Research and Communications in Mathematics and Mathematical Sciences
Vol. 16, Issue 1, 2024, Pages 21-84
ISSN 2319-6939
Published Online on May 17, 2024

http://jyotiacademicpress.org

STABILITY OF DIFFERENCE ANALOGUES OF NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS: A SURVEY OF SOME KNOWN RESULTS

LEONID SHAIKHET

Department of Mathematics
Ariel University
Ariel 40700
Israel
e-mail: leonid.shaikhet@usa.net

Abstract

Functional differential equations arise in the modelling of hereditary systems such as ecological and biological systems, chemical and mechanical systems and many-many other. The long-term behaviour and stability of such systems are important areas for investigation. Analytical solutions of functional differential equations are generally unavailable and a lot of different numerical methods are adopted for obtaining approximate solutions. A quite natural question appears: “Do numerical solutions preserve the stability properties of the exact solution?” Thus, to use numerical investigation of functional differential equations it is very important to know if the considered difference analogue of the original differential equation has the reliability to preserve some general properties of this equation, in particular, property of stability. Here the ability of difference analogues of the nonlinear integro-differential equation of convolution type to preserve the property

2020 Mathematics Subject Classification: 30A30, 39A50, 60G52, 60H20, 65C30.

Keywords and phrases: difference analogue, discrete and continuous time, Lyapunov functional, Ito’s stochastic differential equation, stability, controlled inverted pendulum, Nicholson’s blowflies equation.

Communicated by Cemil Tunc.

Received January 8, 2024; Revised March 20, 2024

of stability of solutions is studied. Several difference analogues are considered both with discrete and with continuous time. Besides, difference analogues of the considered differential equation under stochastic perturbations are studied too. For stability investigation, we employ the general method of constructing Lyapunov functionals. It is shown how the obtained research can be applied to the various mathematical models.

1. Introduction

Functional differential equations arise in the modelling of hereditary systems such as ecological and biological systems, medical and sociological, chemical and mechanical systems and many-many other. The long-term behaviour and stability of such systems are important areas for investigation. Analytical solutions of functional differential equations are generally unavailable and different numerical methods are adopted for obtaining approximate solutions.

A quite natural question appears: "Do numerical solutions preserve the stability properties of the exact solution?" Thus, to use numerical investigation of functional differential equations it is very important to know if the considered difference analogue of the original differential equation has the reliability to preserve some general properties of this equation, in particular, property of stability (see, for instance, [8, 25, 26, 28]).

The investigation focuses on the ability of difference analogues of the nonlinear integro-differential equation of a convolution type

$$\dot{x}(t) = \int_0^t K(t-s)f(x(s))ds$$

to preserve the stability property of solutions. Specifically, we consider the nonlinear integro-differential equation of a convolution type with the exponential kernel

$$\dot{x}(t) = -k\int_0^t e^{-\lambda(t-s)}f(x(s))ds. \tag{1.1}$$

Note the fact that integro-differential equations are very popular in research (see, for instance, [31-35] and the references therein).

It is supposed that in the Equation (1.1) $k > 0, \lambda > 0$ and the function $f(x)$ is presented in the form

$$f(x) = \sum_{i=1}^{m} \alpha_i x^{\nu_i}, \quad \nu_i = \frac{2p_i + 1}{2q_i + 1}, \quad p_i \geq q_i \geq 0, \tag{1.2}$$

where $\alpha_i > 0$, p_i and q_i are integers.

Lemma 1.1. *The zero solution of the Equation* (1.1) *is stable.*

Proof. Putting $x_1(t) = x(t)$, $x_2(t) = \dot{x}(t)$, present the Equation (1.1) in the form of the system of two equations

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -kf(x_1(t)) - \lambda x_2(t). \end{aligned} \tag{1.3}$$

The function

$$V(t) = k\sum_{i=1}^{m} \frac{\alpha_i}{\mu_i} x_1^{2\mu_i}(t) + x_2^2(t), \quad \mu_i = \frac{1}{2}(\nu_i + 1) = \frac{p_i + q_i + 1}{2q_i + 1} \geq 1, \tag{1.4}$$

is a Lyapunov function for the system (1.3) since $V(t) > 0$ for $x_1^2(t) + x_2^2(t) > 0$, and via (1.2), (1.4)

$$\begin{aligned} \dot{V}(t) &= k\sum_{i=1}^{m} \frac{\alpha_i}{\mu_i} 2\mu_i x_1^{2\mu_i - 1} x_2 + 2x_2(-kf(x_1) - \lambda x_2) \\ &= -2\lambda x_2^2(t) + 2kx_2\left(\sum_{i=1}^{m} \alpha_i x_1^{2\mu_i - 1} - f(x_1)\right) \\ &= -2\lambda x_2^2(t) < 0, \end{aligned}$$

unless $x_2(t) = 0$. The proof is completed via the Lyapunov-type theorem [27]. □

Below several difference analogues of the Equation (1.1) are considered both with discrete and with continuous time. Besides, difference analogues of the considered differential equation under stochastic perturbations are studied too. For stability investigation, the general method of Lyapunov functionals construction is used [26, 27]. It is shown how the obtained research can be applied to the known mathematical models.

The following auxiliary statement will be used below.

Lemma 1.2 ([26])**.** *Arbitrary positive numbers* $a, b, \alpha, \beta, \gamma$ *satisfy the inequality*

$$a^{\alpha}b^{\beta} \leq \frac{\alpha}{\alpha+\beta}a^{\alpha+\beta}\gamma^{\beta} + \frac{\beta}{\alpha+\beta}b^{\alpha+\beta}\gamma^{-\alpha}.$$

Equality is reached for $\gamma = ba^{-1}$.

2. Some Difference Analogues with Discrete Time

It is obvious that one differential equation can have several difference analogues according to the choice of a numerical scheme. However, not all of these analogues need to be asymptotically stable. The problem is to determine which methods can be employed with the expectation that the difference analogue will preserve the qualitative behaviour of the solutions of the original problem. In particular, how one may construct a difference analogue of continuous asymptotically stable system that will be asymptotically stable too.

In this section, three possible schemes are proposed for the construction of difference analogues of the integro-differential equation

$$\dot{x}(t) = -k\int_0^t e^{-\lambda(t-s)}x^3(s)ds, \tag{2.1}$$

that is a particular case of the Equation (1.1), (1.2) with $m = 1$, $\alpha_1 = 1$, $p_1 = 1$, $q_1 = 0$. Conditions for the asymptotic stability of the zero solution of these difference analogues are obtained.

Scheme 1. Divide the interval $[0, t]$ into $i+1$ intervals of length $\Delta > 0$. In this way $t = (i+1)\Delta$, $s = j\Delta$, $j = 0, 1, \ldots, i, i+1$, $x_j = x(j\Delta)$. Using the left-hand difference derivative $(x_{i+1} - x_i)/\Delta$ for representation of $\dot{x}(t)$ in the point $t = (i+1)\Delta$ as a result we obtain the difference analogue of the Equation (2.1) in the form:

$$x_{i+1} = x_i - k\Delta^2 \sum_{j=0}^{i} e^{-\lambda\Delta(i+1-j)} x_j^3, \quad i = 0, 1, \ldots. \tag{2.2}$$

Denoting $a = e^{-\lambda\Delta}$, we transform the right-hand side of the Equation (2.2) in the following way:

$$x_{i+1} = x_i - ak\Delta^2 x_i^3 + a\left(-k\Delta^2 \sum_{j=0}^{i-1} e^{-\lambda\Delta(i-j)} x_j^3\right).$$

Applying (2.2) for $i-1$ leads to

$$x_1 = x_0 - ak\Delta^2 x_0^3,$$

$$x_{i+1} = x_i - ak\Delta^2 x_i^3 + a(x_i - x_{i-1}), \quad i = 1, 2, \ldots. \tag{2.3}$$

The Equation (2.3) has four parameters: k, λ, Δ, x_0. Putting

$$y_i = \frac{x_i}{x_0}, \quad \tau = \lambda\Delta, \quad \gamma = k\frac{x_0^2}{\lambda^2}, \quad a = e^{-\tau}, \quad b = \gamma\tau^2, \tag{2.4}$$

we finally obtain the equation with two parameters a and b only

$$y_0 = 1, \quad y_1 = 1 - ab,$$

$$y_{i+1} = y_i - aby_i^3 + a(y_i - y_{i-1}), \quad i = 1, 2, \ldots. \tag{2.5}$$

Scheme 2. Divide the interval $[0, t]$ into i intervals of length $\Delta > 0$. In this way $t = i\Delta$, $s = j\Delta$, $j = 0, 1, \ldots, i$, $x_j = x(j\Delta)$. Using the right-hand difference derivative $(x_{i+1} - x_i)/\Delta$ for representation of $\dot{x}(t)$ in the point $t = i\Delta$, as a result similarly to (2.2) we obtain the difference analogue of the Equation (2.1) in the form:

$$x_{i+1} = x_i - k\Delta^2 \sum_{j=1}^{i} e^{-\lambda\Delta(i-j)} x_j^3, \quad i = 0, 1, \ldots,$$

or

$$x_1 = x_0,$$

$$x_{i+1} = x_i - k\Delta^2 x_i^3 + e^{-\lambda\Delta}(x_i - x_{i-1}), \quad i = 1, 2, \ldots. \tag{2.6}$$

Following the same approach as above and using (2.4), one can represent the difference Equation (2.6) in the two-parameters form

$$y_1 = y_0 = 1,$$

$$y_{i+1} = y_i - by_i^3 + a(y_i - y_{i-1}), \quad i = 1, 2, \ldots. \tag{2.7}$$

Scheme 3. Divide the interval $[0, t]$ into $i + 1$ intervals of length $\Delta > 0$. In this way $t = (i + 1)\Delta$, $s = j\Delta$, $j = 0, 1, \ldots, i, i + 1$, $x_j = x(j\Delta)$. Using the left-hand difference derivative $(x_{i+1} - x_i)/\Delta$ for representation of $\dot{x}(t)$ in the point $t = (i + 1)\Delta$, as a result analogously to (2.2), we obtain the difference equation in the form:

$$x_{i+1} = x_i - k\Delta^2 \sum_{j=1}^{i+1} e^{-\lambda\Delta(i+1-j)} x_j^3, \quad i = 0, 1, \ldots,$$

or

$$x_1 = x_0 - k\Delta^2 x_1^3,$$

$$x_{i+1} = x_i - k\Delta^2 x_{i+1}^3 + e^{-\lambda\Delta}(x_i - x_{i-1}), \quad i = 1, 2, \ldots. \tag{2.8}$$

Using (2.4), in the two-parameters form we obtain

$$y_0 = 1, \quad y_1 = 1 - by_1^3,$$

$$y_{i+1} = y_i - by_{i+1}^3 + a(y_i - y_{i-1}), \quad i = 1, 2, \ldots. \tag{2.9}$$

Remark 2.1. If $b = 0$ then the Equations (2.5), (2.7) and (2.9) coincide with the equation $y_{i+1} = (1 + a)y_i - ay_{i-1}$, which has a stable but not an asymptotically stable zero solution [26].

3. Lyapunov Functionals Construction

Describe here in detail the construction of Lyapunov functionals V_i with the condition $\hat{\Delta}V_i = V_{i+1} - V_i < 0$ for the Equations (2.5), (2.7) and (2.9). (Note that the operator $\hat{\Delta}$ is used here and everywhere below in difference from the step of discretization Δ). Via the general method of Lyapunov functionals construction [26, 27], we have the following four steps:

3.1. The Equation (2.5)

Step 1. Represent the Equation (2.5) in the form

$$y_{i+1} = F_{1i} + \hat{\Delta}F_i, \quad i = 0, 1, \ldots,$$

where

$$F_{1i} = y_i - aby_i^3, \quad i = 0, 1, \ldots,$$

$$F_0 = 0, \quad \hat{\Delta}F_0 = ay_0,$$

$$F_i = ay_{i-1}, \quad \hat{\Delta}F_1 = a(y_i - y_{i-1}), \quad i = 1, 2, \ldots.$$

Step 2. Consider the auxiliary difference equation without delay

$$z_0 = 1, \quad z_{i+1} = z_i - abz_i^3, \quad i = 0, 1, \dots. \tag{3.1}$$

The function $v(z) = z^2$ is a Lyapunov function for this equation. In fact, using (3.1), we get

$$\hat{\Delta} v_i = v(z_{i+1}) - v(z_i) = z_{i+1}^2 - z_i^2$$

$$= (z_i - abz_i^3)^2 - z_i^2 = (ab)^2 z_i^4\left(z_i^2 - \frac{1}{ab}\right).$$

Since $z_0 = 1$ then via the condition $ab < 2$ we have $\Delta v_i < 0$ for all $i = 0, 1, \dots$ providing $z_i \neq 0$.

Step 3. We will construct the Lyapunov functional V_i for the Equation (2.5) in the form $V_i = V_{1i} + V_{2i}$, where

$$V_{10} = v(y_0) = y_0^2, \quad V_{1i} = v(y_i - F_i) = (y_i - ay_{i-1})^2, \quad i = 1, 2, \dots,$$

and the additional functional V_{2i} will be defined below.

Calculating $\hat{\Delta} V_{1i}, i = 1, 2, \dots,$ via (2.5) and Lemma 1.2 $(y_i^3 y_{i-1} \leq \frac{3}{4} y_i^4 + \frac{1}{4} y_{i-1}^4)$, we obtain

$$\hat{\Delta} V_{1i} = (y_{i+1} - ay_i)^2 - (y_i - ay_{i-1})^2$$

$$= (y_i - aby_i^3 - ay_{i-1})^2 - (y_i - ay_{i-1})^2$$

$$= (ab)^2 y_i^6 - 2aby_i^4 + 2a^2by_i^3 y_{i-1}$$

$$\leq (ab)^2 y_i^6 - a^2b\left(\frac{2}{a} - \frac{3}{2}\right) y_i^4 + \frac{1}{2} a^2by_{i-1}^4. \tag{3.2}$$

Similarly for $i = 0$, we have

$$\hat{\Delta} V_{10} = y_1^2 - y_0^2 = (1 - ab)^2 - 1$$

$$= -(ab)^2\left(\frac{2}{ab} - 1\right) < -(ab)^2(g_1(\tau) - 1),$$

where via (2.4)

$$g_1(\tau) = \frac{2(e^{\tau} - 1)}{\gamma\tau^2} = \frac{2(1-a)}{ab} < \frac{2}{ab}. \tag{3.3}$$

Step 4. Choosing the additional functional V_{2i} in the form

$$V_{20} = 0, \quad V_{2i} = \frac{1}{2}a^2 b y_{i-1}^4 \text{ with } \hat{\Delta}V_{2i} = \frac{1}{2}a^2 b(y_i^4 - y_{i-1}^4), \quad i = 1, 2, \ldots, \tag{3.4}$$

for the functional $V_i = V_{1i} + V_{2i}$ via (3.2), (3.4) and (3.3), we get

$$\hat{\Delta}V_i \le (ab)^2 y_i^6 - 2aby_i^4 + 2a^2by_i^4$$

$$= -(ab)^2 y_i^4(g_1(\tau) - y_i^2), \quad i = 0, 1, \ldots. \tag{3.5}$$

From (3.3) it follows also that

$$\lim_{\tau\to 0} g_1(\tau) = \infty, \quad \lim_{\tau\to\infty} g_1(\tau) = \infty,$$

$$\inf_{\tau\ge 0} g_1(\tau) = g_1(\tau_0) = \frac{2}{(2-\tau_0)\tau_0\gamma} \approx \frac{3.088}{\gamma},$$

where $\tau_0 \approx 1.594$ is the root of the equation

$$2 = (2-\tau)e^{\tau}. \tag{3.6}$$

Let us suppose that the sequence y_i^2 is bounded and there exists $\tau > 0$ such that

$$y_i^2 < g_1(\tau), \quad i = 0, 1, \ldots. \tag{3.7}$$

In this way, $\Delta V_i < 0$ for all $i = 0, 1, \ldots$ while $y_i \ne 0$. If the sequence y_i^2 is bounded by $g_1(\tau_0)$, where τ_0 is the root of the Equation (3.6), then

(3.7) is correct for all $\tau > 0$. If y_i^2 is bounded by some $M > g_1(\tau_0)$, then (3.7) is correct for $\tau \in (0, \tau_1) \cup (\tau_2, \infty)$, where τ_1 and τ_2 are two positive roots of the equation

$$2(e^{\tau} - 1) = M\gamma\tau^2. \tag{3.8}$$

3.2. The Equation (2.7)

The corresponding analysis for the Equation (2.7) proceeds as follows:

Step 1. We choose $F_{1i} = y_i - by_i^3$ and $F_i = ay_{i-1}$.

Step 2. This step is the same as for the Equation (2.5) by the condition $b < 2$.

Step 3. Via (2.7), one can show that

$$\hat{\Delta}V_{10} = (y_1 - ay_0)^2 - y_0^2 = a^2 - 2a = a(a-2) < 0,$$

$$\hat{\Delta}V_{1i} \le b^2 y_i^6 - b\left(2 - \frac{3a}{2}\right)y_i^4 + \frac{1}{2}aby_{i-1}^4, \quad i = 1, 2, \ldots.$$

Step 4. Put

$$V_{20} = 0, \quad V_{2i} = \frac{1}{2}aby_{i-1}^4, \quad i = 1, 2, \ldots.$$

Then for the functional $V_i = V_{1i} + V_{2i}$, we have

$$\hat{\Delta}V_i \le -b^2 y_i^4 (g_2(\tau) - y_i^2),$$

where via (2.4)

$$g_2(\tau) = \frac{2(1 - e^{-\tau})}{\gamma\tau^2} = \frac{2(1-a)}{b} < \frac{2}{b}. \tag{3.9}$$

The function $g_2(\tau)$ is a strictly decreasing one for $\tau > 0$ and

$$\lim_{\tau \to 0} g_2(\tau) = \infty, \quad \lim_{\tau \to \infty} g_2(\tau) = 0.$$

Thus, if the sequence y_i^2 is bounded by some $M > 0$ then the condition $y_i^2 < g_2(\tau)$, $i = 0, 1, \ldots,$ (and therefore $\Delta V_i < 0$) is correct for $\tau \in (0, \tau_0)$, where τ_0 is the positive root of the equation

$$2(1 - e^{-\tau}) = M\gamma\tau^2. \tag{3.10}$$

3.3. The Equation (2.9)

Finally for the Equation (2.9) we have:

Step 1. Choose

$$F_{1i} = y_i - by_i^3, \quad F_0 = -by_0^3 = -b,$$

$$F_i = ay_{i-1} - by_i^3, \quad \hat{\Delta}F_i = a(y_i - y_{i-1}) - b(y_{i+1}^3 - y_i^3), \quad i = 1, 2, \ldots.$$

Step 2. This step is the same as for the Equation (2.7).

Step 3. Via (2.9), one can show that

$$\begin{aligned}\hat{\Delta}V_{10} &= (y_1 - F_1)^2 - (y_0 - F_0)^2 \\ &= (1 - a)^2 - (1 + b)^2 \\ &= -a(2 - a) - b(2 + b) < 0,\end{aligned}$$

and via Lemma 1.2,

$$\begin{aligned}\hat{\Delta}V_{1i} &= (y_{i+1} - F_{i+1})^2 - (y_i - F_i)^2 \\ &= (y_{i+1} - ay_i + by_{i+1}^3)^2 - (y_i - ay_{i-1} + by_i^3)^2 \\ &= (y_i - ay_{i-1})^2 - (y_i - ay_{i-1} + by_i^3)^2 \\ &= -2by_i^4 - b^2y_i^6 + 2aby_i^3y_{i-1} \\ &\le -2by_i^4 + 2aby_i^3y_{i-1} \\ &\le -by_i^4\left(2 - \frac{3a}{2}\right) + \frac{1}{2}aby_{i-1}^4, \quad i = 1, 2, \ldots.\end{aligned}$$

Step 4. Put

$$V_{20} = 0, \quad V_{2i} = \frac{1}{2} aby_{i-1}^4, \quad i = 1, 2, \ldots.$$

Thus, for the functional $V_i = V_{1i} + V_{2i}$ for all $\tau > 0$ we obtain $\Delta V_i \leq -2b(1-a)y_i^4 < 0.$

4. Proofs of Asymptotic Stability

Here we can show how the functional V_i constructed above can be used to give desired conclusion. We give here the analysis for the Equation (2.5).

From (3.5) and (3.7) it follows

$$\sum_{j=0}^{i} \hat{\Delta} V_j = V_{i+1} - V_0 \leq -(ab)^2 \sum_{j=0}^{i} y_j^4 (g_1(\tau) - y_j^2) < 0. \tag{4.1}$$

Therefore, $0 \leq V_{i+1} \leq V_0 = y_0^2 = 1$. Moreover, $V_{2,i+1} = \frac{1}{2} a^2 by_i^4 \leq V_{i+1} \leq 1$. From here via (2.4)

$$x_i^2 \leq \sqrt{\frac{2}{b}} \frac{x_0^2}{a} = \sqrt{\frac{2}{\gamma\lambda^2\Delta^2}} \frac{x_0^2}{a} = \sqrt{\frac{2}{k}} \frac{|x_0|}{a\Delta}.$$

So, for any $\varepsilon > 0$ there exists $\delta = \sqrt{\frac{k}{2}} a\Delta\varepsilon^2$ such that $|x_i| < \varepsilon$, $i > 0$, if $|x_0| < \delta$. In other words, we have shown that the zero solution of the Equation (2.5) is stable.

Besides, from (4.1) and $V_{i+1} \geq 0$, it follows that

$$\sum_{j=0}^{\infty} y_j^4 (g_1(\tau) - y_j^2) \leq \frac{V_0}{(ab)^2}. \tag{4.2}$$

The convergence of the series in the left-hand part of (4.2) implies that

$$\lim_{i\to\infty} y_i^4 (g_1(\tau) - y_i^2) = 0.$$

It means that either $\lim_{i\to\infty} y_i^4 = 0$ or $\lim_{i\to\infty} y_i^2 = g_1(\tau)$. In any case the limit of y_i by $i \to \infty$ there exists. From (2.5), it follows that $\lim_{i\to\infty} y_i = 0$. Via (2.4) the solution of the Equation (2.3) satisfies the condition $\lim_{i\to\infty} x_i = 0$. The proof is completed. □

Remark 4.1. A similar argument applies to solutions of the Equations (2.7) and (2.9).

We summarize our conclusions by the following way. Assume that k, λ, x_0 are given and we investigate the solutions x_i of the Equations (2.3), (2.6) and (2.8) for a fixed values of $\Delta > 0$.

Theorem 4.1. *If the solution* x_i *of the Equation* (2.3) *satisfies the condition* $x_i^2 \le g_1(\tau_0)x_0^2$, *where* τ_0 *is the root of the Equation* (3.6), *then* $x_i \to 0$ *regardless of the step size* $\Delta > 0$. *If the solution* x_i *of the Equation* (2.3) *satisfies the condition* $x_i^2 \le Mx_0^2$ *for some* $M > g_1(k_0)$ *then* $x_i \to 0$ *for all* $\Delta \in \left(0, \frac{\tau_1}{\lambda}\right) \cup \left(\frac{\tau_2}{\lambda}, \infty\right)$, *where* τ_1 *and* τ_2 *are the roots of the Equation* (3.8).

If the solution x_i *of the Equation* (2.6) *satisfies the condition* $x_i^2 \le Mx_0^2$ *for some* $M > 0$ *then* $x_i \to 0$ *for all* $\Delta \in (0, \frac{\tau_0}{\lambda})$, *where* τ_0 *is the root of the Equation* (3.10).

The solution x_i *of the Equation* (2.8) *converges to zero for all* $\Delta > 0$.

Remark 4.2 In the statements of Theorem 4.1 we have considered the behaviour of bounded solutions of the discrete equations. We can observe that unbounded solutions may arise with particular combinations of x_0, Δ, λ. Our calculations indicate that if $g_1(\tau) > 1$, then the solution of the Equation (2.5) satisfies the condition

$|y_i| < 1,\ i = 1, 2, \ldots.$ In Figures 1-3 one can see the behaviour of the solution of the Equation (2.5) with the different values of the parameters τ, γ and function $g_1(\tau)$: Figure 1 $(\tau = 0.1, \gamma = 6, g_1(\tau) = 3.51)$, Figure 2 $(\tau = 0.01, \gamma = 15, g_1(\tau) = 13.4)$, Figure 3 $(\tau = 0.1, \gamma = 20, g_1(\tau) = 1.05)$. In Figure 4, it is shown that by $\tau = 0.24, \gamma = 25$ the solutions of the Equations (2.5) (number 1 $g_1(\tau) = 0.38)$ and (2.9) (number 3) converge to zero, but the solution of the Equation (2.7) (number 2, $g_2(\tau) = 0.3)$ goes to infinity.

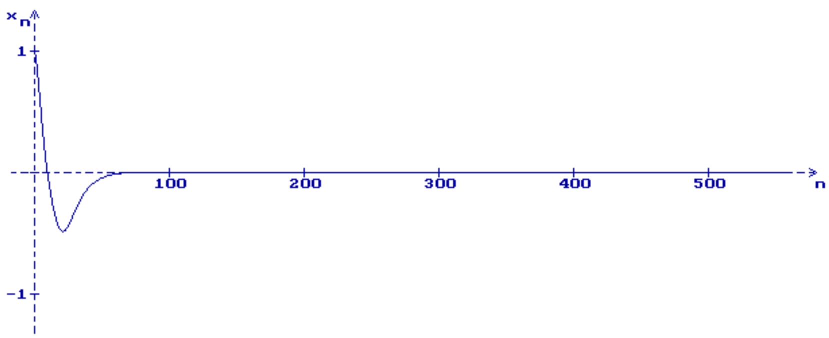

Figure 1. The solution of the Equation (2.5) with $\tau = 0.1, \gamma = 6$ and $g_1(\tau) = 3.51$.

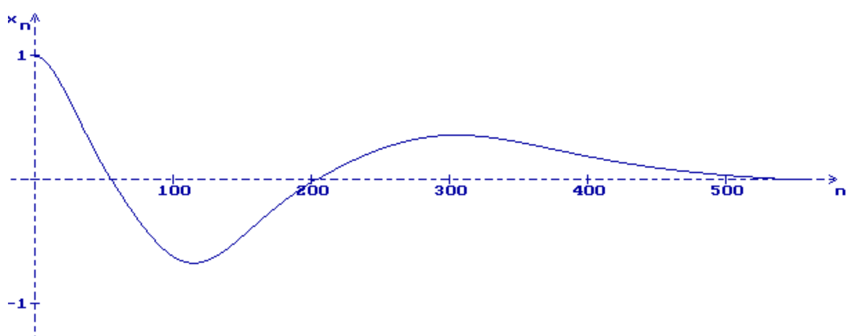

Figure 2. The solution of the Equation (2.5) with $\tau = 0.01, \gamma = 15$ and $g_1(\tau) = 13.4$.

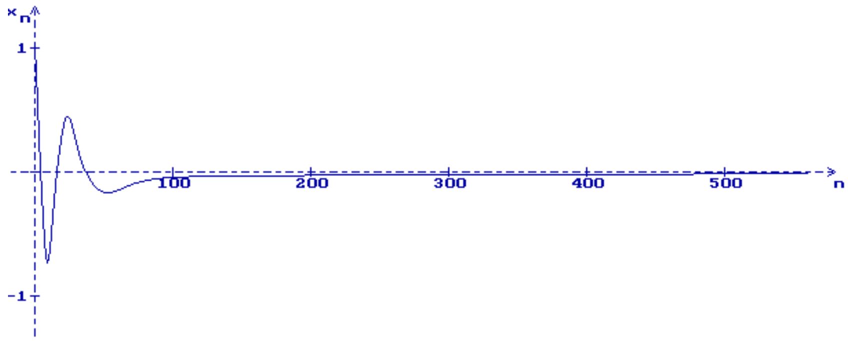

Figure 3. The solution of the Equation (2.5) with $\tau = 0.1$, $\gamma = 20$ and $g_1(\tau) = 1.05$.

Remark 4.3. Note that the difference schemes considered above can be constructed and for more general nonlinear integro-differential equation

$$\dot{x}(t) = -k\int_0^t e^{-\lambda(t-s)}x^r(s)ds, \quad k, \lambda > 0, \tag{4.3}$$

where r is an arbitrary odd number. For instance, the equations of the type of difference analogues (2.3) and (2.6), respectively for the Equation (4.3) are

$$x_1 = x_0 - akh^2x_0^r, \quad x_{i+1} = x_i - ak\Delta^2 x_i^r + a(x_i - x_{i-1}), \quad i = 1, 2, \dots,$$

and

$$y_0 = 1, \quad y_1 = 1 - \gamma a\Delta^2, \quad y_{i+1} = y_i - abx_i^r + a(y_i - y_{i-1}), \quad i = 1, 2, \dots,$$

where $\gamma = k\dfrac{y_0^{r-1}}{\lambda^2}$ and y_i, τ, a, b are defined in (2.4). By that the functional

$$V_i = (y_i - ay_{i-1})^2 + \frac{2a^2b}{r+1}y_{i-1}^{r+1}, \quad i = 1, 2, \dots,$$

satisfied the condition $\Delta V_i \le -(ab)^2 y_i^{r+1}(g_1(\tau) - y_i^{r-1})$ with $g_1(\tau)$ defined in (3.3).

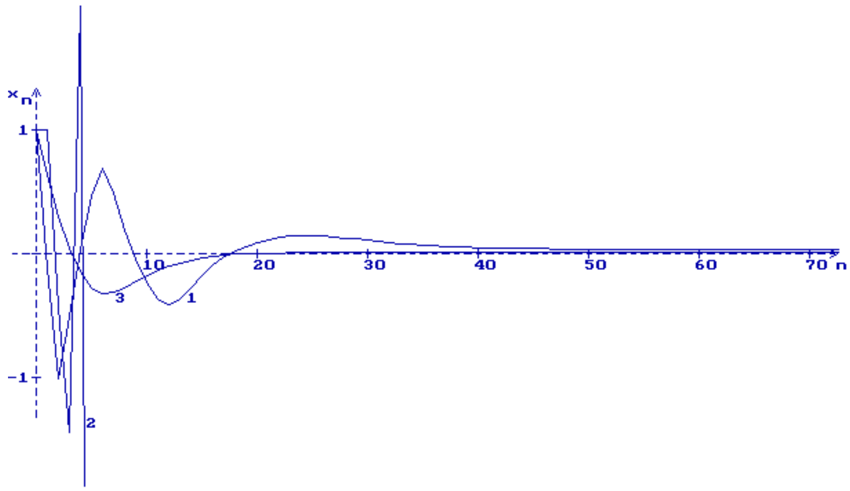

Figure 4. The solutions of the Equations (2.5) (number 1), (2.7) (number 2) and (2.9) (number 3) with $\tau = 0.24,$ $\gamma = 25,$ $g_1(\tau) = 0.38,$ $g_2(\tau) = 0.3,$

5. Difference Analogue with Continuous Time

To construct the difference analogue of the Equation (1.1) with continuous time rewrite this equation in the equivalent form

$$\dot{x}(t) = -k\int_0^t e^{-\lambda s} f(x(t-s))ds. \tag{5.1}$$

Let Δ be a small enough positive number. Using the Equation (1.1) for $t \in [0, \Delta)$ and (5.1) for $t \ge \Delta,$ we can construct a difference analogue in the form of the following difference equation with continuous time:

$$x(t) = x(0) - kt^2 e^{-\lambda t} f(x(0)), \quad t \in [0, \Delta),$$

$$x(t + \Delta) = x(t) - k\Delta^2 F(t), \quad t \ge 0,$$

$$F(t)=\sum_{j=0}^{\left[\frac{t}{\Delta}\right]} e^{-\lambda\Delta j} f(x(t-j\Delta)), \tag{5.2}$$

where $[t]$ is the integer part of the number t. If $t \in [0, \Delta)$ then $F(t) = f(x(t))$. For $t \geq \Delta$ transform $F(t)$ by the following way:

$$\begin{aligned} F(t) &= f(x(t)) + \sum_{j=1}^{\left[\frac{t}{\Delta}\right]} e^{-\lambda\Delta j} f(x(t-j\Delta)) \\ &= f(x(t)) + \sum_{j=0}^{\left[\frac{t}{\Delta}\right]-1} e^{-\lambda\Delta(j+1)} f(x(t-(j+1)\Delta)) \\ &= f(x(t)) + e^{-\lambda\Delta} \sum_{j=0}^{\left[\frac{t-\Delta}{\Delta}\right]} e^{-\lambda\Delta j} f(x(t-\Delta-j\Delta)) \\ &= f(x(t)) + e^{-\lambda\Delta} f(t-\Delta). \end{aligned} \tag{5.3}$$

From (5.2), it follows that

$$F(t) = -\frac{x(t+\Delta)-x(t)}{k\,\Delta^2}, \quad F(t-\Delta) = -\frac{x(t)-x(t-\Delta)}{k\,\Delta^2}.$$

Substituting the obtained $F(t)$ and $F(t-\Delta)$ into (5.3), we transform the Equation (5.2) to the form

$$x(t+\Delta) = x(t) - k\,\Delta^2 f(x(t)) + e^{-\lambda\Delta}(x(t) - x(t-\Delta)), \quad t > \Delta. \tag{5.4}$$

The process $x(t)$ is defined by the Equation (5.4) for $t > t_0 = 2\Delta$ with the initial condition

$$x(\theta) = \phi(\theta), \quad \theta \in [t_0 - 2\Delta, t_0] = [0, 2\Delta], \tag{5.5}$$

where

$$\phi(\theta) = \begin{cases} x(0) - k\theta^2 e^{-\lambda\theta} f(x(0)), & \theta \in [t_0 - 2\Delta, t_0 - \Delta) = [0, \Delta), \\ \phi(\theta - \Delta) - k\Delta^2 f(\phi(\theta - \Delta)), & \theta \in [t_0 - \Delta, t_0] = [\Delta, 2\Delta]. \end{cases}$$

Note that via (1.2) the order of nonlinearity of the Equation (5.4) is, generally speaking, more than one.

Definition 5.1. The solution of the Equation (5.4) with the initial condition (5.5) is called asymptotically quasitrivial if $\lim_{j\to\infty} x(t + j\Delta) = 0$ for each $t \in [t_0, t_0 + \Delta)$.

Definition 5.2. The zero solution of the Equation (5.4) is called stable if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$, such that $|x(t)| < \varepsilon$, for all $t \geq t_0$ if $\|\phi\| = \sup_{\theta\in[t_0-2\Delta,\, t_0]} |\phi(\theta)| < \delta$.

Definition 5.3. The zero solution of the Equation (5.4) is called locally asymptotically quasistable if it is stable and for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$, such that the solution of the Equation (5.4) is asymptotically quasitrivial for each initial condition (5.5) such that $\|\phi\| < \delta$.

Theorem 5.1. *For a small enough* $\Delta > 0$ *an each bounded solution of the Equation* (5.4) *with the initial condition* (5.5) *is asymptotically quasitrivial.*

Proof. Using the procedure of Lyapunov functionals construction [26], we will construct a Lyapunov functional for the Equation (5.4) in the form $V(t) = V_1(t) + V_2(t)$, where

$$V_1(t) = (x(t) - e^{-\lambda\Delta} x(t - \Delta))^2, \quad t \geq t_0, \tag{5.6}$$

is the Lyapunov functional for the auxiliary linear difference equation (the linear part of the Equation (5.4))

$$x(t + \Delta) = x(t) + e^{-\lambda\Delta}(x(t) - x(t - \Delta)), \quad t > \Delta. \tag{5.7}$$

Indeed, for the Equation (5.7) we have $\hat{\Delta}V_1(t) = 0$. It means that the zero solution of the Equation (5.7) is stable but not asymptotically quasistable [26].

Calculating $\hat{\Delta}V_1(t) = V_1(t+\Delta) - V_1(t)$ for the Equation (5.4) via (5.6), we obtain

$$\begin{aligned}
\hat{\Delta}V_1(t) &= (x(t+\Delta) - e^{-\lambda\Delta}x(t))^2 - (x(t) - e^{-\lambda\Delta}x(t-\Delta))^2 \\
&= (x(t) - e^{-\lambda\Delta}x(t-\Delta) - k\Delta^2 f(x(t)))^2 - (x(t) - e^{-\lambda\Delta}x(t-\Delta))^2 \\
&= k^2\Delta^4 f^2(x(t)) - 2k\Delta^2 f(x(t))x(t) + 2k\Delta^2 e^{-\lambda\Delta} f(x(t))x(t-\Delta). \qquad (5.8)
\end{aligned}$$

Via (1.2) and Lemma 1.2, we have

$$\begin{aligned}
\hat{\Delta}V_1(t) &\le k^2\Delta^4 f^2(x(t)) - 2k\Delta^2 f(x(t))x(t) \\
&\quad + 2k\Delta^2 e^{-\lambda\Delta}\sum_{i=1}^{m}\alpha_i\left(\frac{\nu_i}{\nu_i+1}x^{\nu_i+1}(t) + \frac{1}{\nu_i+1}x^{\nu_i+1}(t-\Delta)\right). \qquad (5.9)
\end{aligned}$$

Put

$$V_2(t) = 2k\Delta^2 e^{-\lambda\Delta}\sum_{i=1}^{m}\frac{\alpha_i}{\nu_i+1}x^{\nu_i+1}(t-\Delta), \quad t \ge t_0. \qquad (5.10)$$

From (1.4), it follows that $\nu_i + 1 = 2\mu_i$. So, $V_2(t) \ge 0$. Via (5.9), (5.10) and (1.2), for the functional $V(t) = V_1(t) + V_2(t)$ we obtain

$$\begin{aligned}
\hat{\Delta}V(t) &\le k^2\Delta^4 f^2(x(t)) - 2k\Delta^2(1 - e^{-\lambda\Delta})f_1(x(t)) \\
&\le -\beta_1(\Delta)f_1(x(t))(\beta_2(\Delta) - f_2(x(t))), \quad t \ge t_0, \qquad (5.11)
\end{aligned}$$

where

$$\beta_1(\Delta) = k^2\Delta^4, \quad \beta_2(\Delta) = \frac{2(1 - e^{-\lambda\Delta})}{k\Delta^2}, \quad f_1(x) = f(x)x,$$

$$f_2(x) = f(x)x^{-1}, \quad x \neq 0. \tag{5.12}$$

Suppose that there exists $\Delta_0 > 0$, such that the solution of the Equation (5.4) is uniformly bounded for $\Delta \in [0, \Delta_0]$, i.e., $|x(t)| \leq M$, $t \geq t_0$. Since $f_2(x)$ is a non-decreasing for $x \geq 0$ function and $\lim_{\Delta\to 0} \beta_2(\Delta) = \infty$, then there exists a small enough $\Delta > 0$, such that $f_2(x(t)) \leq f_2(M) < \beta_2(\Delta)$. From this and (5.11) it follows:

$$\hat{\Delta}V(t) \leq -\gamma_1(\Delta)f_1(x(t)), \quad t \geq t_0, \tag{5.13}$$

where $\gamma_1(\Delta) = \beta_1(\Delta)(\beta_2(\Delta) - f_2(M)) > 0$. Rewrite (5.13) for $t + j\Delta$, i.e.,

$$\hat{\Delta}V(t + j\Delta) \leq -\gamma_1(\Delta)f_1(x(t + j\Delta)), \quad t \geq t_0, \quad j = 0, 1, \dots,$$

and summing it from $j = 0$ to $j = i - 1$, we obtain

$$V(t + i\Delta) - V(t) \leq -\gamma_1(\Delta)\sum_{j=0}^{i-1} f_1(x(t + j\Delta)), \quad t \geq t_0. \tag{5.14}$$

From this it follows

$$\gamma_1(\Delta)\sum_{j=0}^{\infty} f_1(x(t + j\Delta)) \leq V(t) < \infty, \quad t \geq t_0.$$

Therefore, $\lim_{j\to\infty} f_1(x(t + j\Delta)) = 0$ for each $t \geq t_0$. Due to (5.12),

$$0 \leq \alpha_1 x^{\nu_1+1}(t + j\Delta) \leq f_1(x(t + j\Delta)), \quad t \geq t_0.$$

So, $\lim_{j\to\infty}|x(t + j\Delta)| = 0$ for each $t \geq t_0$, i.e., the solution of the Equation (5.4) is asymptotically quasitrivial. The proof is completed. □

Theorem 5.2. *The zero solution of the Equation* (5.4) *is stable.*

Proof. We will use here the functional $V(t)$, that was constructed in the proof of the previous theorem. Via (5.14), we have

$$V(t + i\Delta) \leq V(t), \quad i = 0, 1, \dots, \quad t \geq t_0.$$

Putting $t = t_0 + j\Delta + s$ with $j = \left[\frac{t - t_0}{\Delta}\right]$ and $s \in [0, \Delta)$, we obtain

$$V(t_0 + (j + i)\Delta + s) \le V(t) = V(t_0 + j\Delta + s) \le V(t_0 + s). \qquad (5.15)$$

From (5.6) and $\|\phi\| = \sup_{s \in [t_0 - \Delta, t_0)} |\phi(s)|$, we have

$$V_1(t_0 + s) = (x(t_0 + s) - e^{-\lambda\Delta}\phi(t_0 + s - \Delta))^2$$

$$\le 2(|x(t_0 + s)|^2 + e^{-2\lambda\Delta}\|\phi\|^2). \qquad (5.16)$$

From the Equation (5.4), (5.5) for $t = \Delta + s$ and $t_0 = 2\Delta$ it follows that $t_0 + s - \Delta = \Delta + s \in [\Delta, t_0)$ and

$$|x(t_0 + s)| \le (1 + e^{-\lambda\Delta})|\phi(\Delta + s)| + k\Delta^2 |f(\phi(\Delta + s))| + e^{-\lambda\Delta}|\phi(s)|.$$

Due to (1.2)

$$|f(\phi(\theta))| \le \sum_{i=1}^{m} \alpha_i |\phi(\theta)|^{\nu_i} \le C_2 \|\phi\|^{\nu} \qquad \theta \in [0, t_0],$$

where

$$C_2 = \sum_{i=1}^{m} \alpha_i, \qquad \nu = \begin{cases} 1 & \text{if} \quad \|\phi\| \le 1, \\ \max\limits_{i=1,\ldots,m} \nu_i & \text{if} \quad \|\phi\| > 1. \end{cases}$$

Therefore,

$$|x(t_0 + s)| \le C_3 \|\phi\|^{\nu}, \quad C_3 = 1 + 2e^{-\lambda\Delta} + k\Delta^2 C_2,$$

and using (5.16), we obtain

$$V_1(t_0 + s) \le 2(C_3^2 \|\phi\|^{2\nu} + e^{-2\lambda\Delta}\|\phi\|^2). \qquad (5.17)$$

From (5.10), it follows that

$$V_2(t_0 + s) = 2k\Delta^2 e^{-\lambda\Delta} \sum_{i=1}^{m} \frac{\alpha_i}{\nu_i + 1} \phi^{\nu_i + 1}(t_0 + s - \Delta) \le C_1 \|\phi\|^{\nu + 1}, \qquad (5.18)$$

where

$$C_1 = 2k\Delta^2 e^{-\lambda\Delta} \sum_{i=1}^{m} \frac{\alpha_i}{\nu_i + 1}.$$

From (5.15), (5.17) and (5.18) for the functional $V(t) = V_1(t) + V_2(t)$ it follows the inequality

$$V(t) \le V(t_0 + s) \le C_0 \|\phi\|^{2\nu}, \quad t \ge t_0, \quad C_0 = C_1 + 2(C_3^2 + e^{-2\lambda\Delta}). \quad (5.19)$$

Via (5.12), (5.14) and (5.19), we obtain

$$\gamma_1(\Delta)\alpha_1 |x(t)|^{\nu_1+1} \le \gamma_1(\Delta) f_1(x(t))$$

$$\le \gamma_1(\Delta) \sum_{j=0}^{i-1} f_1(x(t + j\Delta)) \le V(t) \le C_0 \|\phi\|^{2\nu}, \quad t \ge t_0.$$

So, for arbitrary $\varepsilon > 0$ there exists a $\delta = (C_0^{-1}\gamma_1(\Delta)\alpha_1\varepsilon^{\nu_1+1})^{\frac{1}{2\nu}} > 0$, such that $|x(t)| < \varepsilon$ if $\|\phi\| < \delta$. The proof is completed. □

Corollary 5.1. *From Theorems* 5.1 *and* 5.2 *it follows that for a small enough* $\Delta > 0$ *the zero solution of the Equation* (5.4) *is locally asymptotically quasistable.*

6. Stability Conditions for Stochastic Differential Equations

To begin with, let us consider some simple examples with the possibility to get stability conditions for stochastic differential equations using its difference analogues. Some known mathematical models under stochastic perturbations will be considered in the next sections.

Example 6.1. Consider the scalar Ito stochastic differential equation of neutral type

$$\dot{x}(t) + ax(t) + bx(t-h) + c\dot{x}(t-h) + \sigma x(t-\tau)\dot{w}(t) = 0, \quad t \ge 0, \quad (6.1)$$

and its difference analogue via the Euler-Maruyama scheme [19]

$$x_{i+1} = (1 - \Delta a)x_i - cx_{i-k+1} + (c - \Delta b)x_{i-k}$$
$$+ \sigma\sqrt{\Delta}x_{i-m}\xi_{i+1}, \quad i = 0, 1, \dots. \tag{6.2}$$

Here $w(t)$ is the standard Wiener process [10, 27]

$$\Delta > 0, \quad t_i = i\Delta, \quad x_i = x(t_i), \quad k = \frac{h}{\Delta}, \quad m = \frac{\tau}{\Delta},$$
$$\xi_{i+1} = \frac{w(t_{i+1}) - w(t_i)}{\sqrt{\Delta}}, \quad \mathbf{E}\xi_i = 0, \quad \mathbf{E}\xi_i^2 = 1, \quad i = 0, 1, \dots, \tag{6.3}$$

and it is supposed that k and m are integers.

Remark 6.1. It is known [10, 27] that the Wiener process is not differentiable. Therefore, the Equation (6.1) and all stochastic differential equations below are understanding in the form of differentials.

In [26], two following sufficient conditions for asymptotic mean square stability of the zero solution of the difference Equation (6.2) are obtained:

$$(1 - \Delta(a + b))^2 + 2\Delta|a + b|(|b|(h - \Delta) + |c - \Delta b|) + \Delta\sigma^2 < 1,$$

and

$$\Delta(a + b)^2 + 2|a + b|(|b|(h - \Delta) + |c - \Delta b|) + \sigma^2 < 2(a + b). \tag{6.4}$$

Let $\Delta \to 0$. Then from (6.4), the known [27] sufficient condition for asymptotic mean square stability of the zero solution of the differential Equation (6.1) follows:

$$(a + b)(1 - |b|h - |c|) > \frac{1}{2}\sigma^2, \quad |b|h + |c| < 1. \tag{6.5}$$

Example 6.2. Consider the scalar stochastic integro-differential equation

$$\dot{x}(t) = ax(t) + b\int_{t-h}^{t} x(s)\mathrm{d}s + \sigma x(t-\tau)\dot{w}(t), \tag{6.6}$$

where $w(t)$ is the standard Wiener process. Using (6.3), the Euler-Maruyama scheme and θ-method ($\theta \in [0, 1]$) for a difference representation of the integral, consider a difference analogue of (6.6) in the form

$$x_{i+1} = [1 + \Delta a + \Delta^2 b(1-\theta)]x_i + \Delta^2 b\left(\sum_{j=1}^{k-1} x_{i-j} + \theta x_{i-k}\right) + \sigma\sqrt{\Delta}x_{i-m}\xi_{i+1}. \tag{6.7}$$

Via [26], we obtain two sufficient conditions for asymptotic mean square stability of the zero solution of the difference Equation (6.7)

$$(1 + \Delta(a + bh))^2 + \Delta h|b|\,|a + bh|(\Delta(2\theta - 1) + h) + \Delta\sigma^2 < 1,$$

and

$$\Delta(a + bh)^2 + h|b|\,|a + bh|\,(\Delta(2\theta - 1) + h) + \sigma^2 < 2|a + bh|. \tag{6.8}$$

Let $\Delta \to 0$. Then from the Equation (6.8) the known [27] sufficient condition for asymptotic mean square stability of the zero solution of the differential Equation (6.6) follows:

$$|a + bh|\left(1 - \frac{1}{2}|b|h^2\right) > \frac{1}{2}\sigma^2, \quad a + bh < 0. \tag{6.9}$$

Remark 6.2. One can see that if the condition (6.5) (or (6.9)) holds, then the inequality (6.4) (or (6.8)) at the same time is a sufficient condition on the step of discretization Δ by which the difference analogue (6.2) (or (6.7)) saves the stability property of the solution of the initial differential equation (6.1) (or (6.6)).

7. Difference Analogue of the Mathematical Model of the Controlled Inverted Pendulum

7.1. Mathematical model of the controlled inverted pendulum

The problem of stabilizing the controlled inverted pendulum has been very popular among researchers for many years (see [1, 2, 4-6, 12, 13, 15, 16, 20, 21, 23, 24, 26, 27, 29] and references therein). The linearized mathematical model of the controlled inverted pendulum can be described by the second-order linear differential equation

$$\ddot{x}(t) - ax(t) = u(t), \quad a > 0, \quad t \geq 0. \tag{7.1}$$

The classical way of stabilization [13] uses the control $u(t) = -b_1 x(t) - b_2 \dot{x}(t)$, $b_1 > a$, $b_2 > 0$. But this type of control, which represents instantaneous feedback, is quite difficult to realize because usually it is necessary to have some finite time to make measurements of the coordinates and velocities, to treat the results of the measurements and to implement them in the control action.

Another way is supposed that the control $u(t)$ does not depend on the velocity but it depends on the previous values of the trajectory $x(s)$, $s \leq t$, and has the form [27]

$$u(t) = \int_0^\infty dK(\tau) x(t - \tau). \tag{7.2}$$

The kernel $K(\tau)$ in (7.2) is a function of bounded variation on $[0, \infty]$ and the integral is understood in the Stieltjes sense. It means in particular that both distributed and discrete delays can be used depending on the concrete choice of the kernel $K(\tau)$.

The initial condition for the system of (7.1), (7.2) has the form

$$x(s) = \varphi(s), \quad \dot{x}(s) = \dot{\varphi}(s), \quad s \leq 0, \tag{7.3}$$

where $\varphi(s)$ is a given continuously differentiable function.

It is supposed also that the Equation (7.1) is under the influence of stochastic perturbations of the type of white noise in the form

$$\ddot{x}(t) - (a + \sigma\dot{w}(t))x(t) = u(t), \tag{7.4}$$

where $w(t)$ is the standard Wiener process and σ is a constant.

Put $x_1(t) = x(t)$, $x_2(t) = \dot{x}(t)$. Then (7.2)-(7.4) can be represented in the form of the system of Ito's stochastic differential equations [10]

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= ax_1(t) + \int_0^\infty dK(\tau)x_1(t-\tau) + \sigma x_1(t)\dot{w}(t), \end{aligned} \tag{7.5}$$

with the initial condition $x_1(s) = \varphi(s)$, $x_2(s) = \dot{\varphi}(s)$, $s \le 0$.

Put

$$\begin{aligned} k_i &= \int_0^\infty \tau^i dK(\tau), \quad i = 0, 1, \\ k_2 &= \int_0^\infty \tau^2 |dK(\tau)|, \quad a_1 = -(a + k_0). \end{aligned} \tag{7.6}$$

The following theorem gives a sufficient stability condition for the system (7.5).

Theorem 7.1 ([27]). *Let*

$$a_1 > 0, \quad k_1 > 0, \tag{7.7}$$

$$\sigma^2 < 2a_1\left(k_1 - k_2\sqrt{\frac{a_1}{2(2-k_2)}}\right). \tag{7.8}$$

Then the zero solution of the system (7.5) *is asymptotically mean square stable.*

Note that the inequalities (7.7) are the necessary conditions for asymptotic mean square stability of the zero solution of the system (7.5) but the inequality (7.8) is a sufficient one only. Besides, for the condition (7.8) k_2 has to satisfy the inequality $k_2 < \sqrt{k^2 + 4k} - k < 2$, where $k = k_1^2/a_1$ [27].

Below, the mathematical model of the controlled inverted pendulum (7.1)-(7.3) is considered in the following simple form:

$$\ddot{x}(t) - (a + \sigma\dot{w}(t))x(t) = b_1 x(t - h_1) + b_2 x(t - h_2), \quad t \geq 0. \tag{7.9}$$

Here $a > 0, b_1, b_2, h_1 > 0$ and $h_2 > 0$ are given arbitrary numbers. From (7.6), it follows that for (7.9)

$$k_0 = b_1 + b_2, \quad k_1 = b_1 h_1 + b_2 h_2,$$

$$k_2 = |b_1|h_1^2 + |b_2|h_2^2, \quad a_1 = -(a + b_1 + b_2). \tag{7.10}$$

The main conclusion of our investigation here can be formulated in the following way: if the conditions (7.7), (7.8) and (7.10) hold, then the zero solution of the Equation (7.9) is asymptotically mean square stable. Additionally, there exists a sufficiently small step of discretization of this equation that the zero solution of the corresponding difference analogue is asymptotically mean square stable too. Below a difference analogue is constructed and an estimation of the step of discretization is obtained.

7.2. Construction of a difference analogue

Transform the differential Equation (7.9) into the system of two equations

$$\dot{x}(t) = y(t), \quad \dot{y}(t) = ax(t) + \sum_{l=1}^{2} b_l x(t - h_l) + \sigma x(t)\dot{w}(t), \tag{7.11}$$

To construct a difference analogue of the system (7.11), put

$$\Delta > 0, \quad t_i = i\Delta, \quad x_i = x(t_i), \quad m_1 = \frac{h_1}{\Delta}, \quad m_2 = \frac{h_2}{\Delta},$$

$$\xi_{i+1} = \frac{w(t_{i+1}) - w(t_i)}{\sqrt{\Delta}}, \quad \mathbf{E}\xi_i = 0, \quad \mathbf{E}\xi_i^2 = 1, \quad i = 0, 1, \dots \tag{7.12}$$

(it is supposed here that m_1 and m_2 are integers). Via the Euler-Maruyama scheme [19], a difference analogue of the system (7.11) is

$$x_{i+1} = x_i + \Delta y_i,$$

$$y_{i+1} = y_i + \Delta\left(ax_i + \sum_{l=1}^{2} b_l x_{i-m_l}\right) + \sigma\sqrt{\Delta}x_i\xi_{i+1}. \tag{7.13}$$

From the first equation of the system (7.13), we have

$$x_i = x_{i-m_l} + \Delta\sum_{j=1}^{m_l} y_{i-j}, \quad l = 1, 2. \tag{7.14}$$

From this and (7.10) it follows that

$$\sum_{l=1}^{2} b_l x_{i-m_l} = k_0 x_i - \Delta\sum_{l=1}^{2} b_l \sum_{j=1}^{m_l} y_{i-j}. \tag{7.15}$$

Substituting (7.15) into the second equation of the system (7.13) and using (7.10), we obtain

$$y_{i+1} = y_i - \Delta a_1 x_i - \Delta^2\sum_{l=1}^{2} b_l \sum_{j=1}^{m_l} y_{i-j} + \sigma\sqrt{\Delta}x_i\xi_{i+1}. \tag{7.16}$$

Put

$$F_i = \Delta^2\sum_{l=1}^{2} b_l \sum_{j=1}^{m_l} (m_l + 1 - j)y_{i-j}. \tag{7.17}$$

Calculating $\hat{\Delta}F_i = F_{i+1} - F_i$ and using (7.17), (7.10) and (7.12), we have

$$\hat{\Delta}F_i = \Delta^2 \sum_{l=1}^{2} b_l \left(\sum_{j=1}^{m_l} (m_l + 1 - j) y_{i+1-j} - \sum_{j=1}^{m_i} (m_l + 1 - j) y_{i-j} \right)$$

$$= \Delta^2 \sum_{l=1}^{2} b_l \left(m_l y_i - \sum_{j=1}^{m_l} y_{i-j} \right) = \Delta k_1 y_i - \Delta^2 \sum_{l=1}^{2} b_l \sum_{j=1}^{m_l} y_{i-j}.$$

From this and (7.16), it follows that

$$y_{i+1} = - \Delta a_1 x_i + (1 - k_1 \Delta) y_i + \hat{\Delta}F_i + \sigma\sqrt{\Delta} x_i \xi_{i+1}.$$

So, the system (7.13) can be presented in the matrix form

$$z(i+1) = Az(i) + \hat{\Delta}F(i) + Bz(i)\xi_{i+1}, \tag{7.18}$$

where

$$z(i) = \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \quad F(i) = \begin{pmatrix} 0 \\ F_i \end{pmatrix}, \quad A = \begin{pmatrix} 1 & \Delta \\ - a_1 \Delta & 1 - k_1 \Delta \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ \sigma\sqrt{\Delta} & 0 \end{pmatrix}. \tag{7.19}$$

7.3. Stability conditions for the auxiliary equation

Following the general method of Lyapunov functionals construction [26, 27], first consider the auxiliary equation without memory

$$z(i+1) = Az(i) + Bz(i)\xi_{i+1}, \tag{7.20}$$

and the function $v_i = z'(i)Dz(i)$, where the matrix D is a positive definite solution of the matrix equation

$$A'DA - D = - C, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}, \quad c > 0, \tag{7.21}$$

with the elements [26]

$$d_{11} = \frac{a_1 \Delta + k_1}{2a_1 \Delta} + \frac{2 - k_1 \Delta + a_1 \Delta^2}{2} a_1 d_{22},$$

$$d_{12} = \frac{1}{2a_1\Delta} + \frac{a_1\Delta}{2}d_{22}, \quad d_{22} = \frac{2 - k_1\Delta + a_1\Delta^2 + 2a_1 c}{a_1\Delta(k_1 - a_1\Delta)(4 - 2k_1\Delta + a_1\Delta^2)}. \tag{7.22}$$

Calculating $\mathbf{E}\hat{\Delta}v_i$ via (7.12) and (7.19)-(7.21), we have

$$\begin{aligned}
\mathbf{E}\hat{\Delta}v_i &= \mathbf{E}\left[z'(i+1)Dz(i+1) - v_i\right] \\
&= \mathbf{E}\left[(Az(i) + Bz(i)\xi_{i+1})' D(Az(i) + Bz(i)\xi_{i+1}) - v_i\right] \\
&= \mathbf{E}\left[-z'(i)Cz(i) + z'(i)B'DBz(i)\right] \\
&= -(1 - d_{22}\sigma^2\Delta)\mathbf{E}x_i^2 - c\mathbf{E}y_i^2.
\end{aligned} \tag{7.23}$$

So, if for some $c > 0$, the inequality

$$\sigma^2 < \frac{1}{d_{22}\Delta} \tag{7.24}$$

holds then the zero solution of the Equation (7.20) is asymptotically mean square stable [26].

Note that the Equation (7.20) can also be written in the scalar form

$$x_{i+2} = A_0 x_{i+1} + A_1 x_i + \sigma_0 x_i \xi_{i+1}, \tag{7.25}$$

with

$$A_0 = 2 - \Delta k_1, \quad A_1 = \Delta k_1 - \Delta^2 a_1 - 1, \quad \sigma_0 = \sigma\sqrt{\Delta^3}. \tag{7.26}$$

It is known [26] that the necessary and sufficient conditions for asymptotic mean square stability of the zero solution of the Equation (7.25) are

$$|A_1| < 1, \quad |A_0| < 1 - A_1, \tag{7.27}$$

$$\sigma_0^2 < \frac{(1 + A_1)\left[(1 - A_1)^2 - A_0^2\right]}{1 - A_1}. \tag{7.28}$$

Substituting (7.26) into (7.27), we obtain the system of the inequalities

$$a_1\Delta < k_1, \quad a_1\Delta^2 - 2k_1\Delta + 4 > 0,$$

with the solution

$$0 < \Delta < \begin{cases} a_1^{-1}k_1, & k_1^2 < 4a_1, \\ a_1^{-1}\left(k_1 - \sqrt{k_1^2 - 4a_1}\right), & k_1^2 \geq 4a_1. \end{cases} \tag{7.29}$$

Substituting (7.26) into (7.28), we obtain the condition

$$\sigma^2 < \frac{a_1(k_1 - a_1\Delta)(4 - 2k_1\Delta + a_1\Delta^2)}{2 - k_1\Delta + a_1\Delta^2}. \tag{7.30}$$

So, by the conditions (7.29) and (7.30) the zero solution of the Equation (7.25), (7.26) is asymptotically mean square stable.

From (7.22) and (7.24), it follows that if the condition (7.30) holds then there exists a small enough $c > 0$ such that the condition (7.24) holds too. Thus, the function $v_i = z'(i)Dz(i)$, where the matrix D is a positive definite solution of the matrix equation (7.21), is a Lyapunov function for the auxiliary Equation (7.20).

7.4. Stability conditions for the difference analogue

Let us obtain now a sufficient condition for asymptotic mean square stability of the zero solution of the Equation (7.18). Rewrite this equation in the form

$$z(i+1) - F(i+1) = Az(i) - F(i) + Bz(i)\xi_{i+1}. \tag{7.31}$$

Following the procedure of Lyapunov functionals construction [26], we will construct a Lyapunov functional V_i for the Equation (7.31) in the form $V_i = V_{1i} + V_{2i}$, where

$$V_{1i} = (z(i) - F(i))' D(z(i) - F(i)) \tag{7.32}$$

and the matrix D is a positive definite solution of the matrix equation (7.21) with the elements (7.22). The additional functional V_{2i} will be chosen below.

Calculating $\mathbf{E}\hat{\Delta}V_{1i}$ via (7.12), (7.21), (7.31) and (7.32), similarly to (7.23), we obtain

$$\begin{aligned}\mathbf{E}\hat{\Delta}V_{1i} &= \mathbf{E}[(z(i+1)-F(i+1))'D(z(i+1)-F(i+1))-V_{1i}]\\ &= \mathbf{E}[(Az(i)-F(i)+Bz(i)\xi_{i+1})'D(Az(i)-F(i)+Bz(i)\xi_{i+1})-V_{1i}]\\ &= -(1-d_{22}\sigma^2\Delta)\mathbf{E}x_i^2-c\mathbf{E}y_i^2-2\mathbf{E}F'(i)D(A-I)z(i). \qquad (7.33)\end{aligned}$$

Note that via (7.19)

$$\begin{aligned}&2F'(i)D(A-I)z(i)\\ &= 2\Delta(0\ \ F_i)\begin{pmatrix} d_{11} & d_{12} \\ d_{12} & d_{22}\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -a_1 & -k_1\end{pmatrix}\begin{pmatrix} x_i \\ y_i\end{pmatrix}\\ &= 2\Delta F_i[-a_1d_{22}x_i+(d_{12}-k_1d_{22})y_i]. \qquad (7.34)\end{aligned}$$

Put

$$\alpha = \frac{2-k_1\Delta+a_1\Delta^2+2a_1c}{(k_1-a_1\Delta)(4-2k_1\Delta+a_1\Delta^2)}, \quad \beta = \frac{\Delta+c(2k_1-a_1\Delta)}{(k_1-a_1\Delta)(4-2k_1\Delta+a_1\Delta^2)}. \qquad (7.35)$$

Then via (7.22), (7.35), we have

$$d_{22} = \frac{\alpha}{a_1\Delta}, \qquad (7.36)$$

and

$$\Delta(d_{12}-k_1d_{22}) = \frac{1}{2a_1}(1-\alpha(2k_1-a_1\Delta))$$

$$= \frac{1}{2a_1}\left(1 - \frac{(2k_1 - a_1\Delta)(2 - k_1\Delta + a_1\Delta^2 + 2a_1 c)}{(k_1 - a_1\Delta)(4 - 2k_1\Delta + a_1\Delta^2)}\right) = -\beta.$$

(7.37)

So, via (7.33)-(7.37)

$$\mathbf{E}\hat{\Delta}V_{1i} = -(1 - a_1^{-1}\alpha\sigma^2)\mathbf{E}x_i^2 - c\mathbf{E}y_i^2 - 2\alpha\mathbf{E}x_iF_i - 2\beta\mathbf{E}y_iF_i.$$

Put now

$$q = \frac{1}{2}\sum_{l=1}^{2}|b_l|h_l(h_l + \Delta), \quad S_i = \Delta^2\sum_{l=1}^{2}|b_l|\sum_{j=1}^{m_l}(m_l + 1 - j)y_{i-j}^2. \tag{7.38}$$

Using (7.17), (7.12) and $\lambda > 0$, we have

$$2x_iF_i \le \Delta^2\sum_{l=1}^{2}|b_l|\sum_{j=1}^{m_l}(m_l + 1 - j)\left(\lambda x_i^2 + \frac{1}{\lambda}y_{i-j}^2\right)$$

$$= \lambda q x_i^2 + \frac{S_i}{\lambda},$$

and analogously

$$2y_iF_i \le qy_i^2 + S_i.$$

As a result we obtain

$$\mathbf{E}\hat{\Delta}V_{1i} \le -[1 - \alpha(\lambda q + a_1^{-1}\sigma^2)]\mathbf{E}x_i^2 - (c - \beta q)\mathbf{E}y_i^2 + \rho\mathbf{E}S_i,$$

where

$$\rho = \frac{\alpha}{\lambda} + \beta. \tag{7.39}$$

To neutralize the positive component in the estimation of $\mathbf{E}\hat{\Delta}V_{1i}$ choose the additional functional V_{2i} in the form

$$V_{2i} = \frac{1}{2}\rho\Delta^2\sum_{l=1}^{2}|b_l|\sum_{j=1}^{m_l}(m_l + 1 - j)(m_l + 2 - j)y_{i-j}^2.$$

Then via (7.38)

$$\begin{aligned}\hat{\Delta}V_{2i} &= \frac{1}{2}\rho\Delta^2\sum_{l=1}^{2}|b_l|\sum_{j=1}^{m_l}(m_l + 1 - j)(m_l + 2 - j)y_{i+1-j}^2 - V_{2i}\\ &= \frac{1}{2}\rho\Delta^2\sum_{l=1}^{2}|b_l|\sum_{j=0}^{m_l}(m_l - j)(m_l + 1 - j)y_{i-j}^2 - V_{2i}\\ &= \rho q y_i^2 - \rho S_i,\end{aligned}$$

and for the functional $V_i = V_{1i} + V_{2i}$, we have

$$\mathbf{E}\hat{\Delta}V_i \le -[1 - \alpha(\lambda q + a_1^{-1}\sigma^2)]\mathbf{E}x_i^2 - [c - q(\beta + \rho)\mathbf{E}y_i^2.$$

Using (7.39), we obtain the stability conditions in the form

$$\alpha\left(\lambda q + \frac{\sigma^2}{a_1}\right) < 1, \quad q\left(2\beta + \frac{\alpha}{\lambda}\right) < c. \tag{7.40}$$

For $\lambda > 0$ from (7.40) it follows that

$$\frac{\alpha q}{c - 2\beta q} < \lambda < \frac{a_1 - \alpha\sigma^2}{\alpha q a_1}. \tag{7.41}$$

Thus, if

$$\frac{\alpha q}{c - 2\beta q} < \frac{a_1 - \alpha\sigma^2}{\alpha q a_1}, \tag{7.42}$$

then there exists $\lambda > 0$ such that the conditions (7.41) and therefore the conditions (7.40) hold.

Let us rewrite (7.42) in the form

$$\frac{\alpha^2 q^2}{c - 2\beta q} + \frac{\alpha\sigma^2}{a_1} < 1. \tag{7.43}$$

To stress the dependence of the left-hand part of (7.43) on c put

$$A_0 = (k_1 - a_1\Delta)(4 - 2k_1\Delta + a_1\Delta^2),$$

$$A_1 = 2 - k_1\Delta + a_1\Delta^2, \quad A_2 = 2k_1 - a_1\Delta. \tag{7.44}$$

Then via (7.35)

$$\alpha = \frac{A_1 + 2a_1 c}{A_0}, \quad \beta = \frac{\Delta + A_2 c}{A_0},$$

and (7.43) takes the form

$$\frac{(A_1 + 2a_1 c)^2 q^2}{Qc - 2q\Delta} + 2\sigma^2 c < B, \tag{7.45}$$

where

$$B = A_0 - \frac{A_1}{a_1}\sigma^2, \quad Q = A_0 - 2qA_2. \tag{7.46}$$

Transform (7.45) into the form

$$B_0 c^2 - B_1 c + B_2 < 0 \tag{7.47}$$

with

$$B_0 = 4a_1^2 q^2 + 2Q\sigma^2,$$

$$B_1 = BQ + 4q\sigma^2\Delta - 4A_1 a_1 q^2, \quad B_2 = A_1^2 q^2 + 2qB\Delta. \tag{7.48}$$

Minimization of the left-hand part of (7.47) with respect to c gives

$$B_1^2 > 4B_0 B_2. \tag{7.49}$$

Via (7.48), the inequality (7.49) can be represented as

$$\sigma^4 - 2P_1\sigma^2 + P_2 > 0, \tag{7.50}$$

where

$$P_1 = Ra_1, \quad P_2 = P_1^2 - \frac{8q^2a_1^3R}{Q}, \quad R = \frac{QA_0}{QA_1 + 4qa_1\Delta}. \tag{7.51}$$

From (7.50) and (7.51) by virtue of (7.45) and (7.46), it follows that

$$\sigma^2 < a_1\left(R - 2q\sqrt{\frac{2a_1R}{Q}}\right). \tag{7.52}$$

Note that in the condition (7.52) a_1 is defined in (7.10), q, Q and R are defined in (7.38), (7.46) and (7.51) and depend on Δ. Thus, the following theorem is proven.

Theorem 7.2 *Let the parameters* a, b_1, b_2, σ *and* Δ *of the system* (7.13) *satisfy the conditions* (7.7), (7.8) *and* (7.52). *Then the zero solution of the system* (7.13) *is asymptotically mean square stable.*

Lemma 7.1. *If for some values of the parameters* a, b_1, b_2 *and* σ *the conditions* (7.7) *and* (7.8) *hold, then there exists a small enough* $\Delta > 0$ *such that the condition* (7.52) *holds too.*

Proof. Note that for $\Delta = 0$ via (7.38), (7.10) $2q = k_2$, via (7.44) $A_0 = 4k_1$, $A_1 = 2$, $A_2 = 2k_1$. So, via (7.46) and (7.51), $Q = 2k_1(2 - k_2)$ and $R = 2k_1$. Substituting all these results into (7.52), we obtain that for $\Delta = 0$ the condition (7.52) coincides with (7.8). Since the right-hand part of (7.52) is continuous with respect to Δ in the point $\Delta = 0$, if the condition (7.52) holds for $\Delta = 0$ then it holds for some small enough $\Delta > 0$ too. The proof is completed. □

Corollary 7.1. *If the parameters* a, b_1, b_2 *and* σ *of the system* (7.13) *satisfy the conditions* (7.7) *and* (7.8), *then there exists a small enough* $\Delta > 0$ (*satisfying the condition* (7.52)) *such that the zero solution of the system* (7.13) *is asymptotically mean square stable.*

7.5. Nonlinear model of the controlled inverted pendulum

Consider the problem of stabilization for the nonlinear model of the controlled inverted pendulum

$$\ddot{x}(t) - (a + \sigma\dot{w}(t))\sin x(t) = b_1 x(t - h_1) + b_2 x(t - h_2), \quad t \geq 0, \tag{7.53}$$

with the initial condition (7.3). Similarly to (7.13) the difference analogue of the Equation (7.53) is

$$x_{i+1} = x_i + \Delta y_i,$$

$$y_{i+1} = y_i + \Delta\left(ax_i + af(x_i) + \sum_{l=1}^{2} b_l x_{i-m_l}\right) + \sigma\sqrt{\Delta}(x_i + f(x_i))\xi_{i+1}, \tag{7.54}$$

where $f(x) = \sin x - x$. The system (7.13) is the linear part of the system (7.54) and the order of nonlinearity of the system (7.54) equals 3, since $|f(x)| \leq \frac{1}{6}|x|^3$. Via [26] we obtain the following statement:

Corollary 7.2. *If the parameters* a, b_1, b_2, σ *and* Δ *of the system* (7.54) *satisfy the conditions* (7.7), (7.8) *and* (7.52), *then the zero solution of the system* (7.54) *is stable in probability.*

8. Nicholson's Blowflies Equation

Consider the nonlinear integro-differential equation with exponential nonlinearity and distributed delay

$$\dot{x}(t) = \int_0^\infty x(t - s)e^{-bx(t-s)}dK(s) - cx(t), \tag{8.1}$$

where the delay term is given by the Stieltjes integral.

Putting in (8.1) $dK(s) = a\delta(s - h)ds$, where $\delta(s)$ is Dirac's function, we obtain well known Nicholson's blowflies differential equation, which is one of the most important mathematical models in ecology [22]

$$\dot{x}(t) = ax(t - h)e^{-bx(t-h)} - cx(t). \tag{8.2}$$

It describes the population dynamics of Nicholson's blowflies. Here $x(t)$ is the size of the population at time t, a is the maximum per capita daily egg production rate, $1/b$ is the size at which the population reproduces at the maximum rate, c is the per capita daily adult death rate and h is the generation time.

The Equation (8.2) along with its difference analogues are very popular in research (see, for instance, [3, 7, 9, 11, 14, 17, 18, 22, 26, 27, 30, 36] and a long list of references therein). Below, we consider stability in probability of the positive equilibrium of the Equation (8.2) by stochastic perturbations and also of one discrete analogue of this equation. The capability of a discrete analogue to preserve stability properties of the original differential equation is studied. All theoretical results are verified by some numerical simulation. Besides it is shown that numerical simulation of the solution of difference analogue allows one to define more exactly a bound of stability region obtained by the sufficient stability condition.

The following method for investigation of the stability is used. The considered nonlinear equation (8.2) is exposed to stochastic perturbations and is linearized in the neighbourhood of the positive equilibrium. The conditions for asymptotic mean square stability of the zero solution of the corresponding linear equation are obtained. Since the order of nonlinearity is higher than one these conditions are sufficient ones (both for continuous and discrete time [26, 27]) for stability in probability of the initial nonlinear equation by stochastic perturbations. This method was used already for investigation of the stability of different nonlinear biological systems with delays: SIR epidemic model, predator-prey model and many others [26, 27].

Note that a generalization of the obtained below results for the Equation (8.1) is an open problem.

8.1. Stability condition for the positive equilibrium

In (8.2), it is supposed that the parameters a, b and c are positive. By the conditions $c \geq a > 0$, $b > 0$, (8.2) has the zero equilibrium only, i.e., $x^* = 0$. By the conditions

$$a > c > 0, \quad b > 0, \tag{8.3}$$

the Equation (8.2) has two equilibria: the zero one and a positive one

$$x^* = \frac{1}{b} \ln \frac{a}{c}. \tag{8.4}$$

It is known [27] that the zero equilibrium in the region (8.3) is unstable. The stability condition in the region (8.3) for the positive equilibrium (8.4) by stochastic perturbations is considered below.

It is supposed that the Equation (8.2) is exposed to stochastic perturbations, which are of the type of white noise, are directly proportional to the deviation of the system state $x(t)$ from the equilibrium x^* and influence $\dot{x}(t)$ immediately. So, the Equation (8.2) is transformed into Ito's stochastic differential equation

$$\dot{x}(t) = ax(t-h)e^{-bx(t-h)} - cx(t) + \sigma\left(x(t) - x^*\right)\dot{w}(t). \tag{8.5}$$

Let us center the Equation (8.5) on the positive equilibrium x^* using the new variable $y(t) = x(t) - x^*$. In this way via (8.4), we obtain

$$\dot{y}(t) = -cy(t) + cy(t-h)e^{-by(t-h)} + \frac{c}{b} \ln \frac{a}{c} (e^{-by(t-h)} - 1) + \sigma y(t)\dot{w}(t). \tag{8.6}$$

It is clear that stability of the equilibrium x^* of the Equation (8.5) is equivalent to stability of the zero solution of the Equation (8.6).

Along with (8.6) we will consider the linear part of this equation. Using the representation $e^y = 1 + y + o(y)$ (where $o(y)$ means that $\lim_{y\to 0} \frac{o(y)}{y} = 0$) and neglecting by $o(y)$, we obtain the linear part (process $z(t)$) of (8.6) in the form

$$\dot{z}(t) = -cz(t) - c\left(\ln\frac{a}{c} - 1\right)z(t-h) + \sigma z(t)\dot{w}(t). \tag{8.7}$$

As shown in [27] if the order of nonlinearity of the equation under consideration is more than one then a sufficient condition for asymptotic mean square stability of the linear part of the initial nonlinear equation is also a sufficient condition for stability in probability of the initial nonlinear equation. So, we will investigate sufficient conditions for asymptotic mean square stability of the linear part (8.7) of the nonlinear stochastic differential Equation (8.6).

Lemma 8.1 ([26, 27])**.** *The necessary and sufficient condition for asymptotic mean square stability of the zero solution of the Equation* (8.7) *is*

$$pG < 1, \tag{8.8}$$

where

$$p = \frac{1}{2}\sigma^2, G = \begin{cases} \dfrac{1 + \frac{c}{q}(\ln\frac{a}{c} - 1)\sin(qh)}{c[1 + (\ln\frac{a}{c} - 1)\cosh(qh)]}, & a > c\,\mathrm{e}^2, \qquad q = c\sqrt{\ln\frac{a}{c}(\ln\frac{a}{c} - 2)}, \\ \dfrac{1 + ch}{2c}, & a = c\,\mathrm{e}^2, \\ \dfrac{1 + \frac{c}{q}(\ln\frac{a}{c} - 1)\sinh(qh)}{c[1 + (\ln\frac{a}{c} - 1)\cosh(qh)]}, & c < a < c\,\mathrm{e}^2, \qquad q = c\sqrt{\ln\frac{a}{c}(2 - \ln\frac{a}{c})}. \end{cases}$$

(8.9)

In particular, if $p > 0$, $h = 0$, *then the stability condition* (8.8), (8.9) *takes the form* $c\ln\frac{a}{c} > p$; *if* $p = 0$, $h > 0$, *then the region of stability is bounded by the lines* $c = 0$, $c = a$ *and* $1 + (\ln\frac{a}{c} - 1)\cos(qh) = 0$ *for* $a > c\,e^2$.

The condition (8.8), (8.9) gives us a region (in the space of the parameters (a, c)) for asymptotic mean square stability of the zero solution of the Equation (8.7) (and at the same time regions for stability in probability of the positive equilibrium x^* of the Equation (8.5)). In Figures 5-8 the region of stability given by the condition (8.8), (8.9) is shown in lilac colour.

Remark 8.1. Note that the stability condition (8.8), (8.9) has the following property: if the point (a, c) belongs to the stability region with some p and h, then for arbitrary positive α the point $(a_0, c_0) = (\alpha a, \alpha c)$ belongs to the stability region with $p_0 = \alpha p$ and $h_0 = \alpha^{-1}h$.

8.2. Stability of difference analogue

Consider a difference analogue of the nonlinear Equation (8.6) using the Euler-Maruyama scheme [19]

$$y_{i+1} = (1 - c\Delta)y_i + c\Delta y_{i-k}\,e^{-by_{i-k}} + \frac{c}{b}\ln\frac{a}{c}\,\Delta(e^{-by_{i-k}} - 1) + \sigma\sqrt{\Delta}\,y_i\xi_{i+1}. \tag{8.10}$$

Here k is an integer, $\Delta = \dfrac{h}{k}$ is the step of discretization,

$$t_i = i\Delta, \quad y_i = y(t_i), \quad \xi_{i+1} = \frac{1}{\sqrt{\Delta}}(w(t_{i+1}) - w(t_i)), \quad i = 0, 1, \ldots.$$

In compliance with (8.7) the linear part of (8.10) is

$$z_{i+1} = (1 - c\Delta)z_i + c\Delta\left(1 - \ln\frac{a}{c}\right)z_{i-k} + \sigma\sqrt{\Delta}\,z_i\xi_{i+1}. \tag{8.11}$$

Consider two following sufficient conditions for asymptotic mean square stability of the zero solution of the Equation (8.11) [26]

$$\frac{p}{c} + \left|1 - \ln\frac{1}{c}\right| |1 - c\Delta| + \frac{1}{2} c\Delta \left(1 + \left|1 - \ln\frac{a}{c}\right|^2\right) < 1 \tag{8.12}$$

and

$$\frac{p}{c} + \frac{1}{2} c\Delta \ln^2 \frac{a}{c} < \left(1 - ch\left|1 - \ln\frac{a}{c}\right|\right) \ln\frac{a}{c}. \tag{8.13}$$

The regions for asymptotic mean square stability of the zero solution of the Equation (8.11) (and at the same time regions for stability in probability of the zero solution of the Equation (8.10)), obtained by the conditions (8.12) and (8.13), are shown in the space of the parameters (a, c) for $p = 12$, $h = 0.024$ and $\Delta = 0.004$ (Figure 5), $\Delta = 0.008$ (Figure 6), $\Delta = 0.012$ (Figure 7). The main part (with number 1) of the stability region is obtained via the condition (8.12), the additional part (with number 2) is obtained via the condition (8.13).

Let us show how the sufficient stability conditions (8.12) and (8.13) are close to the necessary and sufficient stability condition. Consider the case $p = 12$, $h = 0.024$, $\Delta = 0.012$, $k = \frac{h}{\Delta} = 2$. Appropriate necessary and sufficient stability condition for the Equation (8.11) is obtained in [26] in the form

$$\frac{p}{c} + \frac{1}{c} c\Delta\left[1 + \left(1 - \ln\frac{a}{c}\right)^2\right] + \frac{(1 - c\Delta)^2 (1 - \ln\frac{a}{c})(1 - c\Delta\ln\frac{a}{c})}{1 - c\Delta(1 - \ln\frac{a}{c})(1 - c\Delta\ln\frac{a}{c})} < 1. \tag{8.14}$$

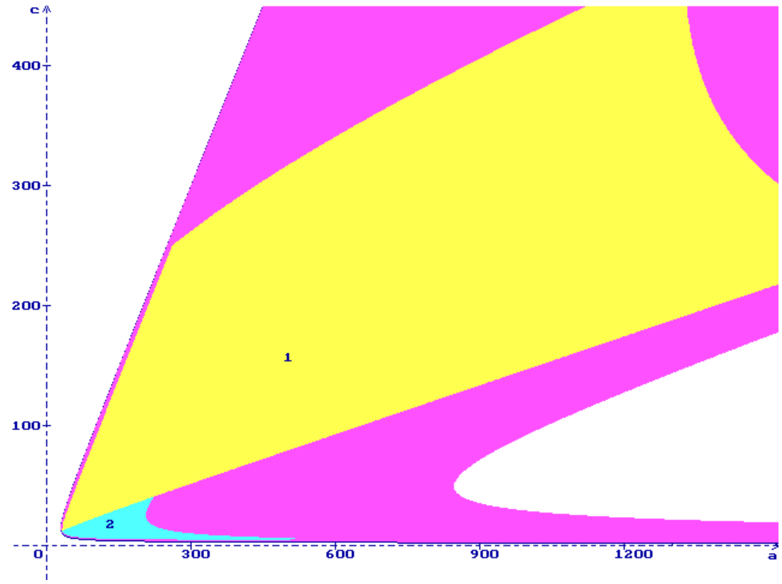

Figure 5. Region of sufficient stability condition for the Equation (8.11): $p = 12$, $h = 0.024$ and $\Delta = 0.004$.

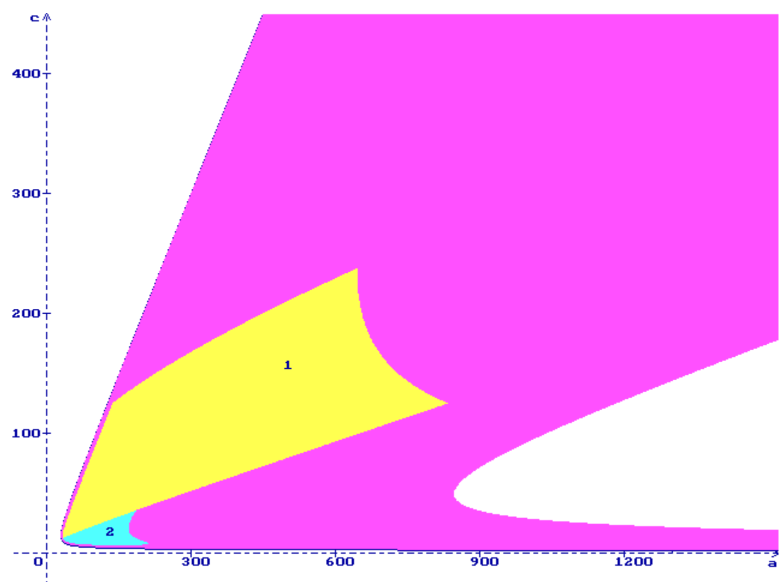

Figure 6. Region of sufficient stability condition for the Equation (8.11): $p = 12$, $h = 0.024$ and $\Delta = 0.008$.

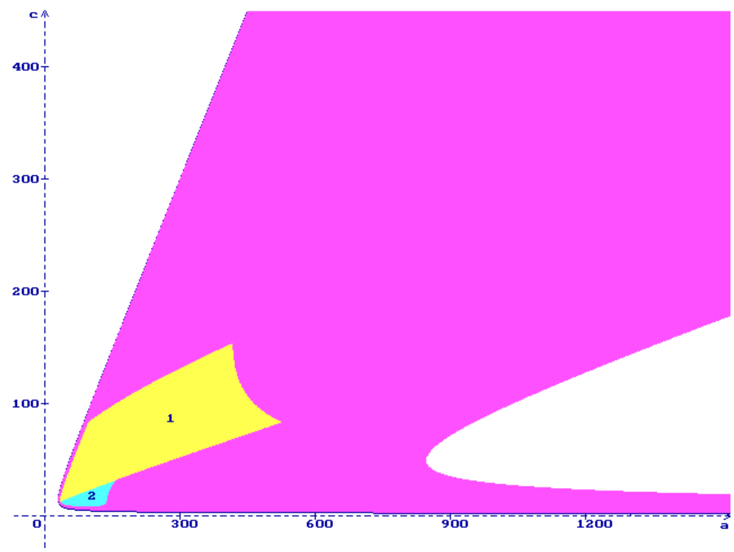

Figure 7. Region of sufficient stability condition for the Equation (8.11): $p = 12$, $h = 0.024$ and $\Delta = 0.012$.

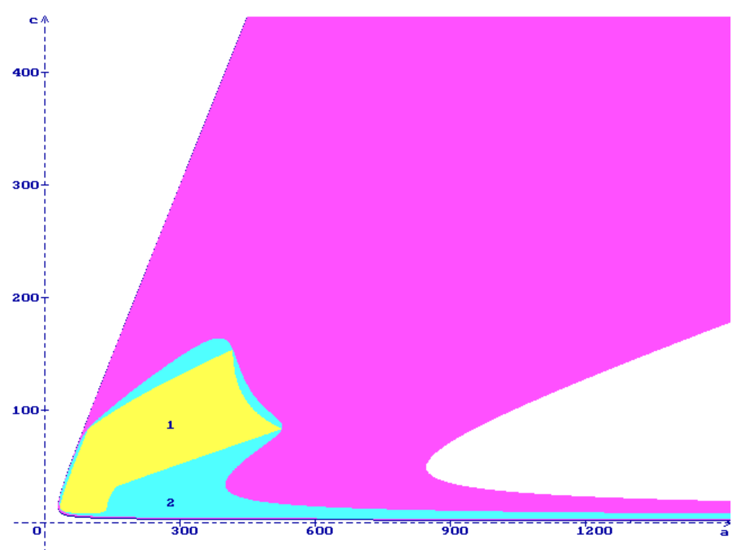

Figure 8. Region of sufficient stability condition and necessary and sufficient stability condition for the Equation (8.11): $p = 12$, $h = 0.024$ and $\Delta = 0.012$.

In Figure 8, the stability region, obtained via the sufficient stability conditions (8.12) and (8.13) (number 1), is shown inside the stability region, obtained via the necessary and sufficient stability condition (8.14) (number 2).

Remark 8.2. The conditions (8.12) and (8.13) can be represented in the explicit form, respectively;

$$c\,e^{1+G_0} > a > c\,e^{1-G_0}, \quad G_0 = \frac{\sqrt{1-\sigma^2\Delta} - |1-c\Delta|}{c\Delta}, \tag{8.15}$$

and

$$c\,e^{G_3} > a > \begin{cases} c\,e^{G_1}, & c > c_0, \\ c\,e^{G_2}, & c \le c_0, \end{cases}$$

$$c_0 = \frac{1-\sqrt{1-\sigma^2\Delta}}{\Delta}, \quad G_1 = \frac{ch - 1 + \sqrt{(1-ch)^2 + (2h-\Delta)\sigma^2}}{c(2h-\Delta)},$$

$$G_2 = \frac{ch + 1 - \sqrt{(1+ch)^2 - (2h+\Delta)\sigma^2}}{c(2h+\Delta)},$$

$$G_3 = \frac{ch + 1 + \sqrt{(1+ch)^2 - (2h+\Delta)\sigma^2}}{c(2h+\Delta)}. \tag{8.16}$$

Remark 8.3. The conditions (8.12), (8.13) and (8.14) for arbitrary values of the parameters of the Equation (8.5) allow us to choose the admissible step of discretization Δ by numerical simulation of the stable solution of this equation. For example, in Figure 5, we can see that for simulation of the solution of the Equation (8.5) with $a = 900$, $c = 200$, we can use $\Delta = 0.004$. But taking into account Figures 6 and 7 we cannot be sure that it is possible to use $\Delta = 0.008$ or $\Delta = 0.012$.

Remark 8.4. Note that the stability conditions (8.12) and (8.13) have the following property: if the point (a, c) belongs to the stability region with some p, h and Δ, then for arbitrary positive α the point $(a_0, c_0) = (\alpha a, \alpha c)$ belongs to the stability region with $p_0 = \alpha p$, $h_0 = \alpha^{-1}h$ and $\Delta_0 = \alpha^{-1}\Delta$.

Remark 8.5. In [14, 30], the discrete analogue of the Equation (8.2) was considered in the form (in our notations)

$$x_{i+1} = (1 - c\Delta)x_i + a\Delta x_{i-k}\, e^{-bx_{i-k}}.$$

By the assumption $c\Delta < 1$, the sufficient condition for asymptotic stability of the positive equilibrium (8.4) was obtained in [14]:

$$[(1 - c\Delta)^{-(k+1)} - 1]\left(\frac{a}{c} - 1\right) < 1, \tag{8.17}$$

and improved in [30]:

$$[(1 - c\Delta)^{-(k+1)} - 1]\ln\frac{a}{c} \leq 1. \tag{8.18}$$

Note that in the conditions (8.15) and (8.16) the assumption $c\Delta < 1$ does not have to be made. Let us show that even with the assumption $c\Delta < 1$ the conditions (8.15) and (8.16) (in deterministic case, i.e., by $\sigma^2 = 0$) are better than (8.18).

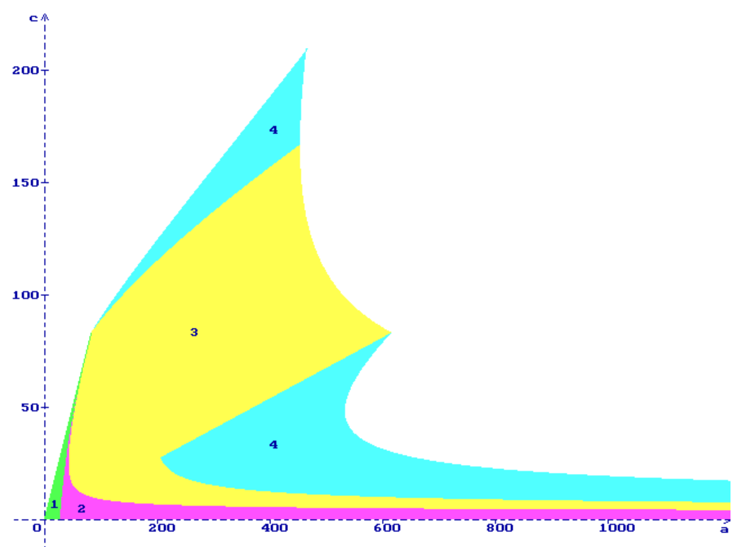

Figure 9. Region of sufficient stability condition and necessary and sufficient stability condition for the Equation (8.11): $p = 0$, $h = 0.024$ and $\Delta = 0.012$.

In fact, if $\sigma^2 = 0$ and $c\Delta < 1$, then the condition (8.15) takes the form $a < c\,e^2$. Representing (8.18) as

$$a \le c\,e^{[(1-c\Delta)^{-(k+1)}-1]^{-1}}, \tag{8.19}$$

one can see that (8.15) is better than (8.19) if $((1 - c\Delta)^{-(k+1)} - 1)^{-1} \le 2$ or $c\Delta \ge 1 - \left(\frac{2}{3}\right)^{\frac{1}{k+1}}$.

Let us show that the condition (8.16) is better than (8.18) for $c\Delta \in (0, 1)$. In fact, if $\sigma^2 = 0$, then condition (8.16) takes the form

$$a < c\,e^{\frac{ch+1}{c(h+0.5\Delta)}}. \tag{8.20}$$

Via (8.19) and (8.20) it is enough to show that

$$\frac{1}{(1 - c\Delta)^{-(k+1)} - 1} \le \frac{ch + 1}{c(h + 0.5\Delta)}$$

or the function

$$f(c) = \frac{1}{(1 - c\Delta)^{k+1}} - 1 - \frac{c(h + 0.5\Delta)}{ch + 1}$$

is nonnegative for $c\Delta \in [0, 1)$. This is in fact so, since $f(0) = 0$ and via $k\Delta = h$

$$f'(c) = \frac{h + \Delta}{(1 - c\Delta)^{k+2}} - \frac{h + 0.5\Delta}{(ch + 1)^2} \ge 0.$$

In Figure 9, one can see the stability regions for $h = 0.024$ and $\Delta = 0.012$ given by the condition (8.17) (number 1), given by the condition (8.18) (numbers 1 and 2), given by the conditions (8.12), (8.13) (numbers 1, 2 and 3) and given by the condition (8.14) (numbers 1, 2, 3 and 4).

8.3. Numerical analysis in the deterministic case

Consider the Equation (8.5) at first in the deterministic case $(p = 0)$ with delay $h = 0.024$. We will simulate solutions of this equation via its discrete analogue (8.11) with $\Delta = 0.012$. The corresponding stability region is shown in Figure 10. Note that for $p = 0$ the stability region slightly differs from the similar region for $p = 12$ (Figure 8). The initial function is $z(s) = a_0 \cos(s)$, $s \in [-h, 0]$, where a_0 has different values in different points.

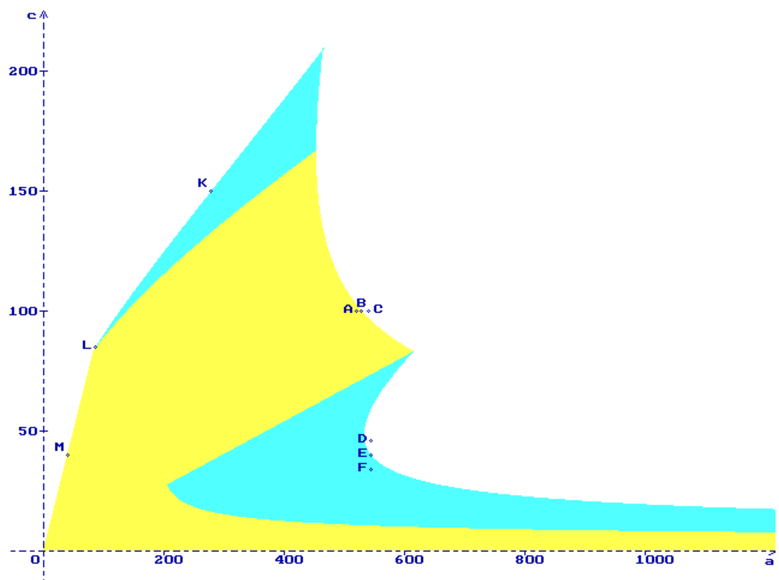

Figure 10. Region of sufficient stability condition and necessary and sufficient stability condition for the Equation (8.11): $p = 0$, $h = 0.024$ and $\Delta = 0.012$.

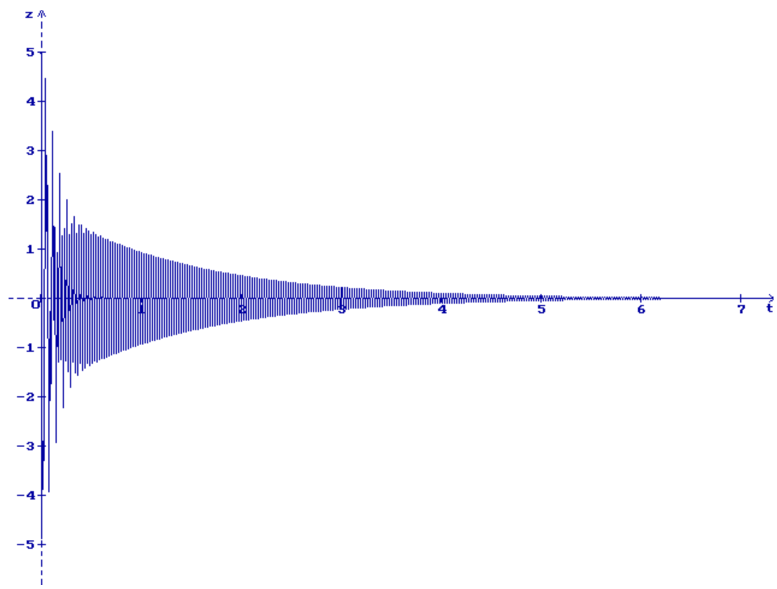

Figure 11. Stable zero solution of the Equation (8.11) in the point $A(520, 100)$, $a_0 = 5$.

In Figure 10, one can see the points A(520, 100), B(529.45, 100), C(540, 100), D(544.5, 46), E(544.5, 40), F(544.5, 34), K(279.9, 150), L(87.5, 85), and M(40, 40). The points A and F belong to the stability region, the points C and D do not belong to the stability region the points B, E, K, L and M are placed on the bound of the stability region. The trajectories of solutions of the Equation (8.11) at points A(520, 100), B(529.45, 100), C(540, 100), are shown in Figures 11, 12 and 13, respectively. One can see that on the bound of stability region (the point B) the solution is bounded, to move a bit outside of the stability region (the point C) gives an unstable solution and to move a bit inside of the stability region (the point A) gives the stable zero solution. A similar picture one can obtain in the points D(544.5, 46) (unstable solution), E(544.5, 40) (bounded solution), F(544.5, 34) (stable zero solution).

The points K, L and M are placed on the bound of the stability region, similarly to the points B and E the solutions of the Equation (8.11) in these points are bounded functions.

For instance, in the Figures 14, 15 and 16 the solutions of the Equation (8.11) are shown respectively in the point $L(87.5, 85)$ (bounded solution) and close to this point the points $L_1(88, 85)$ (the stable zero solution) and $L_2(87, 85)$ (unstable solution). Note also that in the case $b > 0$, $a = c > 0$ the initial Equation (8.2) has only the zero equilibrium and the solution of the Equation (8.11) is a constant: see Figure 17 for the point M.

This fact can be used to construct the exact bound of the stability region in the case when we have a sufficient stability condition only. For example, in the case $p = 0$, $h = 0.024$, $\Delta = 0.008$, the points $P(50, 50)$, $Q(288.65, 170)$, $R(680, 250.079)$, $T(923.63, 125)$, (Figure 18) belong to the bound of the exact stability region. In all these points the solution of the Equation (8.11) is bounded. In particular, in the point P similarly to the point M the solution is a constant.

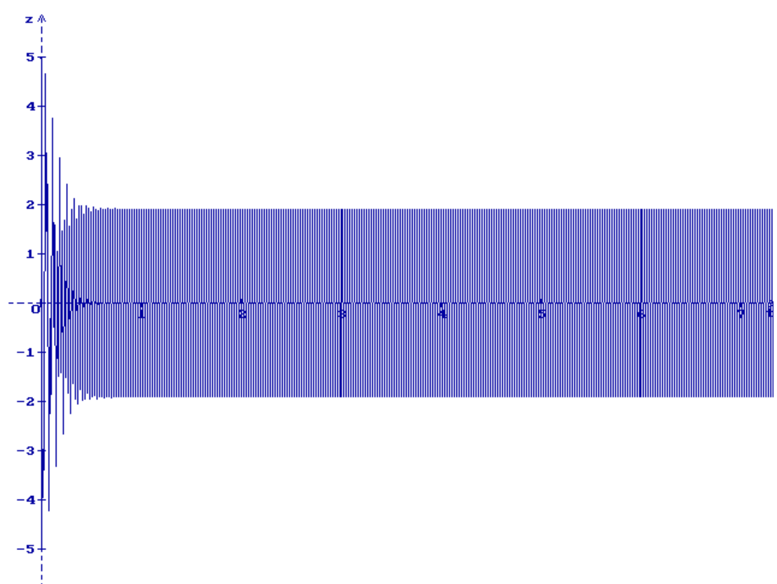

Figure 12. Bounded solution of the Equation (8.11) in the point $B(529.45, 100)$, $a_0 = 5$.

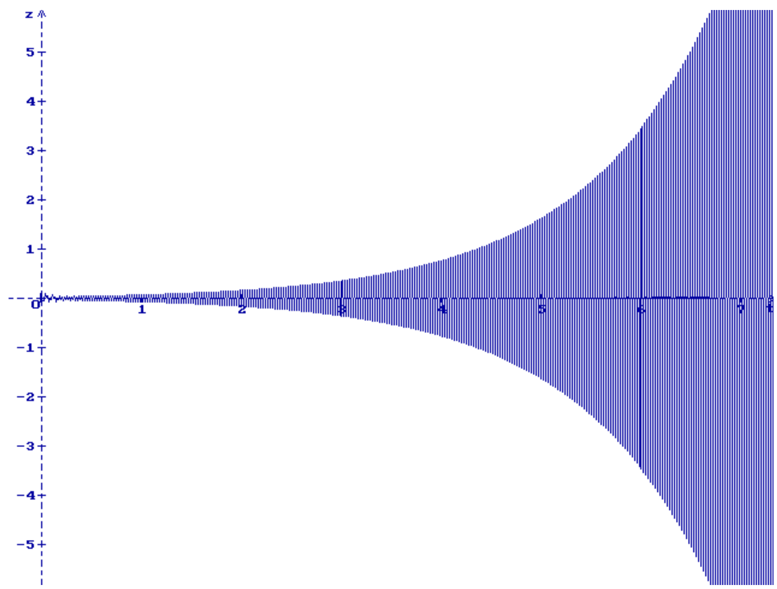

Figure 13. Unstable solution of the Equation (8.11) in the point $C(540,\,100)$, $a_0 = 0.1$.

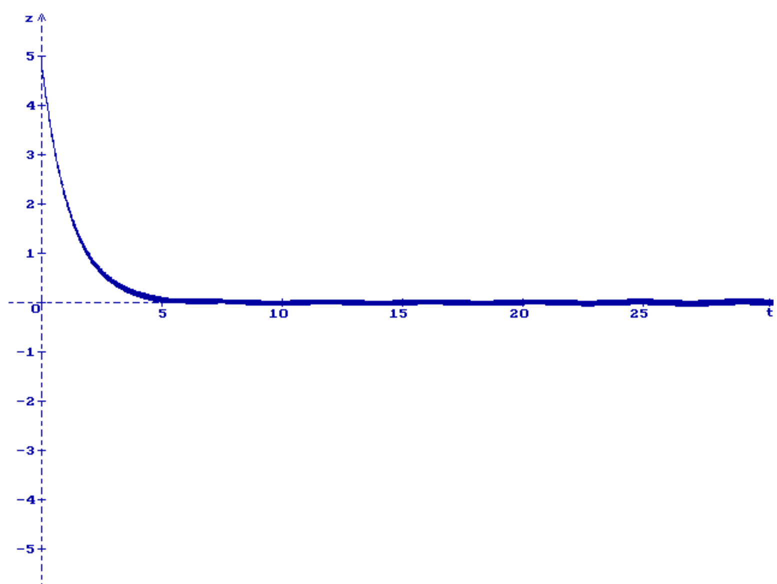

Figure 14. Bounded solution of the Equation (8.11) in the point $L(87.5,\,85)$, $a_0 = 5$.

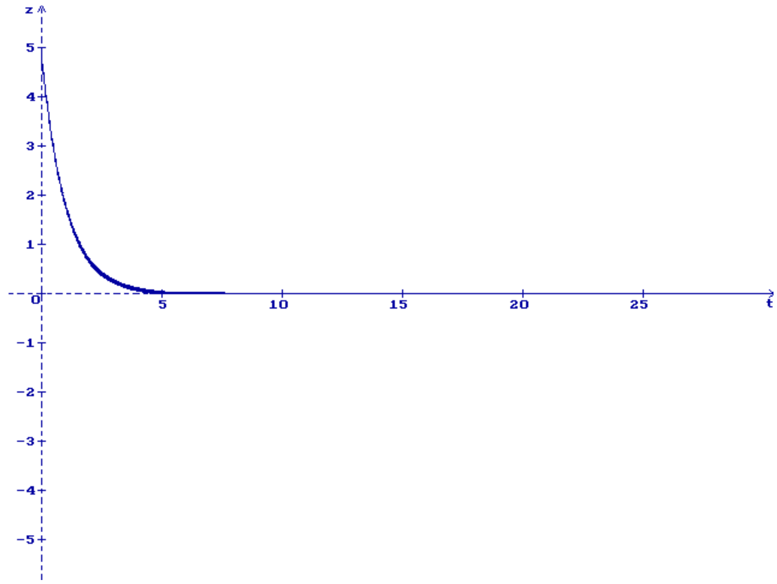

Figure 15. Stable zero solution of the Equation (8.11) in the point $L_1(88, 85)$, $a_0 = 5$.

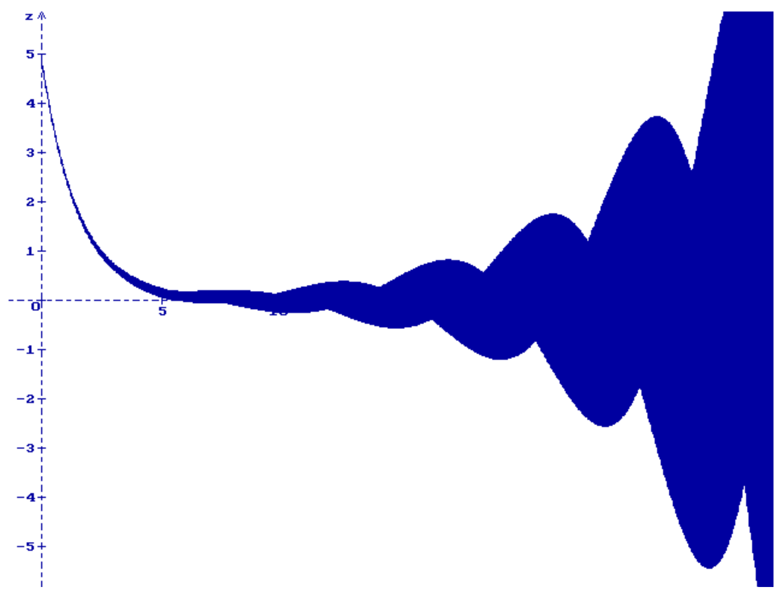

Figure 16. Unstable solution of the Equation (8.11) in the point $L_2(87, 85)$, $a_0 = 3$.

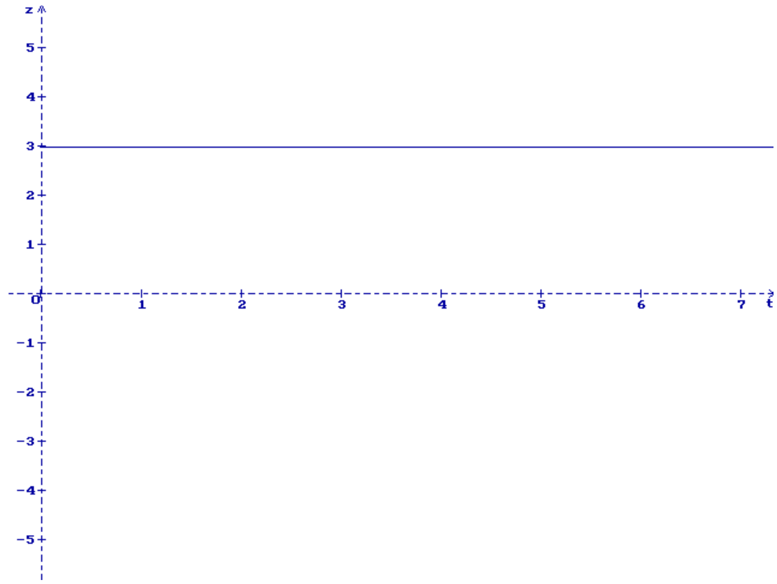

Figure 17. Stable solution of the Equation (8.11) in the point $M(40,\ 40)$, $a_0 = 3$.

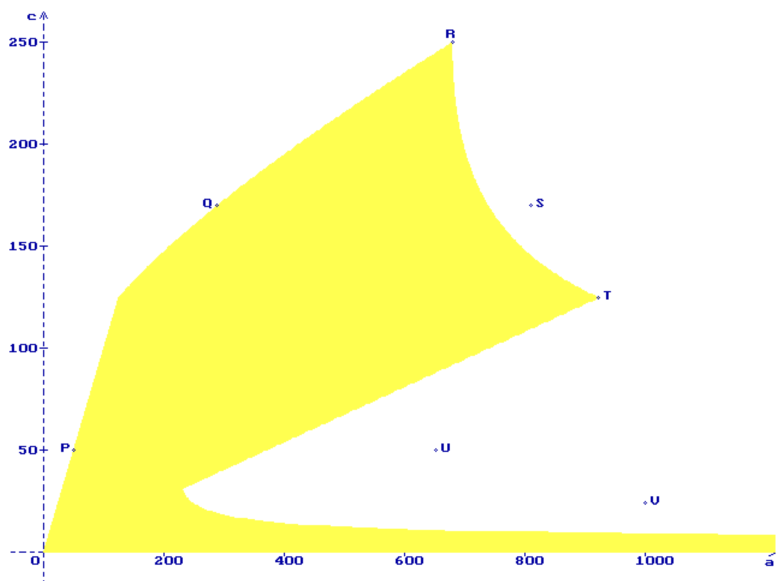

Figure 18. Region of sufficient stability condition for the Equation (8.11): $p = 0$, $h = 0.024$, $\Delta = 0.008$.

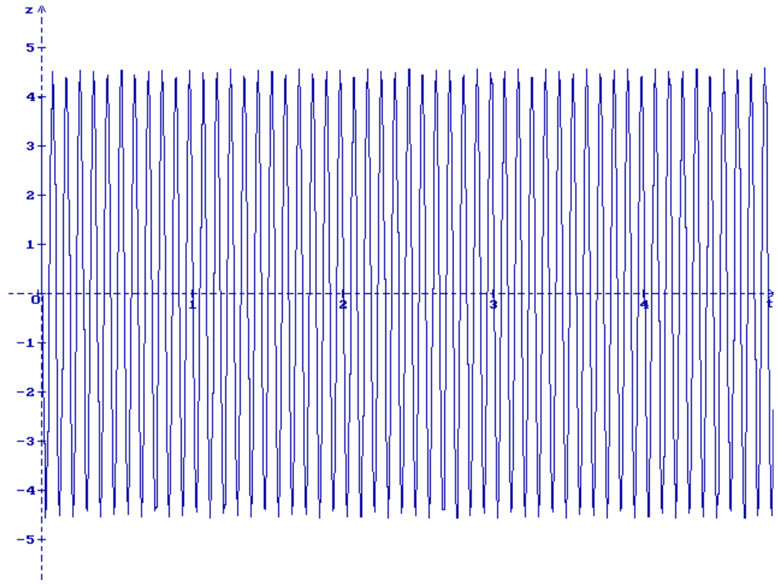

Figure 19. Bounded solution of the Equation (8.11) in the point $V(1000, 24.16)$, $a_0 = 4$.

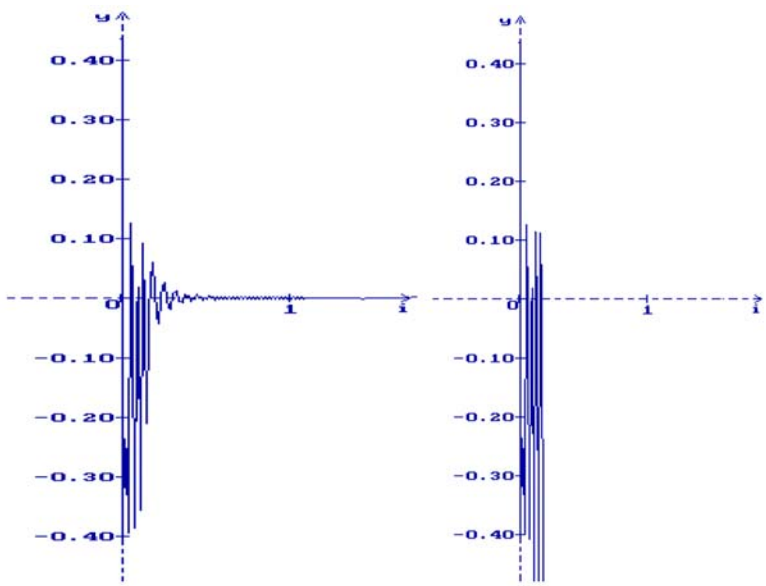

Figure 20. The solution of the Equation (8.10) in the point $A(520, 100)$ (Figure 10) for $a_0 = 0.437$ (left) and for $a_0 = 0.438$ (right).

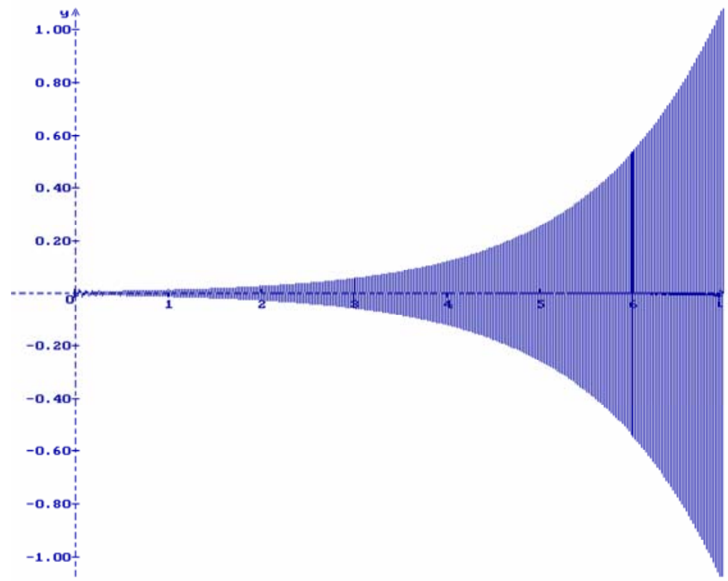

Figure 21. Unstable solution of the Equation (8.10) in the point $C(540, 100)$ (Figure 10) for $a_0 = 0.001$.

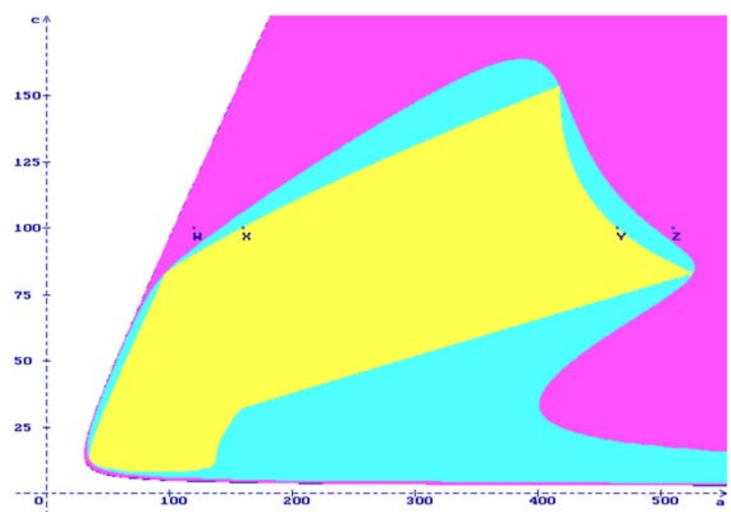

Figure 22. Regions of sufficient stability condition and necessary and sufficient stability condition for the Equation (8.11): $p = 12$, $h = 0.024$ and $\Delta = 0.012$.

Via numerical simulations it was found that in the points $S(810, 170)$, $U(652.6, 50)$, $V(1000, 24.16)$ (Figure 18) the solutions are bounded too (see, for instance, the point V, Figure 19), so, these points also belong to the bound of the exact stability region. If desired, one can get a lot of such points.

Consider now the behaviour of a solution of the nonlinear differential Equation (8.6) in the case $p = 0$. We will simulate solutions of this equation via its discrete analogue (8.10) with $\Delta = 0.012$. If in the point (a, c) the zero solution of the Equation (8.11) is asymptotically stable (it means that for arbitrary initial function the solution of the Equation (8.11) goes to zero) then the zero solution of the Equation (8.10) is stable in the first approximation (it means that for each small enough initial function the solution of the Equation (8.10) goes to zero). On the other hand if the zero solution of the Equation (8.11) is not asymptotically stable, then for arbitrary indefinitely small initial function the solution of the Equation (8.10) does not go to zero.

These facts are illustrated by the following examples. In the point $A(520, 100)$, the zero solution of the Equation (8.11) is asymptotically stable (Figure 11, $a_0 = 5$), so in this point the solution of the Equation (8.10) $(b = 4)$ goes to zero for small enough initial function (Figure 20, $a_0 = 0.437$, left) and quickly enough goes to infinity for a little larger initial function (Figure 20, $a_0 = 0.438$, right). In the point $C(540, 100)$, the zero solution of the Equation (8.11) is not asymptotically stable (Figure 13, $a_0 = 0.1$) and the solution of the Equation (8.10) $(b = 1)$ goes to infinity for an indefinitely small initial function (Figure 21, $a_0 = 0.001$).

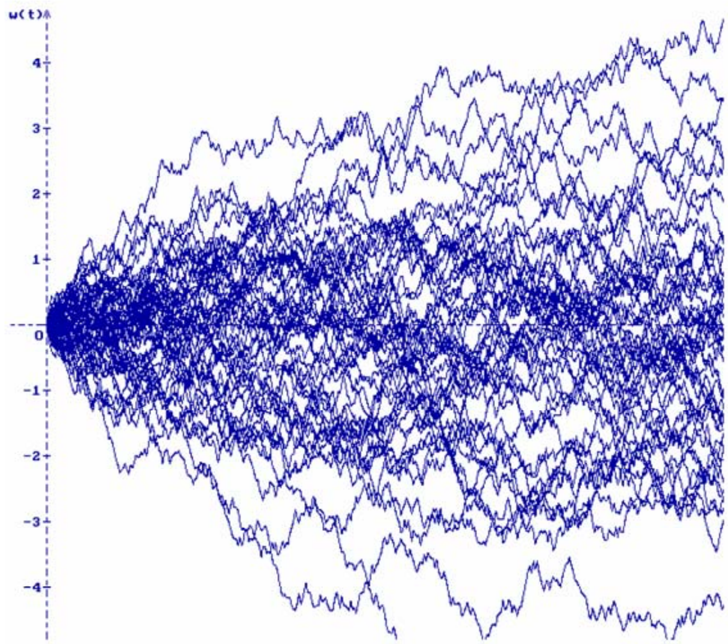

Figure 23. 50 trajectories of the standard Wiener process.

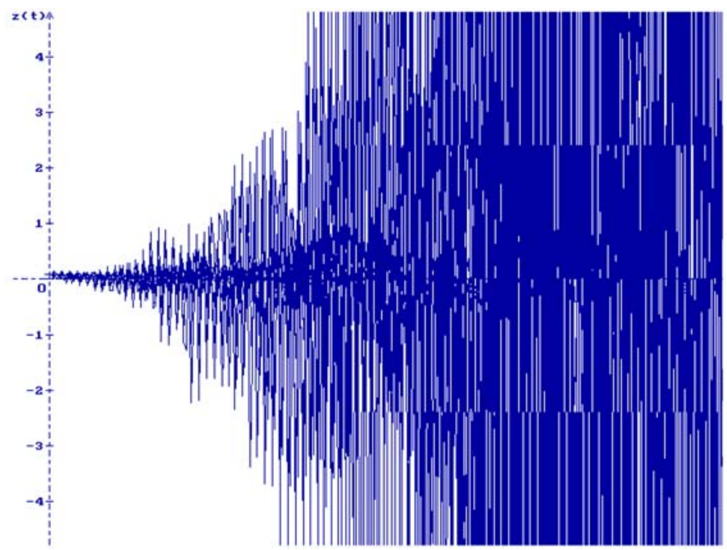

Figure 24. Unstable solution of the Equation (8.11) in the point $W(120, 100)$, $a_0 = 0.1$.

8.4. Numerical analysis in the stochastic case

Consider finally the behaviour of the solution of the Equation (8.7) in the stochastic case with $p = 12$, delay $h = 0.024$ and the initial function $z(s) = a_0 \cos(s)$, $s \in [-h, 0]$. A solution of this equation is simulated here via its discrete analogue (8.11) with $\Delta = 0.012$. The corresponding stability region is shown in Figure 22, which is the increasing copy of Figure 8 with the additional points $X(160, 100)$, $Y(465, 100)$, which belong to the stability region of the Equation (8.11), and the points $W(120, 100)$, $Z(510, 100)$, which do not belong to the stability region of the Equation (8.11).

For numerical simulation of the solution of the Equation (8.11), one uses some special algorithm of numerical simulation of the Wiener process trajectories [27]. Fifty trajectories of the Wiener process obtained via this algorithm are shown in Figure 23. In Figure 24, ten trajectories of the solution of the Equation (8.11) are shown in the point $W(120, 100)$ with $a_0 = 0.1$. The point $W(120, 100)$ belongs to the stability region of the stochastic differential Equation (8.7), but it does not belong to the stability region of its difference analogue (8.11). One can see that each trajectory of the solution of the Equation (8.11) in the point $W(120, 100)$ oscillates and goes to infinity.

A similar picture can be seen in Figure 25 where 100 trajectories of the solution of the Equation (8.11) are shown in the point $Z(510, 100)$ with $a_0 = 0.1$. In Figure 26, 100 trajectories of the solution of the Equation (8.11) are shown in the point $X(160, 100)$ with $a_0 = 8.5$. The point X belongs to the stability region of the Equation (8.11) and all trajectories go to zero. One hundred trajectories of the stable solution of the Equation (8.11) are shown also in Figure 27 in the point $Y(465, 100)$ with $a_0 = 6.5$.

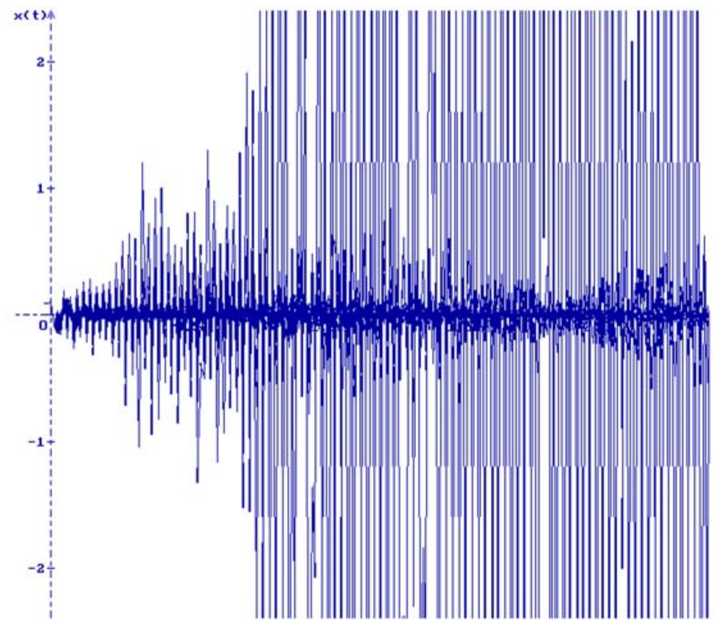

Figure 25. Unstable solution of the Equation (8.11) in the point $Z(510{,}100)$, $a_0 = 0.1$.

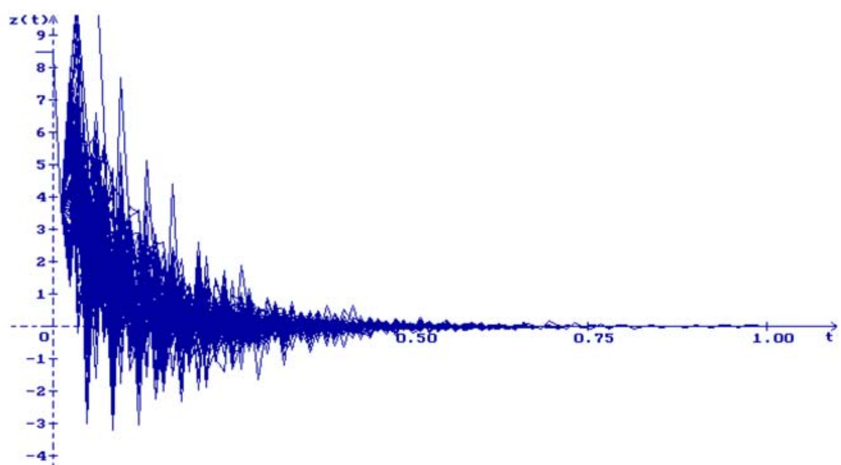

Figure 26. Stable solution of the Equation (8.11) in the point $X(160, 100)$, $a_0 = 8.5$.

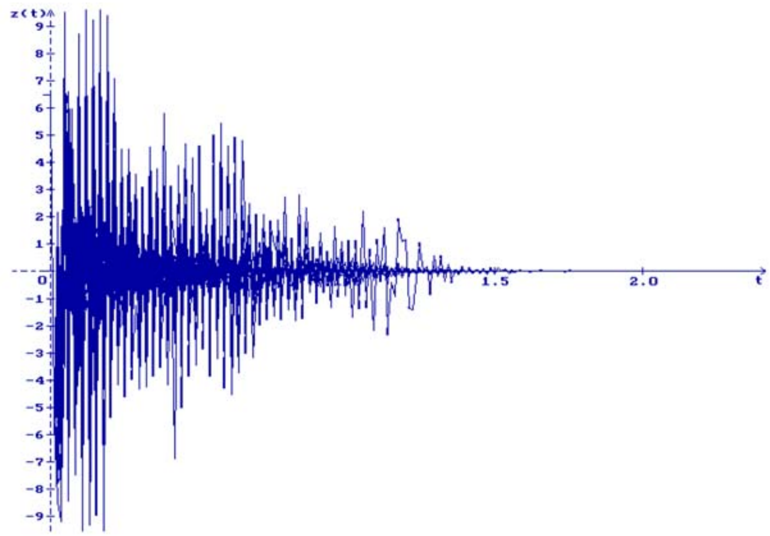

Figure 27. Stable solution of the Equation (8.11) in the point $Y(465, 100)$, $a_0 = 6.5$.

9. Conclusion

The paper is devoted to the important issue of compliance of numerical modeling of the solution of the difference analogue with the original nonlinear integro-differential equation. Various schemes for constructing difference analogues both with discrete and continuous time are considered. To study the stability of difference analogues, the general method of constructing Lyapunov functionals is used. Numerical examples with some well-known mathematical models demonstrate the effectiveness of the theoretical results and the possibility of their use in various applications.

References

[1] D. J. Acheson, A pendulum theorem, Proceedings of the Royal Society, Series A, London 443(1917) (1993), 239-245.

DOI: https://doi.org/10.1098/rspa.1993.0142

[2] D. J. Acheson and T. Mullin, Upside-down pendulums, Nature 366(6452) (1993), 215-216.

DOI: https://doi.org/10.1038/366215b0

[3] L. Berezansky, E. Braverman and L. Idels, Nicholson's blowflies differential equations revisited: Main results and open problems, Applied Mathematical Modelling 34(6) (2010), 1405-1417.

DOI: https://doi.org/10.1016/j.apm.2009.08.027

[4] J. A. Blackburn, H. J. T. Smithand and N. Gronbech-Jensen, Stability and Hopf bifurcations in an inverted pendulum, American Journal of Physics 60(10) (1992), 903-908.

DOI: https://doi.org/10.1119/1.17011

[5] P. Borne, V. Kolmanovskii and L. Shaikhet, Steady-state solutions of nonlinear model of inverted pendulum, Theory of Stochastic Processes 5(3-4) (1999), 203-209.

[6] P. Borne, V. Kolmanovskii and L. Shaikhet, Stabilization of inverted pendulum by control with delay, Dynamic Systems and Applications 9(4) (2000), 501-515.

[7] N. Bradul and L. Shaikhet, Stability of the positive point of equilibrium of Nicholson's blowflies equation with stochastic perturbations: Numerical analysis, Discrete Dynamics in Nature and Society (2007); Article ID 092959, pages 25.

DOI: https://doi.org/10.1155/2007/92959

[8] N. Bradul and L. Shaikhet, Stability of a difference analogue of the mathematic predator-prey model with stochastic perturbations, Mathematics and Mechanics, Odessa National University 14(20) (2009), 7-23 (in Russian).

[9] X. Ding and W. Li, Stability and bifurcation of numerical discretization Nicholson blowflies equation with delay, Discrete Dynamics in Nature and Society (2006); Article ID 19413, 12 pages.

DOI: https://doi.org/10.1155/DDNS/2006/19413

[10] I. I. Gikhman and A. V. Skorokhod, Stochastic Differential Equations, Springer, Berlin, 1972.

[11] W. Gurney, S. Blythe and R. Nisbet, Nicholson's blowflies revised, Nature 287 (1980), 17-21.

DOI: https://doi.org/10.1038/287017a0

[12] P. Imkeller and Ch. Lederer, Some formulas for Lyapunov exponents and rotation numbers in two dimensions and the stability of the harmonic oscillator and the inverted pendulum, Dynamic Systems 16(1) (2001), 29-61.

DOI: https://doi.org/10.1080/02681110010001289

[13] P. L. Kapitza, Dynamical stability of a pendulum when its point of suspension vibrates, Pergamon Press, Oxford 2 (1965), 714-725.

[14] V. L. Kocic and G. Ladas, Oscillation and global attractivity in a discrete model of Nicholson's blowflies, Applicable Analysis 38(1-2) (1990), 21-31.

DOI: https://doi.org/10.1080/00036819008839952

[15] M. Levi, Stability of the inverted pendulum: A topological explanation, SIAM Review 30(4) (1988), 639-644.

[16] M. Levi and W. Weckesser, Stabilization of the inverted linearized pendulum by high frequency vibrations, SIAM Review 37(2) (1995), 219-223.

[17] C.-K. Lin, C.-T. Lin, Y. Lin and M. Mei, Exponential stability of nonmonotone traveling waves for Nicholson's blowflies equation, SIAM Journal of Mathematical Analysis 46(2) (2014), 1053-1084.

DOI: https://doi.org/10.1137/12090439

[18] B. Liu, Global exponential stability of positive periodic solutions for a delayed Nicholson's blowflies model, Journal of Mathematical Analysis and Applications 412(1) (2014), 212-221.

DOI: https://doi.org/10.1016/j.jmaa.2013.10.049

[19] G. Maruyama, Continuous Markov processes and stochastic equations, Rendiconti del Circolo Matematico di Palermo: Series 2, 4(1) (1955), 48-90.

DOI: https://doi.org/10.1007/BF02846028

[20] G. J. Mata and E. Pestana, Effective Hamiltonian and dynamic stability of the inverted pendulum, European Journal of Physics 25(6) (2004), 717-721.

DOI: https://doi.org/10.1088/0143-0807/25/6/003

[21] R. Mitchell, Stability of the inverted pendulum subjected to almost periodic and stochastic base motion: An application of the method of averaging, International Journal of Non-Linear Mechanics 7(1) (1972), 101-123.

DOI: https://doi.org/10.1016/0020-7462(72)90025-X

[22] A. J. Nicholson, An outline of the dynamics of animal populations, Australian Journal of Zoology 2(1) (1954), 9-65.

DOI: https://doi.org/10.1071/ZO9540009

[23] A. I. Ovseyevich, The stability of an inverted pendulum when there are rapid random oscillations of the suspension point, Journal of Applied Mathematics and Mechanics 70(5) (2006), 762-768.

DOI: https://doi.org/10.1016/j.jappmathmech.2006.11.010

[24] L. Shaikhet, Stability of difference analogue of linear mathematical inverted pendulum, Discrete Dynamics in Nature and Society (2005); Article ID 149487, pp. 215-226.

DOI: https://doi.org/10.1155/DDNS.2005.215

[25] L. Shaikhet, About stability of a difference analogue of a nonlinear integro-differential equation of convolution type, Applied Mathematics Letters 19(11) (2006), 1216-1221.

DOI: https://doi.org/10.1016/j.aml.2006.01.004

[26] L. Shaikhet, Lyapunov Functionals and Stability of Stochastic Difference Equations, Springer Science & Business Media, 2011.

DOI: https://doi.org/10.1007/978-0-85729-685-6

[27] L. Shaikhet, Lyapunov Functionals and Stability of Stochastic Functional Differential Equations, Springer Science & Business Media, 2013.

DOI: https://doi.org/10.1007/978-3-319-00101-2

[28] L. Shaikhet and J. Roberts, Reliability of difference analogues to preserve stability properties of stochastic Volterra integro-differential equations, Advances in Difference Equations (2006); Article ID 073897, 22 pages.

DOI: https://doi.org/10.1155/ADE/2006/73897

[29] R. Sharp, Y.-H. Tsai and B. Engquist, Multiple time scale numerical methods for the inverted pendulum problem, In Book: Multiscale Methods in Science and Engineering, Lecture Notes in Computational Science and Engineering, Berlin: Springer 44 (2005), 241-261.

DOI: https://doi.org/10.1007/3-540-26444-2_13

[30] JW-H. So and J. S. Yu, On the stability and uniform persistence of a discrete model of Nicholson's blowflies, Journal of Mathematical Analysis and Applications 193(1) (1995), 233-244.

DOI: https://doi.org/10.1006/jmaa.1995.1231

[31] R. O. A. Taie and D. A. M. Bakhit, Some new results on the uniform asymptotic stability for Volterra integro-differential equations with delays, Mediterranean Journal of Mathematics 20 (2023); Article 280.

DOI: https://doi.org/10.1007/s00009-023-02489-w

[32] C. Tunc and O. Tunc, On the stability, integrability and boundedness analyses of systems of integro-differential equations with time-delay retardation, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales, Serie A: Matematicas 115(3) (2021); Article 115.

DOI: https://doi.org/10.1007/s13398-021-01058-8

[33] C. Tunc and O. Tunc, On the fundamental analyses of solutions to nonlinear integro-differential equations of the second order, Mathematics 10(22) (2022); Article 4235.

DOI: https://doi.org/10.3390/math10224235

[34] C. Tunc, O. Tunc and J. C. Yao, On the new qualitative results in integro-differential equations with Caputo fractional derivative and multiple kernels and delays, Journal of Nonlinear and Convex Analysis 23(11) (2022), 2577-2591

[35] C. Tunc, O. Tunc, C. F. Wen and J. C. Yao, On the qualitative analyses solutions of new mathematical models of integro-differential equations with infinite delay, Mathematical Methods in the Applied Sciences 46(13) (2023), 14087-14103.

DOI: https://doi.org/10.1002/mma.9306

[36] B. G. Zhang and H. X. Xu, A note on the global attractivity of a discrete model of Nicholson's blowflies, Discrete Dynamics in Nature and Society 3 (1999); Article ID 362607, 51-55.

DOI: https://doi.org/10.1155/S1026022699000072

■