* \mathbf{v}^\gamma}, Y = \frac{\partial^*}{\partial^* \mathbf{y}^i}, Z = \frac{\partial^*}{\partial^* \mathbf{y}^j}\}, \ldots$ and $\{X = \frac{\partial^*}{\partial^* \mathbf{v}^\gamma}, Y = \frac{\partial^*}{\partial^* \mathbf{v}^\alpha}, Z = \frac{\partial^*}{\partial^* \mathbf{v}^\gamma}\}$, respectively, and using (122), (123), Proposition 3.2 and Corollary 4.4. In a similar way obtain (124), (125), (126), (136), and (144). Finally, (130), (131), (132), (145), (146) and (147) is a consequence of Proposition 3.2 and (121). □

Remark 4.6. By using Lemma 4, we infer that the warped Schouten-Van Kampen connection ∇° is a metric connection, but the warped Vrăceanu connection ∇ is not metric.

Corollary 4.7. *If*

$$\mathbf{g}_{ij\|*k} = 0, \quad \mathbf{g}_{ij\|*\gamma} = 0, \tag{148}$$

$$\mathbf{g}_{\alpha\beta\|*k} = 0, \quad \mathbf{g}_{\alpha\beta\|*\gamma} = 0, \tag{149}$$

$$\mathbf{g}_{i\beta\|*k} = 0, \quad \mathbf{g}_{i\beta\|*\gamma} = 0, \tag{150}$$

then the Vranceanu connection ∇ *is a vertical metric connection. Also if*

$$\widetilde{\mathbf{g}}_{ij|*k} = 0, \quad \widetilde{\mathbf{g}}_{ij|*\gamma} = 0, \tag{151}$$

$$\widetilde{\mathbf{g}}_{\alpha\beta|*k} = 0, \quad \widetilde{\mathbf{g}}_{\alpha\beta|*\gamma} = 0, \tag{152}$$

$$\widetilde{\mathbf{g}}_{i\beta|*k} = 0, \quad \widetilde{\mathbf{g}}_{i\beta|*\gamma} = 0, \tag{153}$$

then ∇ *is a horizontal metric connection.*

5. Main Results

Suppose that $M = (M_1 \times_f M_2, F)$, then according to (46), (47), (48) and (49), we define

$$^{*}\mathbf{R} = ({}^{*}\mathbf{R}^{a}_{b\ c}) = \left({}^{*}\mathbf{R}^{i}_{j\ k}, {}^{*}\mathbf{R}^{i}_{j\ \gamma}, {}^{*}\mathbf{R}^{i}_{\beta\ k}, {}^{*}\mathbf{R}^{i}_{\beta\ \gamma}, {}^{*}\mathbf{R}^{\alpha}_{j\ k}, {}^{*}\mathbf{R}^{\alpha}_{j\ \gamma}, {}^{*}\mathbf{R}^{\alpha}_{\beta\ k}, {}^{*}\mathbf{R}^{\alpha}_{\beta\ \gamma}\right), \tag{154}$$

and called **the warped curvature tensor**.

Theorem 5.1. *Let* $M = (M_1 \times_f M_2, F)$ *and* $f : M_1 \to \mathbf{R}_+$ *be non-constant* C^{∞} *function. If M is a Riemannian manifold, then M has the warped curvature* $^{*}\mathbf{R} = 0$ *if and only if, the Schouten-Van Kampen and Vranceanu connections* ∇° *and* ∇ *defined by the Levi-Civita connection on* $(TM^{\circ}, {}^{*}\mathbf{G})$ *are coincide, that is,* $\nabla^{\circ} = \nabla$.

Proof. By using Proposition 2.4, $M = (M_1 \times_f M_2)$ is a Riemannian manifold, if and only if (M_1, F_1) and (M_2, F_2) are Riemannian manifolds. Hence, according to (57) and (67), we infer that $C = 0$, if and only if $\mathcal{C}^i_{j\,k} = 0$ and $\mathfrak{C}^{\alpha}_{\beta\,\gamma} = 0$. Then by using Propositions 4.2 and 4.3 the proof is complete. □

Now we consider $(TM^\circ, {}^*\mathbf{G})$ as the Riemannian manifold, here $M = M_1 \times_f M_2$ and ${}^*\mathcal{F}_v$ is the vertical foliation (see [1, 3]) on it. Then from Proposition 3.2, we deduced that ${}^*\mathcal{F}_v$ is *totally geodesic* if and only if,

$$\begin{cases} \mathbf{K}^{k}_{i\,j} := -\frac{1}{2}\rho_{ij|h}\rho^{hk} = 0, \\ \mathbf{K}^{k}_{\alpha\,\beta} := -\frac{1}{2}\sigma_{\alpha\beta|h}\rho^{hk} = 0, \\ \mathbf{K}^{\gamma}_{\alpha\,\beta} := -\frac{1}{2}\sigma_{\alpha\beta|\lambda}\frac{1}{f^2}\sigma^{\lambda\gamma} = 0. \end{cases} \tag{155}$$

Similarly, by using Proposition 3.2 and (122), ${}^*\mathbf{G}$ is *bundle-like* for ${}^*\mathcal{F}_v$ if and only

$$\begin{cases} \dfrac{\partial^* \mathbf{g}_{ij}}{\partial^* \mathbf{y}^k} = 0, & \dfrac{\partial^* \mathbf{g}_{ij}}{\partial^* \mathbf{v}^\gamma} = 0, \\ \dfrac{\partial^* \mathbf{g}_{\alpha\beta}}{\partial^* \mathbf{y}^k} = 0, & \dfrac{\partial^* \mathbf{g}_{\alpha\beta}}{\partial^* \mathbf{v}^\gamma} = 0, \\ \dfrac{\partial^* \mathbf{g}_{i\beta}}{\partial^* \mathbf{y}^k} = 0, & \dfrac{\partial^* \mathbf{g}_{i\beta}}{\partial^* \mathbf{v}^\gamma} = 0. \end{cases} \tag{156}$$

Using relations (155) and (156), the following theorem is obtained.

Theorem 5.2. *Let* $(M = M_1 \times_f M_2, F)$ *be a warped Finsler manifold and let* ${}^*\mathcal{F}_v$ *be the vertical foliation on M. Then we have the following assertions*:

(i) ${}^*\mathbf{G}$ *is bundle-like for* ${}^*\mathcal{F}_v$, *if and only if the Vranceanu connection is a vertical metric connection.*

(ii) ${}^*\mathcal{F}_v$ *is totally geodesic if and only if the Vranceanu connection is a horizontal metric connection.*

Corollary 5.3. *Let* $(M = M_1 \times_f M_2, F)$ *be a warped Finsler manifold and let* ${}^*\mathcal{F}_v$ *be the vertical foliation on M. Let us consider* G_1 *and* G_2 *are the Sasaki-Matsumoto metric on* $T^\circ M_1$ *and* $T^\circ M_2$, *respectively. Let* F_1 *and* F_2 *be the vertical foliations on* M_1 *and* M_2, *respectively.*

(i) *If* ${}^*\mathbf{G}$ *is bundle-like for* ${}^*\mathcal{F}_v$, *then* G_1 *and* G_2 *is not of necessity bundle-like for* F_1 *and* F_2. *Conversely, if* G_1 *and* G_2 *are bundle-like for* F_1 *and* F_2, *respectively, then* ${}^*\mathbf{G}$ *is not of necessity bundle-like for* ${}^*\mathcal{F}_v$.

(ii) *If* ${}^*\mathcal{F}_v$ *is totally geodesic, then* G_1 *and* G_2 *are not of necessity totally geodesics. Conversely, if* G_1 *and* G_2 *are totally geodesics, then* ${}^*\mathcal{F}_v$ *is not of necessity totally geodesic.*

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