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BEYOND BETA AND VASICEK: A COMPARATIVE ANALYSIS OF CONTINUOUS DISTRIBUTIONS ON (0, 1)

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Abstract

Distribution families on the unit interval play an important role in many statistical applications, especially in the field of finance. In the course of the recent years, it became law to some extent to use the Beta and the Kumaraswamy distribution, respectively, if loss rates are assumed to be stochastic and to use the Vasicek distribution as the corresponding pendant for default rates. On the other hand, a deeper look into the general statistical literature reveals several possible alternatives which are not so familiar in the financial community. Against this background and with view to possible model risk, we provide a comparative analysis of twelve two-parametric distribution families.

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1. Introduction

To deal with the uncertainty of the future, input parameters of nearly all statistical models have to rely on plausible distributional assumptions. Especially in the area of credit risk management, probability distributions on the unit interval $[0, 1]$ play an important role, uppermost in the context of LGDs und PDs[1].

Loss given default (LGD) is the proportion of the counterparty's exposure that will be lost if a default occurs. Consequently, uncertainty regarding the actual LGD is an important source of credit portfolio risk. The recovery rate (RR) is defined as its complement with respect to the full, i.e., one minus LGD. Referring to the distributional assumption of LGDs or RRs, Gupton and Stein [9] state that LGDs or RRs have a density that theoretically should be best described by a Beta distribution since the support is limited to the interval between zero and one with various shapes (e.g., U-shaped, J-shaped or uniform) governed by two parameters[2]. In contrast, Höcht [11] advocated the less prominent Kumaraswamy distribution.

Probability of Default (PD) describes the likelihood of a default over a particular time horizon, typically one year. It provides an estimate that a borrower will be unable to meet its debt obligations and is a key parameter used in the calculation of economic or regulatory capital for a financial institution. Typically, the Vasicek distribution, which arises in Vasicek's portfolio model by determining the PD in a downturn situation, is chosen, see Vasicek [29, 30].

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[1]Beyond LGD, probability distributions arise as link functions in the context of credit portfolio modeling. Above that, distribution functions on [0, 1] can be used as weighting functions (see, e.g., Jones [16]) in order to generate exible distributions on $\mathbb{R}$ which might be also used in market risk or operational risk.

[2]In order to resemble bimodality, Hlawatsch and Ostrowksi [10] approximate the LGD by a mixture of Beta distributions.

Beyond PD and LGD, Tasche [24] stated that when fitting different distributions to the same mean and standard deviation, the Vasicek, Kumaraswamy and Beta distributions do not differ considerably. The Vasicek distribution however, is not well-known and it is difficult to locate literature on its implementation. Vasicek distributions also require numerous inputs of different variables which complicates implementation. The Kumaraswamy and Beta methods are simpler, but the Kumaraswamy distribution has some implementation problems as moment matching for these distributions requires complex numerical solving of a two-dimensional optimisation problem[3].

As mentioned above, the PD and LGD literature primarily focuses on the Beta, the Kumaraswamy and the Vasicek distribution. However, in the general statistical literature, a number of other flexible distributional models appear which might serve as alternative. Against this background, the outline of this paper is as follows: Section 2 briefly summarizes the candidate functions classifying them as simple, standard (from a finance perspective), exotic and new ones (to the best of our knowledge). In Section 3, we present a detailed analysis in order to compare the flexibility of these distributions. Section 4 concludes.

2. Flexible Two-Parameter Distributions on [0, 1]

To keep within reasonable bounds, we restrict ourselves to the discussion of flexible *two-parametric* distribution families which we divide into four subfamilies, termed as simple ones, classical ones, exotic or less popular ones (at least in the financial literature) and new ones forth on. Extensions of these families to three or more parameter are already available to some extent but beyond the scope of this work. Although the goodness-of-fit might be improved applying these families, analytical tractability and convenient statistical properties often get lost.

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[3]Frye [4] derive the LGD (distribution) function under the assumption that the conditional default rate has a Vasicek distribution.

2.1. The simple ones

Uniform distribution: For a detailed treatment of the uniform distribution on the unit interval (Notaion $X \sim U(0, 1)$), we refer to Johnson et al. [14] and Evans et al. [3]. Without any doubt, it is the most famous but also simplest one of those we consider in the sequel and has density $f_{UNI}(x) = 1$ for $0 \leq x \leq 1$. The uniform distribution is used if only minimum and maximum (here 0 and 1) of a random variable X are known.

Triangular distribution: The triangular distribution is typically used as a subjective description of a population for which there is only limited sample data, and especially in cases where the relationship between variables is known but data is scarce. It is based on a knowledge of the minimum and maximum and an "inspired guess" as to the modal value. For these reasons, the triangle distribution has been called a *lack of knowledge* distribution. In the standard case on $[0, 1]$, the density is

$$f_{TRI}(x) = \begin{cases} \dfrac{2x}{m}, & 0 \leq x \leq m, \\ \dfrac{2(1-x)}{1-m}, & m < x \leq 1. \end{cases}$$

Kotz and van Dorp [18] deal with the ML estimation of m.

Trapezoidal distribution: A straightforward generalization is the trapezoidal distribution with density

$$f_{TRA}(x) = \begin{cases} \dfrac{2x}{m_1(m_2 - m_1 + 1)}, & 0 \leq x \leq m_1, \\ \dfrac{2}{m_2 - m_1 + 1}, & m_1 < x \leq m_2, \\ \dfrac{2(1-x)}{(1-m_2)(m_2 - m_1 + 1)}, & m_2 < x \leq 1, \end{cases}$$

which reduces to the triangle distribution on $[0, 1]$ if $m_1 = m_2$. For a detailed discussion of this class, see also Kotz and van Dorp [18].

2.2. The standard ones

Beta distribution: Note that if $X_1, \ldots, X_n$ are independent and identically distributed random variables from $U(0, 1)$ and if $X_{(r)}$ denotes the r-th order statistics of this sample, then the pdf of $X_{(r)}$ is a Beta distribution with parameters $a = r$ and $b = n - r + 1$. Note that the general Beta distribution has density

$$f_U(x, \alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad x \in [0, 1], \quad \alpha, \beta > 0, \tag{1}$$

and $B(\alpha, \beta)$ denotes the Beta function of the second kind (see, e.g., Jones [15]).

Kumaraswamy distribution: The Kumaraswamy distribution was introduced by Kumaraswamy [19] in 1980. Referring to Kazemi et al. [17], it can be characterized by the following representation: Consider an iid random sample from a uniform variable U on $(0, 1)$ denoted by $\mathcal{X} = \{U_{11}, \ldots, U_{1m}, \ldots, U_{n1}, \ldots, U_{nm}\}$. The cumulative distribution function of

$$X = \min_{1 \le i \le n} \max_{1 \le j \le m} U_{ij}$$

is given by $F_X(x) = 1 - (1 - x^m)^n$. Therefore, it is also called the MinMax distribution, see Lawrence and McQueston [20]. Alternatively, given a generalized exponential Y_{GEX} (see Gupta & Kundu [8]), then $Z = \exp(-Y_{GEX})$ has a Kumaraswamy law under a simple re-parameterization.

Vasicek distribution: The Vasicek distribution dates back to Vasicek [29], [30], [31] and was intensively discussed by Tasche [24]. It arises in the context of credit risk as an approximative distribution of loss rates in large, homogeneous portfolios. Its density on $[0, 1]$ reads

$$f(x; \varrho, p) = \sqrt{\frac{1-\varrho}{\varrho}} \exp\left(\frac{1}{2}\left(\Phi^{-1}(x)^2 - \left(\frac{\sqrt{1-\varrho}\,\Phi^{-1}(x) - \Phi^{-1}(p)}{\sqrt{\varrho}}\right)^2\right)\right), \; 0 < \rho, \; p < 1,$$

and allows for different and flexible shapes.

2.3. The exotic ones

Generalized Topp Leone distribution: Topp and Leone [25] proposed this distribution as an alternative to the Beta distribution. The generalized Topp-Leone family of distributions which was recovered by Nadarajah and Kotz [22] is generated from a slope distribution with density $f(x) = \alpha x - 2(\alpha - 1)x$, $\alpha \in [0, 2]$ by elevating the corresponding cdf to a power $\beta > 0$, see also Vicari et al. [32].

Generalized Biparabolic distribution: Similar to the triangular distribution, the *standard* biparabolic distribution is obtained from the minimum (a), most likely (m), and maximum (b). It can be constructed as follows, see Figure 1 the values a, m, and b determine the parabolas $f_1(x)$ from $(0, 0)$ and the vertex (m, h), and the parabola $f_2(x)$ using the point $(1, 0)$ and the vertex (m, h). Alternatively, using the construction scheme suggested by van Dorp and Kotz [28], the density reads as

$$f_{BP}(x;\ m,\ p(y)) = \begin{cases} p(x / m), & 0 \le x \le m \\ p\left(\dfrac{1 - x}{1 - m}\right), & m \le x \le 1 \end{cases} \quad \text{with } p(x) = -\frac{3}{2}(x^2 - 2x).$$

The *generalized biparabolic distribution* (GBP) of García et al. [5] arises from the generator function

$$p(x;\ \theta) = \frac{(2\theta + 1)(\theta + 1)}{-3\theta - 1}(x^{2\theta} - 2x^{\theta}), \quad \theta \ge 0.$$

It holds that $f_{GBP}(0;\ m,\ \theta) = f_{GBP}(1;\ m,\ \theta) = 0$ regardless of θ and m.

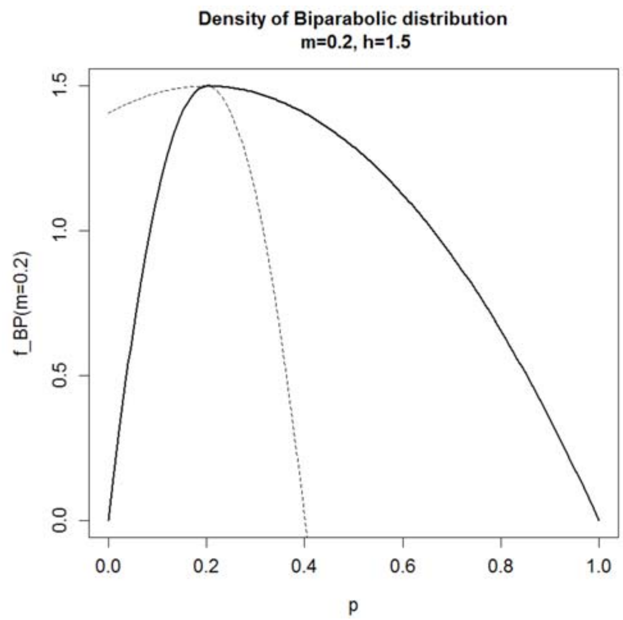


Figure 1. Construction of the standard biparabolic distribution.

Two-sided power distribution: Let U_1, U_2 be iid $U(0, 1)$ variables. Then the variable $Z = (1 - m)\min\{U_1, U_2\} + m\max\{U_1, U_2\}$ has a two sided power distribution with density given in Table 6. Please note that $\min\{U_1, U_2\}$ and $\max\{U_1, U_2\}$ have right and left triangle distribution. For details on the TSP distribution which can be seen as nonsmooth alternative to the Beta distribution, we refer to van Dorp and Kotz [27].

Negative Log-Gamma distribution: The name of a Negative Log-Gamma variable derives from its construction scheme $Z = \exp(-Y_G)$ which shows that the negative logarithm of a NLG variable is Gamma distributed with parameter $\alpha, \beta > 0$. Its density is of the form

$$f_{NLG}(x; \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{(1/\beta)-1}(-\ln x)^{\alpha-1}, \quad \alpha, \beta > 0.$$

In contrast to its competitors, both cdf and quantile function are not available in closed form. For details on the NLG which is often applied in reliability analysis, we refer to Martz and Waller [21], p. 242 and Allella et al. [1].

Log-Lindley distribution: Recently, Gómez-Déniz et al. [7] introduced the Log-Lindley distribution. Starting from a special case $(\alpha = 1, \eta = \theta, \text{ and } \lambda = 1/\gamma)$ of the generalized Lindley of Zakerzadeh and Dolati [33] on the positive axis, a LL variable is defined as $Z = \exp(-Y_{GenLindley})$. It has density

$$f_{LLI}(x; \eta, \lambda) = \frac{\eta^2}{1 + \lambda\eta}(\lambda - \ln x)x^{\eta-1}, \quad \eta > 0, \lambda \geq 0,$$

and admits a closed form cdf, and allows for increasing, decreasing, and unimodal densities.

Johnson-SB distribution: Dating back to 1949, the Johnson SB distribution (see Johnson et al. [13], p. 34 and Stuart and Ord [23], p. 240 for details) results from a transformed normal distribution. Its cdf can be briefly written as

$$F_{JSB}(x; \gamma, \delta) = \Phi(F_{\text{Log}}^{-1}(x; \gamma, \delta)), \quad \delta > 0, \gamma \in \mathbb{R}, \tag{2}$$

where F_{Log}^{-1} denotes the quantile function of a logistic variable with location γ and scale δ. In contrast to all the other distributions we consider, the JSB distribution allows for bimodal densities and $f_{JSB}(0; \gamma, \delta) = f_{JSB}(1; \gamma, \delta) = 0$ regardless of γ and δ. Hence, such densities take somewhat of a U-shaped form.

Logistic uniform distribution: Recently, Torabi and Montazeri [26] discussed, amongst others, the LU distribution. Similar to the JSB-construction in (2), its cdf is defined as

$$F_{LU}(x; m, s) = F_{\text{Log}}(F_{\text{Log}}^{-1}(x; 0, 1); m, s), \quad m \in \mathbb{R}, s > 0, \tag{3}$$

such that all generating functions (pdf, cdf, and quantile function) are in closed form. Above that, LU densities can be unimodal or anti-unimodal, increasing or decreasing, symmetric or skewed.

2.4. The new ones

Negative Log-Champernowne: In line with NLG and LLI, the Negative Log-Champernowne distribution denotes a random variable $Z = \exp(-Y_{Champ})$, where Y_{Champ} in turn denotes the Champernowne distribution [2] on $(0, \infty)$ which was originally introduced as a model for income distributions but also comes to application in the area of operational risk. Its density reads as

$$f(x;\, \alpha,\, M) = \frac{\alpha M^{\alpha} x^{\alpha-1}}{(x^{\alpha} + M^{\alpha})^2}, \quad M, \alpha > 0.$$

To our best knowledge, the NLC distribution has not been mentioned yet. Its pdf, cdf, and qf admit a closed-form solution presented in Table 6.

Negative Log-Weibull: Similar to the NLG, one can define a new probability law on $[0, 1]$ by $Z = \exp(-W)$, where W has a classical Weibull distribution. Again, pdf, cdf, and qf are available in closed-form. Note that the Weibull pdf reads as

$$f(x;\, a,\, b) = ab(ax)^{b-1} \exp(-(ax)^{b}).$$

Details on the NLG are also presented in Table 6 (given in last section).

3. A Comparative Analysis

For reason of simplicity, we use the following abbreviations: Uniform (UNI), triangle (TRI), trapezoidal (TRA), Beta (BETA), Kumaraswamy (KUM), Vasicek (VAS), Johnson-SB (JSB), Logistic uniform (LUN), Log-Lindley (LLI), Negative Log-Gamma (NLG), Generalized Biparabolic (GBP), Two-sided power (TSP), Generalized Topp-Leone (GTL), Negative Log-Champernowne (NLC), and Negative Log-Weibull (NLW) distribution. The corresponding parameter domain is summarized in Table 1 below.

Table 1. Distributions on $[0, 1]$ under consideration

Abbreviation	Name	θ_1	θ_2
UNI	Uniform		
TRI	Triangle	$m \in [0, 1]$	
TRA	Trapezoidal	$m_1 \in [0, m_2]$	$m_2 \in [m_1, 1]$
BETA	Beta	$\alpha > 0$	$\beta > 0$
KUM	Kumaraswamy	$\alpha > 0$	$\beta > 0$
VAS	Vasicek	$0 < p < 1$	$0 < \rho < 1$
JSB	Johnson S_B	$\gamma \in \mathbb{R}$	$\delta > 0$
LUN	Logistic uniform	$m \in \mathbb{R}$	$s > 0$
LLI	LogLindley	$\lambda \geq 0$	$\eta > 0$
GBP	Generalized biparabolic	$m \in [0, 1]$	$\theta > 0$
TSP	Two-sided Power	$m \in [0, 1]$	$\alpha > 0$
GTL	Generalized Topp-Leone	$\alpha \in (0, 2]$	$\beta > 0$
NLG	Negative LogGamma	$\alpha > 0$	$\beta > 0$
NLC	Negative LogChampernowne	$M > 0$	$\alpha > 0$
NLW	Negative Log-Weibull	$a > 0$	$b > 0$

3.1. Statistical properties: An overview

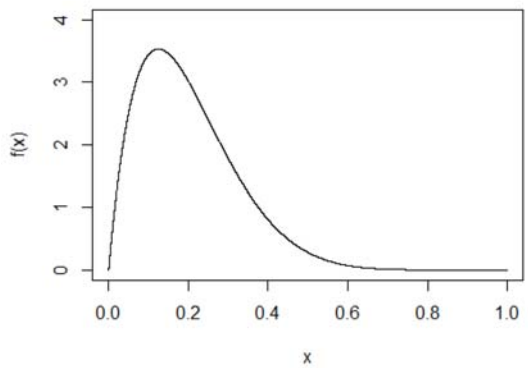
Table 2 below summarizes the possible shape(s) of the densities under consideration together with the corresponding domain of parameters. Neglecting the simple ones, most of the distribution families include unimodal, anti-unimodal, increasing, and decreasing densities. There are certain restrictions for LLI or GBP, for instance.

Table 2. Shapes of the probability densities

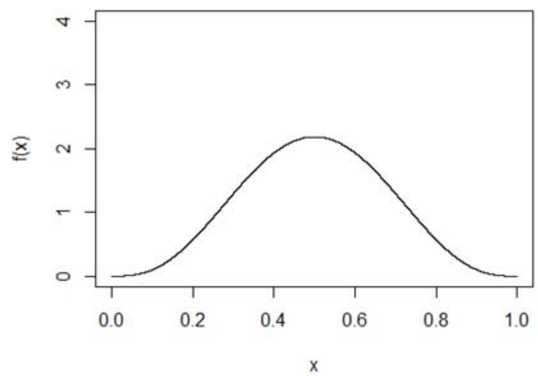
$X \sim$	Unimodal	Anti-unimodal, U-shape	Increasing	Decreasing
TRI	$0 \le m \le 1$	–	–	–
TRA	$m_1 = m_2$	–	–	–
$BETA(\alpha, \beta)$	$\alpha > 1, \beta > 1$	$\alpha < 1, \beta < 1$	$\alpha \ge 1, \beta \le 1$	$\alpha \le 1, \beta \ge 1$
$KUM(\alpha, \beta)$	$\alpha > 1, \beta > 1$	$\alpha < 1, \beta < 1$	$\alpha \ge 1, \beta \le 1$	$\alpha \le 1, \beta \ge 1$
$VAS(\varrho, p)$	$\varrho < \frac{1}{2}$	$\varrho > \frac{1}{2}$	$\varrho = \frac{1}{2}, p > \frac{1}{2}$	$\varrho = \frac{1}{2}, p < \frac{1}{2}$
$JSB(\gamma, \delta)$	see [12]	–	–	–
$LUN(m, s)$	$s < 1$	$s > 1$	$m < 0, s = 1$	$m > 0, s = 1$
$LLI(\lambda, \eta)$	$\eta > 1, \lambda(\eta - 1) < 1$	–	$\eta > 1, \lambda(\eta - 1) \ge 1$	$\eta \le 1$
$GBP(m, \theta)$	$\alpha \ne 1$	–	–	–
$TSP(m, \alpha)$	$\alpha > 1, m \in [0, 1]$	$0 < m, \alpha < 1$	$m = 1, \alpha > 0$	–
$GTL(\alpha, \beta)$	$\alpha \in (0, 2], \beta > 1$	$\alpha \in (0, 2], \beta < 1$	$\alpha \in (0, 1), \beta \ge 1$	$\alpha \in (1, 2], \beta \le 1$
$NLG(\alpha, \beta)$	$\alpha > 1, \beta < 1$	$\alpha < 1, \beta > 1$	$\alpha \le 1, \beta \le 1$	$\alpha \ge 1, \beta \ge 1$
$NLC(\alpha, M)$	–	$\alpha \ne 1; \alpha = 1, M < 2$	–	$\alpha = 1, M \ge 2$
$NLW(\alpha, \beta)$	$b > 1$	$b < 1$	$\alpha > 1, b = 1$	$\alpha < 1, b = 1$

3.2. Reproducing different shapes

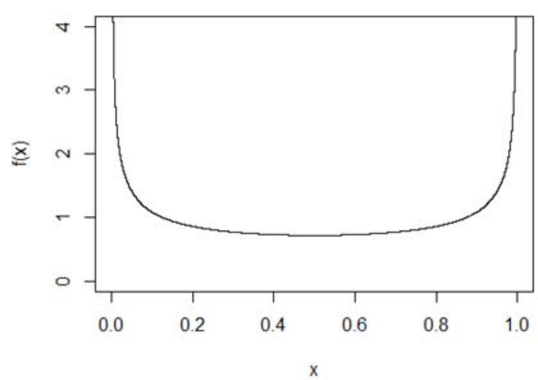
In order to verify the exibility of the underlying distributions, we choose six different shapes of a Beta and a triangular distribution, respectively, which one might come across to: unimodal and skewed, see Figure 2(a), unimodal and symmetric, see Figure 2(b), antiunimodal and symmetric, see Figure 2(c), anti-unimodal and skewed, see Figure 2(d), triangular and symmetric, see Figure 2(e), triangular and skewed, see Figure 2(f).



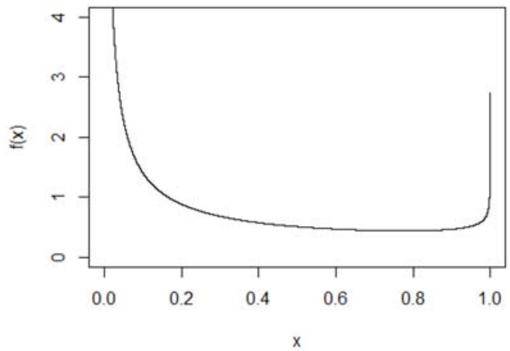
(a) $X \sim BETA\,(2, 8)$



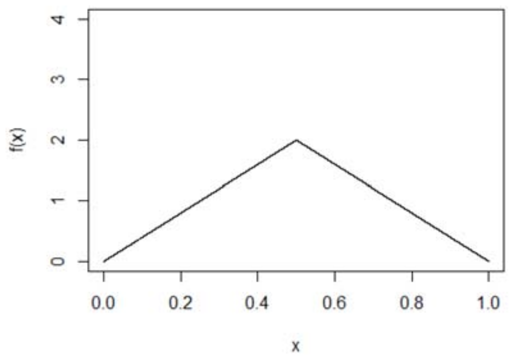
(b) $X \sim BETA\,(4, 4)$



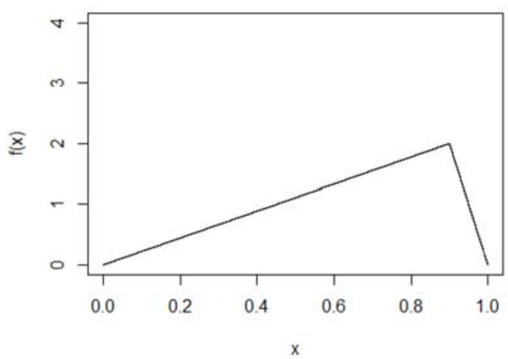
(c) $X \sim BETA(\frac{3}{5}, \frac{3}{5})$



(d) $X \sim BETA(\frac{3}{10}, \frac{4}{5})$



(e) $X \sim TRI(\frac{1}{2})$



(f) $X \sim TRI(\frac{9}{10})$

Figure 2. Densities to be reproduced.

For each distribution, we draw a random sample of length $n = 1000$ several times (number of repetitions: 1000) and fitted the models under consideration using maximum likelihood estimation. In order to compare the goodness-of-fit (see, for instance, Gibbs [6]), we calculated both Kolmogorov-Smirnov distance between the sample(s) and the fitted distribution(s)

$$d_{KS}(X, Y) := \sup \left| F_X(x) - \hat{F}_Y(x) \right|, \quad x \in [0, 1],$$

and Hellinger distance between the original density and the estimated density:

$$d_H(X, Y) := \frac{1}{\sqrt{2}} \left[\int_0^1 \left(\sqrt{f_X(t)} - \sqrt{\hat{f}_Y(x)} \right)^2 dt \right]^{1/2}.$$

Tables 3 and 4 summarizes the mean $(\hat{\mu})$ and the standard deviation $(\hat{\sigma})$ of the repeated experiments. First of all, the results w.r.t. Kolmogorov-Smirnov and Hellinger metric are very similar. Beta shapes are best approximated (in the sense of a small distance) by NLG, KUM or VAS and (only) in the symmetric case also by JSB, LUN and GBP. In contrast, LLI and TSP provide a poor fit. LUN, GBP, and TSP indicate best approximation to triangular densities for the symmetric case, whereas JSB, LUN, and TSP are favourable in the asymmetric case.

Table 3. Distances w.r.t. Kolmogorov-Smirnov metric

$X \sim$		$BETA(2, 8)$	$BETA(4, 4)$	$BETA(\frac{3}{5}, \frac{3}{5})$	$BETA(\frac{3}{10}, \frac{4}{5})$	$TRI(\frac{1}{2})$	$TRI(\frac{9}{10})$
$U(0, 1)$	$\hat{\mu}$	0.537	0.189	0.092	0.375	0.136	0.232
	$\hat{\sigma}$	0.009	0.007	0.010	0.015	0.008	0.013
TRI	$\hat{\mu}$	0.322	0.066	0.263	0.282	0.022	0.025
	$\hat{\sigma}$	0.010	0.007	0.013	0.015	0.006	0.008
TRA	$\hat{\mu}$	0.322	0.066	0.092	0.373	0.021	0.024
	$\hat{\sigma}$	0.010	0.007	0.010	0.019	0.008	0.008
BETA	$\hat{\mu}$	0.019	0.020	0.020	0.021	0.030	0.040
	$\hat{\sigma}$	0.005	0.005	0.005	0.006	0.007	0.007
KUM	$\hat{\mu}$	0.025	0.026	0.021	**0.021**	0.031	0.040
	$\hat{\sigma}$	0.006	0.007	0.006	0.006	0.007	0.007
VAS	$\hat{\mu}$	0.025	**0.020**	**0.020**	0.024	0.030	0.033
	$\hat{\sigma}$	0.007	0.005	0.005	0.006	0.006	0.007
JSB	$\hat{\mu}$	0.045	0.021	0.044	0.093	0.038	**0.022**
	$\hat{\sigma}$	0.008	0.005	0.007	0.010	0.0070	0.005
LUN	$\hat{\mu}$	0.030	0.023	0.021	0.052	**0.020**	0.027
	$\hat{\sigma}$	0.005	0.005	0.005	0.006	0.005	0.005

Table 3. (Continued)

$X \sim$		$BETA(2, 8)$	$BETA(4, 4)$	$BETA(\frac{3}{5}, \frac{3}{5})$	$BETA(\frac{3}{10}, \frac{4}{5})$	$TRI(\frac{1}{2})$	$TRI(\frac{9}{10})$
LLI	$\hat{\mu}$	0.204	0.123	0.115	0.047	0.053	0.049
	$\hat{\sigma}$	0.008	0.009	0.012	0.010	0.010	0.008
GBP	$\hat{\mu}$	0.073	0.020	0.091	0.376	0.025	0.035
	$\hat{\sigma}$	0.014	0.005	0.010	0.015	0.006	0.010
TSP	$\hat{\mu}$	0.328	0.034	0.024	0.044	0.020	0.022
	$\hat{\sigma}$	0.007	0.007	0.006	0.010	0.005	0.011
GTL	$\hat{\mu}$	0.241	0.121	0.088	0.025	0.039	0.048
	$\hat{\sigma}$	0.008	0.076	0.011	0.006	0.009	0.006
NLG	$\hat{\mu}$	**0.024**	**0.020**	0.023	0.022	0.030	0.040
	$\hat{\sigma}$	0.006	0.005	0.006	0.006	0.007	0.007
NLC	$\hat{\mu}$	0.028	0.032	0.071	0.061	0.033	0.038
	$\hat{\sigma}$	0.005	0.005	0.005	0.006	0.005	0.005
NLW	$\hat{\mu}$	0.055	0.042	0.033	0.025	0.047	0.041
	$\hat{\sigma}$	0.007	0.007	0.007	0.006	0.008	0.006

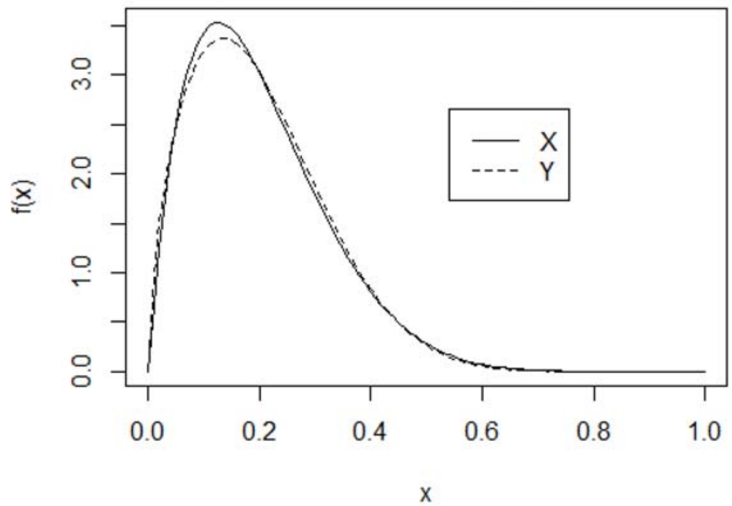
Table 4. Distances w.r.t. Hellinger metric

$X \sim$		$BETA(2, 8)$	$BETA(4, 4)$	$BETA(\frac{3}{5}, \frac{3}{5})$	$BETA(\frac{3}{10}, \frac{4}{5})$	$TRI(\frac{1}{2})$	$TRI(\frac{9}{10})$
UNI	$\hat{\mu}$	0.521	0.359	0.155	0.360	0.239	0.239
	$\hat{\sigma}$	0.000	0.000	0.000	0.000	0.000	0.000
TRI	$\hat{\mu}$	0.332	0.131	0.303	0.334	0.009	0.009
	$\hat{\sigma}$	0.001	0.001	0.001	0.000	0.001	0.001
TRA	$\hat{\mu}$	0.332	0.131	0.155	0.361	0.013	0.012
	$\hat{\sigma}$	0.001	0.001	0.001	0.004	0.010	0.010
BETA	$\hat{\mu}$	0.014	0.014	0.014	0.012	0.037	0.062
	$\hat{\sigma}$	0.007	0.007	0.008	0.008	0.003	0.002
KUM	$\hat{\mu}$	0.030	0.031	0.016	**0.014**	0.038	0.059
	$\hat{\sigma}$	0.004	0.004	0.007	0.007	0.003	0.002
VAS	$\hat{\mu}$	0.032	**0.015**	**0.015**	0.029	0.039	0.051
	$\hat{\sigma}$	0.004	0.007	0.007	0.005	0.003	0.002
JSB	$\hat{\mu}$	0.074	0.023	0.076	0.145	0.060	**0.032**
	$\hat{\sigma}$	0.003	0.005	0.004	0.006	0.003	0.004
LUN	$\hat{\mu}$	0.078	0.040	0.024	0.118	**0.022**	0.051
	$\hat{\sigma}$	0.002	0.003	0.005	0.003	0.005	0.002

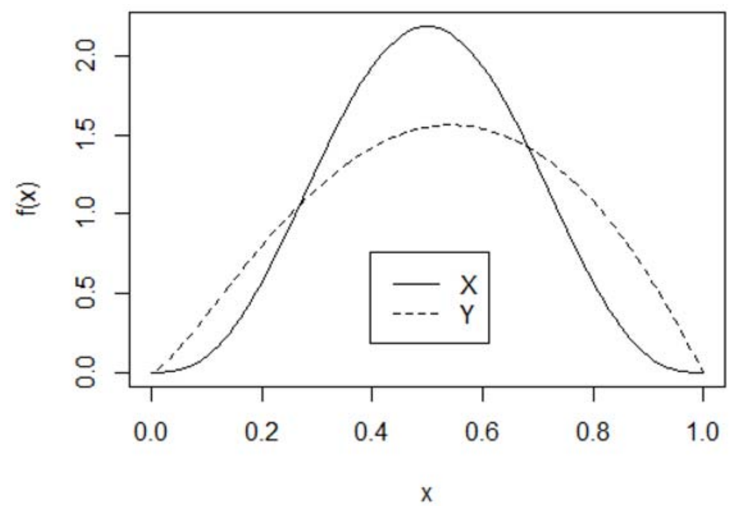
Table 4. (Continued)

$X \sim$		$BETA(2, 8)$	$BETA(4, 4)$	$BETA(\frac{3}{5}, \frac{3}{5})$	$BETA(\frac{3}{10}, \frac{4}{5})$	$TRI(\frac{1}{2})$	$TRI(\frac{9}{10})$
LLI	$\hat{\mu}$	0.294	0.172	0.137	0.051	0.057	0.095
	$\hat{\sigma}$	0.000	0.000	0.001	0.003	0.002	0.003
GBP	$\hat{\mu}$	0.101	0.023	0.155	0.360	0.033	0.033
	$\hat{\sigma}$	0.006	0.005	0.000	0.000	0.003	0.004
TSP	$\hat{\mu}$	0.443	0.053	0.025	0.053	0.015	0.015
	$\hat{\sigma}$	0.000	0.003	0.005	0.005	0.008	0.016
GTL	$\hat{\mu}$	0.322	0.151	0.114	0.035	0.042	0.102
	$\hat{\sigma}$	0.000	0.051	0.001	0.003	0.002	0.002
NLG	$\hat{\mu}$	**0.027**	**0.015**	0.018	0.015	0.039	0.064
	$\hat{\sigma}$	0.004	0.007	0.006	0.007	0.003	0.002
NLC	$\hat{\mu}$	0.113	0.077	0.049	0.024	0.069	0.083
	$\hat{\sigma}$	0.004	0.003	0.003	0.005	0.003	0.002
NLW	$\hat{\mu}$	0.061	0.083	0.177	0.096	0.089	0.101
	$\hat{\sigma}$	0.002	0.002	0.006	0.005	0.002	0.002

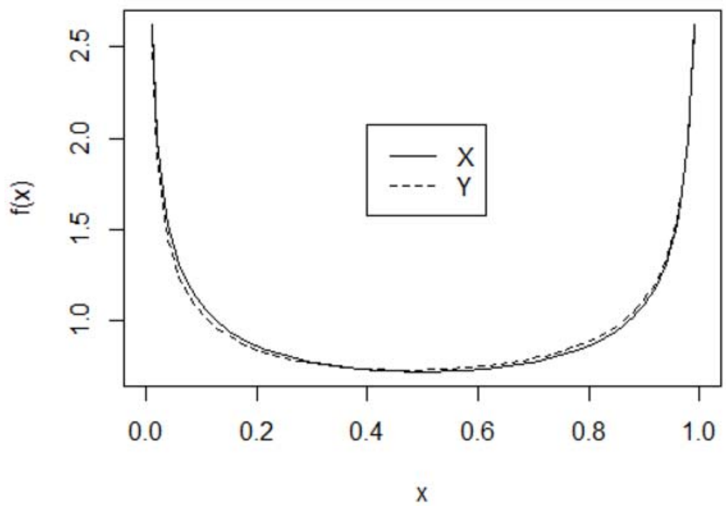
Exemplarily, Figure 3 illustrates 4 situations with a very good fit (see panel (a), (c)) and, in contrast, a poor fit (see panel (b), (d)).



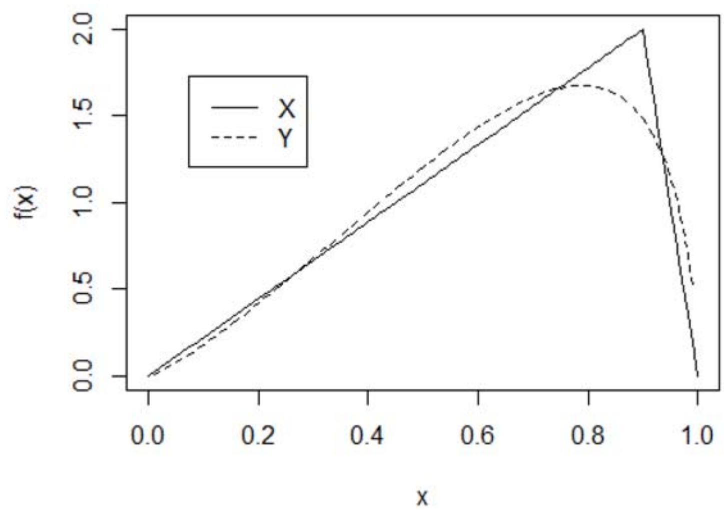
(a) $X \sim BETA(2, 8)$, $Y \sim KUM(1.62, 10.55)$



(b) $X \sim BETA(4, 4)$, $Y \sim LLI(0, 2.64)$



(c) $X \sim BETA(\frac{3}{5}, \frac{3}{5}),\ Y \sim NLG(0.62, 2.01)$

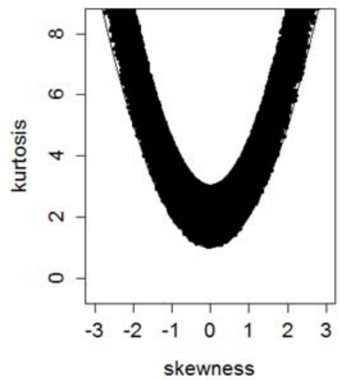


(d) $X \sim TRI(\frac{9}{10}),\ Y \sim VAS(0.33, 0.63)$

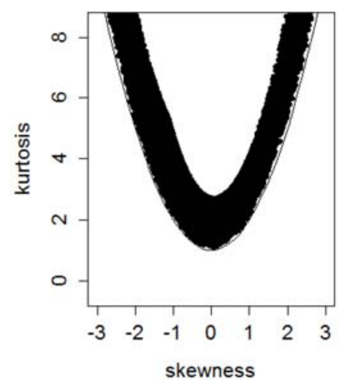
Figure 3. Selected densities of estimated distribution.

3.3. Moment ratio diagrams and $\mu - \sigma$ diagrams

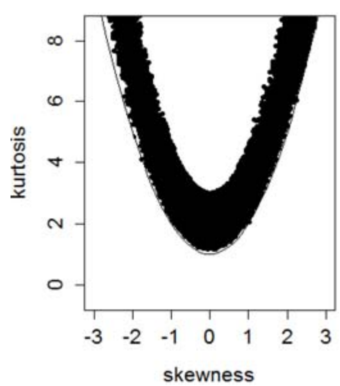
In order to compare flexibility from another perspective, moment ratio diagrams (or skewness and kurtosis plots) are provided in Figure 4. Moment ratio diagrams provide a useful visual assessment of skewness and kurtosis and consists of all possible pairs of third and fourth standardized moments (M_3, M_4) that can be obtained through different combinations of the shape parameters of the underlying distributions, provided their existence. In general, the relation $M_3 < \sqrt{M_4 - 1}$ for $M_4 \geq 1$ holds, i.e., for a given level of kurtosis only a finite range of skewness may be spanned. As a result from Figure 4, a restricted area has to be stated for LLI, GTL and to some extent for GBP, whereas LUN, NLC und TSP seem to be very flexible. Above that, moment ratio diagrams of NLG, VAS, KUM, BETA, and NLW are rather similar. Finally, figure illustrates the area of possible $\mu - \sigma$-combinations (μ: expectation value, sd: standard deviation) for the underlying distributions. Whereas BETA, KUM, VAS, JSB, LUN, NLG, NLC, and NLW are close together, certain restrictions appear for LLI, GBP, TSP, and GTL.



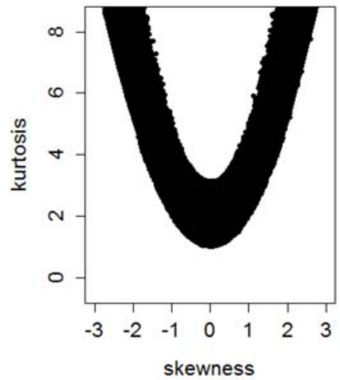
(a) BETA



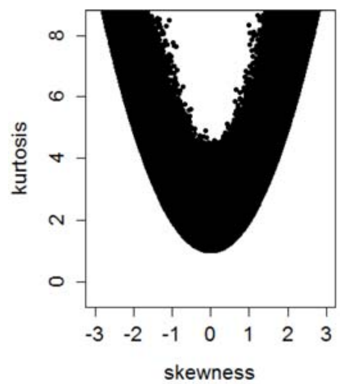
(b) KUM



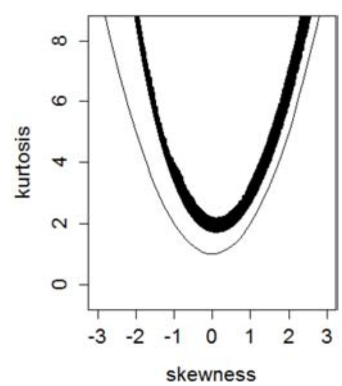
(c) VAS



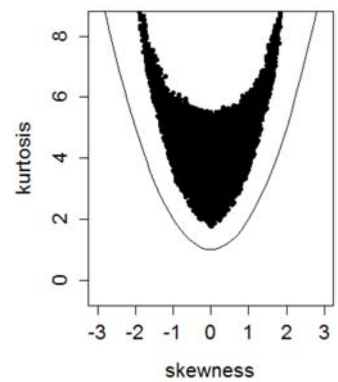
(d) JSB



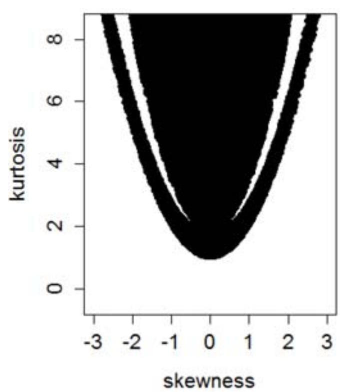
(e) LUN



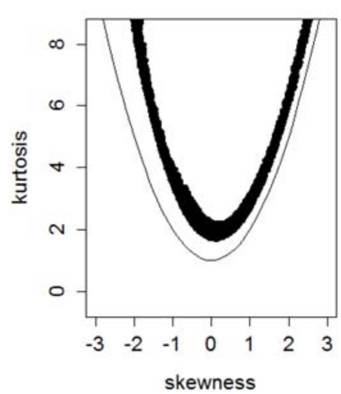
(f) LLI



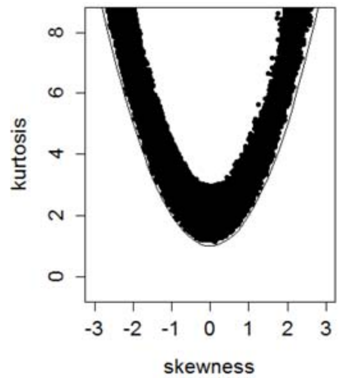
(g) GBP



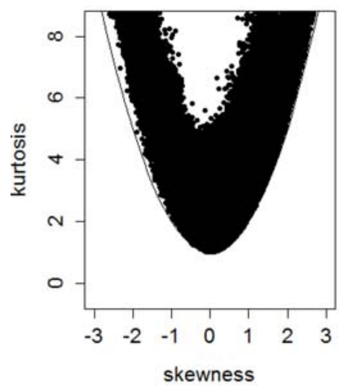
(h) TSP



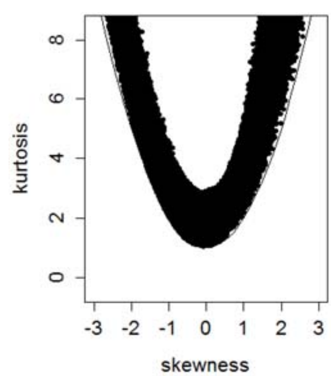
(i) GTL



(j) NLG

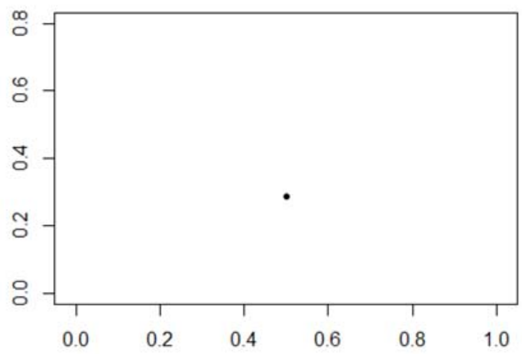


(k) NLC

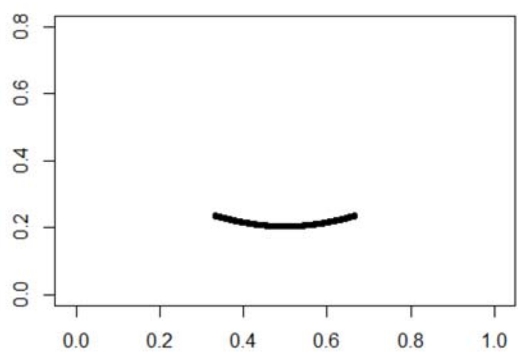


(l) NLW

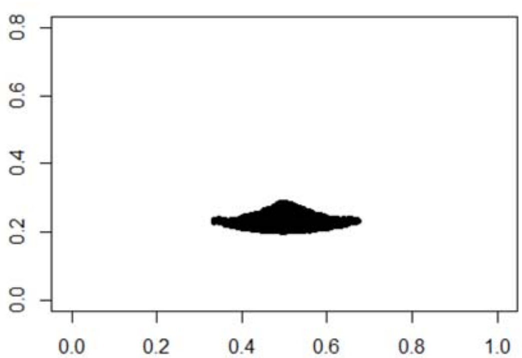
Figure 4. Skewness-kurtosis plots (y-axis: skewness, x-axis: kurtosis).



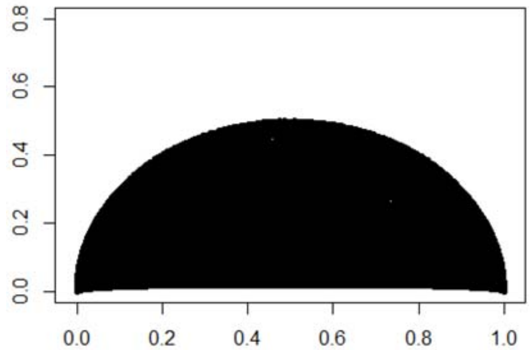
(a) UNI



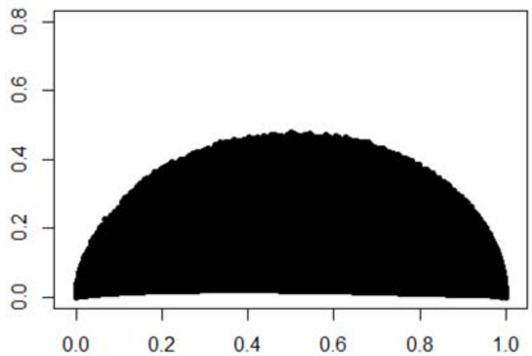
(b) TRI



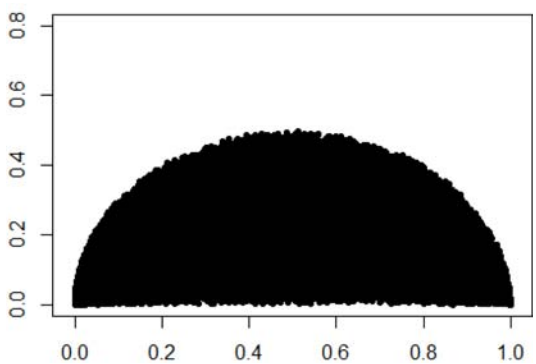
(c) TRA



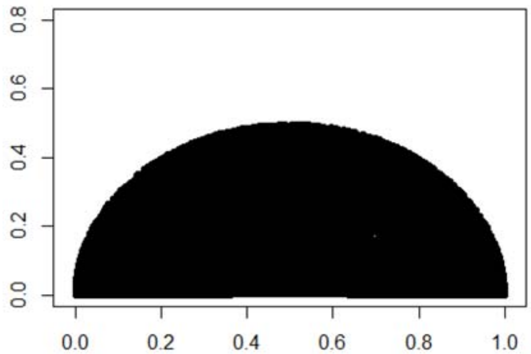
(d) BETA



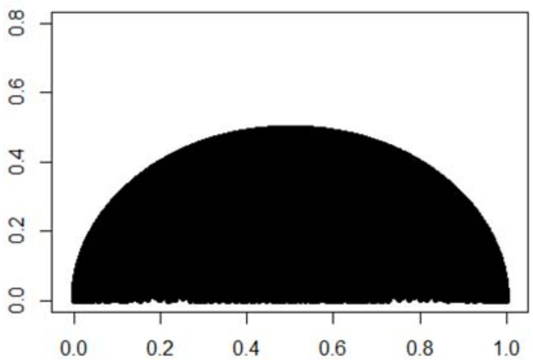
(e) KUM



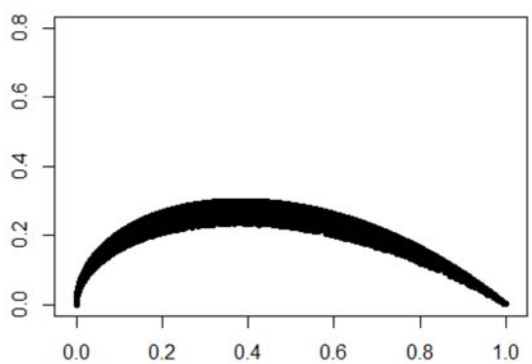
(f) VAS



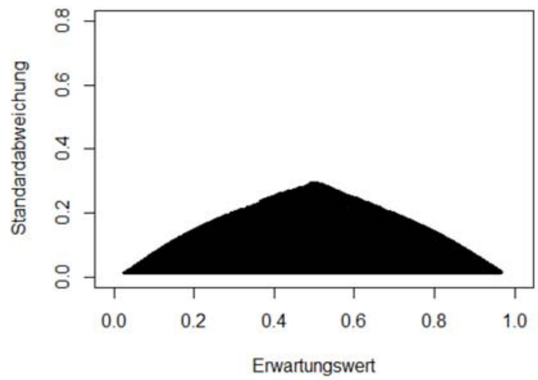
(g) JSB



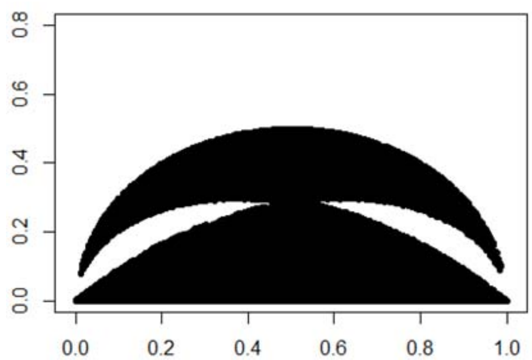
(h) LUN



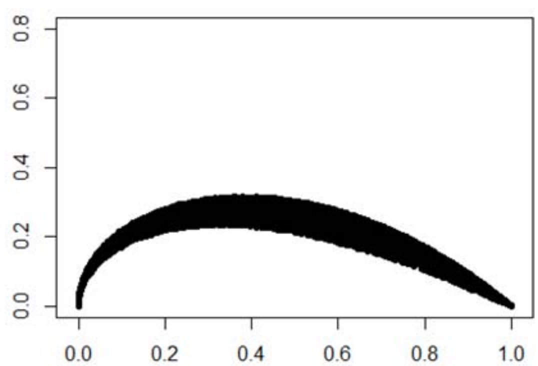
(i) LLI



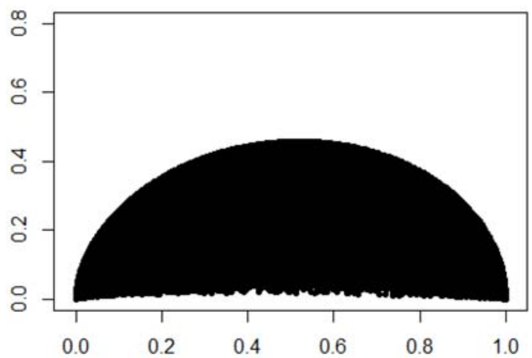
(j) GBP



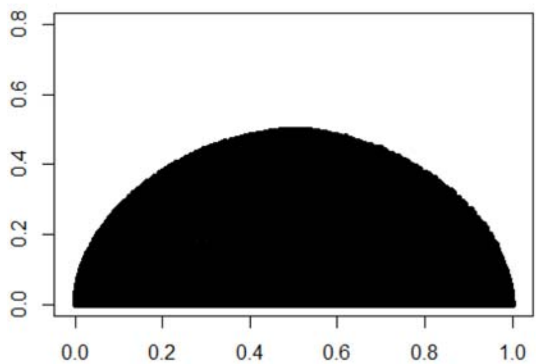
(k) TSP



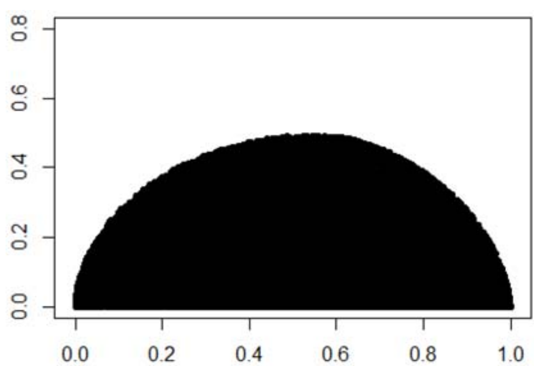
(l) GTL



(m) NLG



(n) NLC



(o) NLW

Figure 5. Possible $\mu - \sigma$ combinations (y-axis: μ, x-axis: σ).

3.4. Performance tests

In order to compare the performance with respect to random number generation, we generated 100,000,000 random variables from all distributions. Without loss of generality, the parameterizations are chosen close to the triangular distributions, as shown in Figure 6. Table 5 summarizes both parameter values and run times.

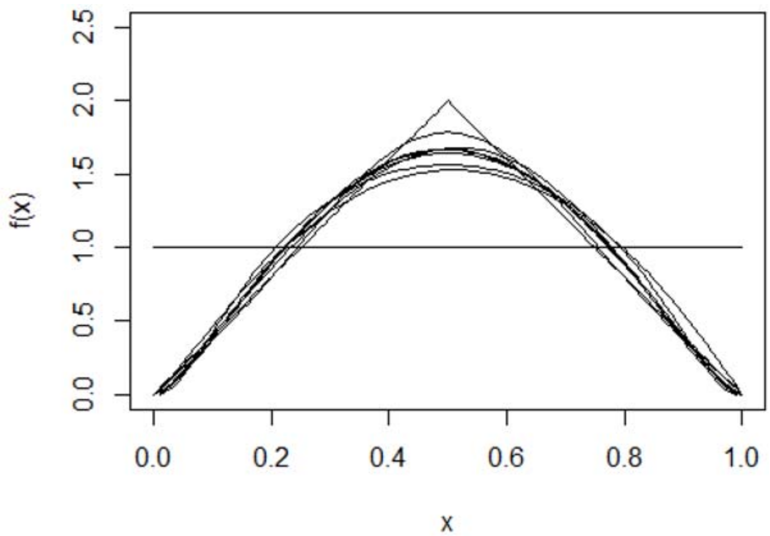


Figure 6. Generation of random numbers-densities.

Table 5. Required time for generation of 100,000,000 random numbers

$X \sim$	Time
UNI(0, 1)	0.17 min
TRI(0.50)	0.90 min
TRA(0.47, 0.53)	1.97 min
BETA(2.43, 2.43)	1.45 min
KUM(2.13, 2.62)	1.79 min
VAS(0.27, 0.50)	1.32 min
JSB(0.00, 0.98)	1.54 min
LUN(0.00, 0.56)	1.07 min
LLI(0.00, 2.48)	18.87 min
GBP(0.50, 1.32)	> 20 min
TSP(0.50, 2.01)	2.97 min
GTL(2.00, 2.43)	1.19 min
NLG(2.37, 0.34)	1.52 min
NLC(0.67, 2.52)	1.49 min
NLW(1.11, 1.60)	1.69 min

It becomes obvious that the random number generation is very time-consuming if the qf is not available in closed form (e.g., LLI and GBP), except for the Beta distribution which seems to be implemented very efficient in R. However, switching to the Vasicek (VAS), the logistic uniform (LUN) or to the generalized Topp-Leone (GTL) distribution reduces run time up to 20 percent.

4. Summary

Traditionally, both the Beta and the Vasicek distribution are chosen as stochastic model for default rates (PD) and/or loss rates (LGD) in the financial literature. However, as shown in this article, there are several alternatives with similar flexible density functions. If, above that, a lot of random numbers are needed (e.g., within a Monte Carlo context) some of these alternatives even outperform the Beta and the Vasicek distribution w.r.t. run time.

Table 6. Density, distribution function and quantile functions

$X \sim$	PDF	CDF	Quantile function
$TRI(m)$	$\begin{cases} \frac{2x}{m}, & 0 \le x \le m, \\ \frac{2(1-x)}{1-m}, & m < x \le 1, \end{cases}$	$\begin{cases} \frac{x^2}{m}, & 0 \le x \le m, \\ 1-\frac{(1-x)^2}{1-m}, & m < x \le 1, \end{cases}$	$\begin{cases} \sqrt{mu}, & 0 \le x \le m, \\ 1-\sqrt{(1-u)(1-m)}, & m < x \le 1, \end{cases}$
$TRA(m_1, m_2)$	$\begin{cases} \frac{2x}{m_1(m_2-m_1+1)}, & 0 \le x \le m_1, \\ \frac{2}{m_2-m_1+1}, & m_1 < x \le m_2, \\ \frac{2(1-x)}{(1-m_2)(m_2-m_1+1)}, & m_2 < x \le 1, \end{cases}$	$\begin{cases} \frac{x^2}{m_1(m_2-m_1+1)}, & 0 \le x \le m_1, \\ \frac{2x-m_1}{m_2-m_1+1}, & m_1 < x \le m_2, \\ 1-\frac{(1-x)^2}{(1-m_2)(m_2-m_1+1)}, & m_2 < x \le 1, \end{cases}$	$\begin{cases} \sqrt{um_1(m_2-m_1+1)}, & 0 \le u \le \frac{m_1}{m_2-m_1+1}, \\ \frac{1}{2}(m_1+u(m_2-m_1+1)), & \frac{m_1}{m_2-m_1+1} < u \le \frac{2m_2-m_1}{m_2-m_1+1}, \\ 1-\sqrt{(1-u)(1-m_2)(m_2-m_1+1)}, & \frac{2m_2-m_1}{m_2-m_1+1} < u \le 1, \end{cases}$
$BETA(\alpha, \beta)$	$\frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$	$\frac{1}{B(\alpha,\beta)} \int_0^x t^{\alpha-1}(1-t)^{\beta-1} dt$	no closed form
$KUM(\alpha, \beta)$	$\alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}$	$1-(1-x^{\alpha})^{\beta}$	$\left(1-(1-u)^{1/\beta}\right)^{1/\alpha}$
$VAS(\varrho, p)$	$\sqrt{\frac{1-\varrho}{\varrho}} \exp\left(\frac{1}{2}\left(\Phi^{-1}(x)^2-\left(\frac{\sqrt{1-\varrho}\Phi^{-1}(x)-\Phi^{-1}(p)}{\sqrt{\varrho}}\right)^2\right)\right)$	$\Phi\left(\frac{\sqrt{1-\varrho}\Phi^{-1}(x)-\Phi^{-1}(p)}{\sqrt{\varrho}}\right)$	$\Phi\left(\frac{\Phi^{-1}(p)+\sqrt{\varrho}\Phi^{-1}(u)}{\sqrt{1-\varrho}}\right)$
$JSB(\gamma, \delta)$	$\frac{\delta}{\sqrt{2\pi}} \frac{1}{x(1-x)} \exp\left(-\frac{1}{2}\left[\gamma+\delta \ln\left[\frac{x}{1-x}\right]\right]^2\right)$	$\Phi\left(\gamma+\delta \ln\left(\frac{x}{1-x}\right)\right)$	$\frac{e^{(\Phi^{-1}(u)-\gamma)/\delta}}{1+e^{(\Phi^{-1}(u)-\gamma)/\delta}}$
$LUN(m, s)$	$\left(e^{m/s}\left(\frac{x}{1-x}\right)^{1/s}\right)\left(sx(1-x)\left(e^{m/s}+\left(\frac{x}{1-x}\right)^{1/s}\right)^2\right)^{-1}$	$\left(1+e^{m/s}\left(\frac{1-x}{x}\right)^{1/s}\right)^{-1}$	$\left(1+e^{-m}\left(\frac{1}{u}-1\right)^{s}\right)^{-1}$

Table 6. (Continued)

$X\sim$	PDF	CDF	Quantile function
$LLI(\lambda,\ \eta)$	$\frac{\eta^2}{1+\lambda\eta}(\lambda-\log x)x^{\eta-1}$	$\frac{x^{\eta}[1+\eta(\lambda-\log x)]}{1+\lambda\eta}$	no closed form
$GBP(m,\ \theta)$	$\frac{(2\theta+1)(\theta+1)}{-3\theta-1}\cdot\begin{cases}\left(\frac{x}{m}\right)^{2\theta}-2\left(\frac{x}{m}\right)^{\theta}, & 0\le x\le m,\\ \left(\frac{1-x}{1-m}\right)^{2\theta}-2\left(\frac{1-x}{1-m}\right)^{\theta}, & m<x\le 1,\end{cases}$	$\begin{cases}\frac{(2\theta+1)(\theta+1)}{-3\theta-1}m\left[\frac{1}{2\theta+1}\left(\frac{x}{m}\right)^{2\theta+1}-\frac{2}{\theta+1}\left(\frac{x}{m}\right)^{\theta+1}\right], & 0\le x\le m,\\ 1+\frac{(2\theta+1)(\theta+1)}{-3\theta-1}(m-1)\left[\frac{1}{2\theta+1}\left(\frac{1-x}{1-m}\right)^{2\theta+1}-\frac{2}{\theta+1}\left(\frac{1-x}{1-m}\right)^{\theta+1}\right], & m<x\le 1,\end{cases}$	no closed form
$TSP(m,\ \alpha)$	$\begin{cases}\alpha\left(\frac{x}{m}\right)^{\alpha-1}, & 0\le x\le m,\\ \alpha\left(\frac{1-x}{1-m}\right)^{\alpha-1}, & m<x\le 1,\end{cases}$	$\begin{cases}m\left(\frac{x}{m}\right)^{\alpha}, & 0\le x\le m,\\ 1-(1-m)\left(\frac{1-x}{1-m}\right)^{\alpha}, & m<x\le 1,\end{cases}$	$\begin{cases}m\left(\frac{u}{m}\right)^{1/\alpha}, & 0\le u\le m,\\ 1-(1-m)\left(\frac{1-u}{1-m}\right)^{1/\alpha}, & m<u\le 1,\end{cases}$
$GTL(\alpha,\ \beta)$	$\beta(\alpha x-(\alpha-1)x^2)^{\beta-1}(\alpha-2(\alpha-1)x)$	$x^{\beta}(\alpha-(\alpha-1)x)^{\beta}$	$\begin{cases}\frac{-\alpha+\sqrt{\alpha^2-4(1-\alpha)(-x)^{1/\beta}}}{2(1-\alpha)}, & \alpha\ne 1,\\ x^{1/\beta}, & \alpha=1,\end{cases}$
$NLG(\alpha,\ \beta)$	$\frac{1}{\Gamma(\alpha)\beta^{\alpha}}x^{(1/\beta)-1}(-\ln x)^{\alpha-1}$	$\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\int_0^x t^{(1/\beta)-1}(-\ln t)^{\alpha-1}dt$	no closed form
$NLC(M,\ \alpha)$	$\frac{\alpha M^{\alpha}.(-\log(x))^{\alpha/1}}{((-\log(x))^{\alpha}+M^{\alpha})^2}$	$1-\frac{(-\log(x))^{\alpha}}{(-\log(x)+M^{\alpha})^{\alpha}}$	$\exp\left(-M(\frac{1-x}{x})^{1/\alpha}\right)$
$NLW(a,\ b)$	$ab\cdot\exp(-(-a\ln(x))^{b})\cdot(-a\ln(x))^{b-1}\cdot x^{-1}$	$\exp(-(-a\ln(x))^{b})$	$\exp(-\frac{1}{\alpha}\cdot(-\ln(p))^{1/b})$.

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