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BOUNDEDNESS OF TOEPLITZ TYPE OPERATOR RELATED TO FRACTIONAL AND SINGULAR INTEGRAL OPERATORS SATISFYING A VARIANT OF HÖRMANDER'S CONDITION

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Abstract

In this paper, some sharp maximal function inequalities for the Toeplitz type operator related to the fractional and singular integral operators satisfying a variant of Hörmander's condition are proved. As an application, we obtain the boundedness of the operator on Lebesgue, Morrey and Triebel-Lizorkin spaces.

1. Introduction

As the development of singular integral operators (see [18, 19]), their commutators have been well studied. In [3, 18, 19], the authors prove that the commutators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [1]) proves a similar result when singular integral operators are replaced by

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the fractional integral operators. In [8, 15], the boundedness for the commutators generated by the singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(R^n)(1 < p < \infty)$ spaces are obtained. In [9, 10], some Toeplitz type operators related to the singular integral operators and strongly singular integral operators are introduced, and the boundedness for the operators generated by BMO and Lipschitz functions are obtained. In [7], some singular integral operators satisfying a variant of Hörmander's condition are introduced, and the boundedness for the operators are obtained (see [7, 22]). The purpose of this paper is to prove the sharp maximal function inequalities for the Toeplitz type operator related to some singular integral operators satisfying a variant of Hörmander's condition. As an application, we obtain the boundedness of the Toeplitz type operator on Lebesgue, Morrey and Triebel-Lizorkin spaces.

2. Preliminaries

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$M^{\#}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x) dx$. We say that f belongs to $BMO(R^n)$ if $M^{\#}(f)$ belongs to $L^\infty(R^n)$ and define $\|f\|_{BMO} = \|M^{\#}(f)\|_{L^\infty}$. It has been known that (see [20])

$$\|f - f_{2^k Q}\|_{BMO} \leq Ck\|f\|_{BMO}.$$

Let

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

For $\eta > 0$, let $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$.

For $0 < \eta < n$ and $0 < r < n/\eta$, set

$$M_{\eta, r}(f)(x) = \sup_{Q \ni x}\left(\frac{1}{|Q|^{1-r\eta/n}}\int_Q |f(y)|^r dy\right)^{1/r}.$$

The A_p weight is defined by (see [6])

$$A_p = \left\{w \in L^1_{\mathrm{loc}}(R^n) : \sup_Q\left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} < \infty\right\},$$

$$1 < p < \infty,$$

$$A_1 = \{w \in L^p_{\mathrm{loc}}(R^n) : M(w)(x) \le Cw(x), \text{ a.e.}\},$$

and

$$A_\infty = \bigcup_{p \ge 1} A_p.$$

For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta, \infty}(R^n)$ be the homogeneous Triebel-Lizorkin space (see [15]).

For $\beta > 0$, the Lipschitz space $Lip_\beta(R^n)$ is the space of functions f such that

$$\|f\|_{Lip_\beta} = \sup_{\substack{x, y \in R^n \\ x \ne y}} \frac{|f(x) - f(y)|}{|x - y|^\beta} < \infty.$$

Definition 1. Let $\Phi = \{\phi_1, \dots, \phi_l\}$ be a finite family of bounded functions in R^n. For any locally integrable function f, the Φ sharp maximal function of f is defined by

$$M_\Phi^\#(f)(x) = \sup_{Q \ni x} \inf_{\{c_1, \dots, c_l\}} \frac{1}{|Q|}\int_Q \left|f(y) - \sum_{i=1}^{l} c_i \phi_i(x_Q - y)\right| dy,$$

where the infimum is taken over all l-tuples $\{c_1, \dots, c_l\}$ of complex numbers and x_Q is the center of Q.

Remark. We note that $M_{\Phi}^{\#} \approx M^{\#}$ if $l = 1$ and $\phi_1 = 1$.

Definition 2. Given a positive and locally integrable function f in R^n, we say that f satisfies the reverse Hölder's condition (write this as $f \in RH_{\infty}(R^n)$), if for any cube Q centered at the origin we have

$$0 < \sup_{x \in Q} f(x) \leq C \frac{1}{|Q|} \int_Q f(y) dy.$$

Definition 3. Let φ be a positive, increasing function on R^+ and there exists a constant $D > 0$ such that

$$\varphi(2t) \leq D\varphi(t) \quad \text{for} \quad t \geq 0.$$

Let f be a locally integrable function on R^n. Set, for $0 \leq \eta < n$ and $1 \leq p < n / \eta$,

$$\|f\|_{L^{p,\varphi}} = \sup_{x \in R^n, d > 0} \left(\frac{1}{\varphi(d)} \int_{Q(x, d)} |f(y)|^p dy \right)^{1/p},$$

where $Q(x, d) = \{y \in R^n : |x - y| < d\}$. The generalized fractional Morrey space is defined by

$$L^{p, \eta, \varphi}(R^n) = \{f \in L^1_{\text{loc}}(R^n) : \|f\|_{L^{p, \eta, \varphi}} < \infty\}.$$

We write $L^{p, \eta, \varphi}(R^n) = L^{p, \varphi}(R^n)$ if $\eta = 0$, which is the generalized Morrey space. If $\varphi(d) = d^n, \eta > 0$, then $L^{p, \varphi}(R^n) = L^{p, \eta}(R^n)$, which is the classical Morrey spaces (see [16, 17]). If $\varphi(d) = 1$, then $L^{p, \varphi}(R^n) = L^p(R^n)$, which is the Lebesgue spaces (see [6]).

As the Morrey space may be considered as an extension of the Lebesgue space, it is natural and important to study the boundedness of the operator on the Morrey spaces (see [2, 4, 5, 11, 14]).

In this paper, we will study some singular integral operators as following (see [22]).

Definition 4. Let $K \in L^2(R^n)$ and satisfy

$$\|\hat{K}\|_{L^\infty} \leq C,$$

$$|K(x)| \leq C|x|^{-n},$$

there exist functions $B_1, \ldots, B_l \in L^1_{\text{loc}}(R^n - \{0\})$ and $\Phi = \{\phi_1, \ldots, \phi_l\} \subset L^\infty(R^n)$ such that $|\det[\phi_j(y_i)]|^2 \in RH_\infty(R^{nl})$, and for a fixed $\delta > 0$ and any $|x| > 2|y| > 0$,

$$|K(x-y) - \sum_{i=1}^{l} B_i(x)\phi_i(y)| \leq C\frac{|y|^\delta}{|x-y|^{n+\delta}}.$$

For $f \in C_0^\infty$, we define the singular integral operator related to the kernel K by

$$T(f)(x) = \int_{R^n} K(x-y)f(y)dy.$$

Moreover, let b be a locally integrable function on R^n. The Toeplitz type operator related to T is defined by

$$T^b = \sum_{j=1}^{m}(T^{j,1}M_b I_\alpha T^{j,2} + T^{j,3}I_\alpha M_b T^{j,4}),$$

where $T^{j,1}$ are T or $\pm I$ (the identity operator), $T^{j,2}$ and $T^{j,4}$ are the bounded linear operators on $L^p(R^n)$ for $1 < p < \infty$, $T^{j,3} = \pm I$, $j = 1, \ldots, m$, $M_b(f) = bf$ and I_α is the fractional integral operator $(0 < \alpha < n)$ (see [2]).

Remark. Note that the classical Calderón-Zygmund singular integral operator satisfies Definition 4 (see [20, 21]). Also note that the commutator $[b, T](f) = bT(f) - T(bf)$ is a particular operator of the Toeplitz type operator T^b. The Toeplitz type operator T^b is the non-trivial generalizations of the commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [18, 19]). The main purpose of this paper is to prove the sharp maximal inequalities for the Toeplitz type operator T^b. As the application, we obtain the L^p-norm inequality, Morrey and Triebel-Lizorkin spaces boundedness for the Toeplitz type operators T^b.

3. Theorems and Lemmas

We shall prove the following theorems.

Theorem 1. *Let T be the singular integral operator as Definition* 4, $0 < \beta < 1$, $0 < \alpha < n$, $1 < s < n/(\alpha + \beta)$ *and* $b \in Lip_\beta(R^n)$. *If* $T^1(g) = 0$ *for any* $g \in L^u(R^n)(1 < u < \infty)$, *then there exists a constant* $C > 0$ *such that, for any* $f \in C_0^\infty(R^n)$ *and* $\tilde{x} \in R^n$,

$$M_\Phi^\#(T^b(f))(\tilde{x}) \leq C\|b\|_{Lip_\beta} \sum_{j=1}^m (M_{\beta, s}(I_\alpha T^{j,2}(f))(\tilde{x}) + M_{\beta+\alpha, s}(T^{j,4}(f))(\tilde{x})).$$

Theorem 2. *Let T be the singular integral operator as Definition* 4, $0 < \beta < \min(1, \delta)$, $0 < \alpha < n$, $1 < s < n/\alpha$ *and* $b \in Lip_\beta(R^n)$. *If* $T^1(g) = 0$ *for any* $g \in L^u(R^n)(1 < u < \infty)$, *then there exists a constant* $C > 0$ *such that, for any* $f \in C_0^\infty(R^n)$ *and* $\tilde{x} \in R^n$,

$$\sup_{Q \ni \tilde{x}} \inf_{\{c_0, c_1, \ldots, c_l\}} \frac{1}{|Q|^{1+\beta/n}} \int_Q \left| T^b(f)(x) - c_0 - \sum_{i=1}^{l} c_i \phi_i(x_0 - x) \right| dx$$

$$\leq C\|b\|_{Lip_\beta} \sum_{j=1}^{m} (M_s(I_\alpha T^{j,2}(f))(\tilde{x}) + M_{\alpha,s}(T^{j,4}(f))(\tilde{x})),$$

where the infimum is taken over all $(l+1)$*-tuples* $\{c_0, c_1, \ldots, c_l\}$ *of complex numbers and* x_0 *is the center of* Q.

Theorem 3. *Let* T *be the singular integral operator as Definition* 4, $1 < s < \infty$, $0 < \alpha < n$ *and* $b \in BMO(R^n)$. *If* $T^1(g) = 0$ *for any* $g \in L^u(R^n)$ $(1 < u < \infty)$, *then there exists a constant* $C > 0$ *such that, for any* $f \in C_0^\infty(R^n)$ *and* $\tilde{x} \in R^n$,

$$M_\Phi^\#(T^b(f))(\tilde{x}) \leq C\|b\|_{BMO} \sum_{j=1}^{m} (M_s(I_\alpha T^{j,2}(f))(\tilde{x}) + M_{\alpha,s}(T^{j,4}(f))(\tilde{x})).$$

Theorem 4. *Let* T *be the singular integral operator as Definition* 4, $0 < \beta < 1$, $0 < \alpha < n$, $1 < p < n/(\alpha+\beta)$, $1/q = 1/p - (\alpha+\beta)/n$ *and* $b \in Lip_\beta(R^n)$. *If* $T^1(g) = 0$ *for any* $g \in L^u(R^n)(1 < u < \infty)$, *then* T^b *is bounded from* $L^p(R^n)$ *to* $L^q(R^n)$.

Theorem 5. *Let* T *be the singular integral operator as Definition* 4, $0 < \beta < 1$, $0 < \alpha < n$, $1 < p < n/(\alpha+\beta)$, $1/q = 1/p - (\alpha+\beta)/n$, $0 < D < 2^n$ *and* $b \in Lip_\beta(R^n)$. *If* $T^1(g) = 0$ *for any* $g \in L^u(R^n)(1 < u < \infty)$, *then* T^b *is bounded from* $L^{p,\alpha+\beta,\varphi}(R^n)$ *to* $L^{q,\varphi}(R^n)$.

Theorem 6. *Let T be the singular integral operator as Definition* 4, $0 < \beta < \min(1, \delta)$, $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ *and* $b \in Lip_\beta(R^n)$. *If* $T^1(g) = 0$ *for any* $g \in L^u(R^n)(1 < u < \infty)$, *then* T^b *is bounded from* $L^p(R^n)$ *to* $\dot{F}_p^{\beta,\infty}(R^n)$.

Theorem 7. *Let T be the singular integral operator as Definition* 4, $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ *and* $b \in BMO(R^n)$. *If* $T^1(g) = 0$ *for any* $g \in L^u(R^n)(1 < u < \infty)$, *then* T^b *is bounded from* $L^p(R^n)$ *to* $L^q(R^n)$.

Theorem 8. *Let T be the singular integral operator as Definition* 4, $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < D < 2^n$ *and* $b \in BMO(R^n)$. *If* $T^1(g) = 0$ *for any* $g \in L^u(R^n)(1 < u < \infty)$, *then* T^b *is bounded from* $L^{p,\alpha,\varphi}(R^n)$ *to* $L^{q,\varphi}(R^n)$.

Corollary. *Let* $[b, T](f) = bT(f) - T(bf)$ *be the commutator generated by the singular integral operator T as Definition* 4 *and b. Then Theorems* 1-8 *hold for* $[b, T]$.

To prove the theorems, we need the following lemmas.

Lemma 1 (see [22])**.** *Let T be the singular integral operator as Definition* 4 *and* $1 < p < \infty$. *Then T is bounded on* $L^p(R^n)$.

Lemma 2 (see [22])**.** *Let* $1 < p < \infty$, $w \in A_\infty$ *and* $\Phi = \{\phi_1, \ldots, \phi_l\} \subset L^\infty(R^n)$ *such that* $|\det[\phi_j(y_i)]|^2 \in RH_\infty(R^{nl})$. *Then*

$$\int_{R^n} M(f)(x)^p w(x)dx \leq C\int_{R^n} M_\Phi^\#(f)(x)^p w(x)dx,$$

for any smooth function f for which the left-hand side is finite.

Lemma 3 (see [15]). *For* $0 < \beta < 1$ *and* $1 < p < \infty$, *we have*

$$\|f\|_{\dot{F}_p^{\beta,\infty}} \approx \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q| dx \right\|_{L^p} \approx \left\| \sup_{Q \ni \cdot} \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - c| dx \right\|_{L^p}.$$

Lemma 4 (see [2, 6]). *Suppose that* $0 < \alpha < n$, $1 \le s < p < n/\alpha$ *and* $1/r = 1/p - \alpha/n$. *Then*

$$\|I_\alpha(f)\|_{L^r} \le C\|f\|_{L^p},$$

and

$$\|M_{\alpha,s}(f)\|_{L^r} \le C\|f\|_{L^p}.$$

Lemma 5. *Let* $1 < p < \infty$, $0 < D < 2^n$ *and* $\Phi = \{\phi_1, \ldots, \phi_l\} \subset L^\infty(R^n)$ *such that* $|\det[\phi_j(y_i)]|^2 \in RH_\infty(R^{nl})$. *Then, for any smooth function f for which the left-hand side is finite*,

$$\|M(f)\|_{L^{p,\varphi}} \le C\|M_\Phi^\#(f)\|_{L^{p,\varphi}}.$$

Proof. For any cube $Q = Q(x_0, d)$ in R^n, we know $M(\chi_Q) \in A_1$ for any cube $Q = Q(x, d)$ by [6]. Noticing that $M(\chi_Q) \le 1$ and $M(\chi_Q)(x) \le d^n/(|x - x_0| - d)^n$ if $x \in Q^c$, by Lemma 2, we have, for $f \in L^{p,\varphi}(R^n)$,

$$\int_Q M(f)(x)^p dx = \int_{R^n} M(f)(x)^p \chi_Q(x) dx$$

$$\le \int_{R^n} M(f)(x)^p M(\chi_Q)(x) dx$$

$$\le C \int_{R^n} M_\Phi^\#(f)(x)|^p M(\chi_Q)(x) dx$$

$$= C\left(\int_Q M_\Phi^\#(f)(x)^p M(\chi_Q)(x) dx + \sum_{k=0}^\infty \int_{2^{k+1}Q \setminus 2^k Q} M_\Phi^\#(f)(x)^p M(\chi_Q)(x) dx \right)$$

$$\leq C\left(\int_Q M_\Phi^\#(f)(x)^p\,dx + \sum_{k=0}^\infty \int_{2^{k+1}Q\setminus 2^kQ} M_\Phi^\#(f)(x)^p\,\frac{|Q|}{|2^{k+1}Q|}\,dx\right)$$

$$\leq C\left(\int_Q M_\Phi^\#(f)(x)^p\,dx + \sum_{k=0}^\infty \int_{2^{k+1}Q} M_\Phi^\#(f)(x)^p\,2^{-kn}\,dy\right)$$

$$\leq C\|M_\Phi^\#(f)\|_{L^{p,\varphi}}^p \sum_{k=0}^\infty 2^{-kn}\,\varphi(2^{k+1}d)$$

$$\leq C\|M_\Phi^\#(f)\|_{L^{p,\varphi}}^p \sum_{k=0}^\infty (2^{-n}D)^k\,\varphi(d)$$

$$\leq C\|M_\Phi^\#(f)\|_{L^{p,\varphi}}^p\,\varphi(d),$$

thus

$$\left(\frac{1}{\varphi(d)}\int_Q M(f)(x)^p\,dx\right)^{1/p} \leq C\left(\frac{1}{\varphi(d)}\int_Q M_\Phi^\#(f)(x)^p\,dx\right)^{1/p},$$

and

$$\|M(f)\|_{L^{p,\varphi}} \leq C\|M_\Phi^\#(f)\|_{L^{p,\varphi}}.$$

This finishes the proof.

Lemma 6. *Let* $0 < \alpha < n$, $0 < D < 2^n$, $1 \leq s < p < n/\alpha$ *and* $1/r = 1/p - \alpha/n$. *Then*

$$\|I_\alpha(f)\|_{L^{r,\varphi}} \leq C\|f\|_{L^{p,\alpha,\varphi}},$$

and

$$\|M_{\alpha,s}(f)\|_{L^{r,\varphi}} \leq C\|f\|_{L^{p,\alpha,\varphi}}.$$

Lemma 7. *Let T be the singular integral operator as Definition* 4. *Then, for* $1 < p < \infty$, $0 < D < 2^n$,

$$\|T(f)\|_{L^{p,\varphi}} \le C\|f\|_{L^{p,\varphi}}.$$

The proofs of two Lemmas are similar to that of Lemma 5 by Lemmas 1 and 4, we omit the details.

4. Proofs of Theorems

Proof of Theorem 1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_Q, the following inequality holds:

$$\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - C_Q\right| dx \le C\|b\|_{Lip_\beta} \sum_{j=1}^m (M_{\beta,s}(I_\alpha T^{j,2}(f))(\tilde{x})$$

$$+ M_{\beta+\alpha,s}(T^{j,4}(f))(\tilde{x})).$$

Without loss of generality, we may assume $T^{j,1}$ are $T (j = 1, \ldots, m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. We write, by $T^1(g) = 0$,

$$T^b(f)(x) = \sum_{j=1}^m T^{j,1} M_b I_\alpha T^{j,2}(f)(x) + \sum_{j=1}^m T^{j,3} I_\alpha M_b T^{j,4}(f)(x) = A_b(x)$$

$$+ B_b(x) = A_{b-b_Q}(x) + B_{b-b_Q}(x),$$

where

$$A_{b-b_Q}(x) = \sum_{j=1}^m T^{j,1} M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{j,2}(f)(x) + \sum_{j=1}^m T^{j,1} M_{(b-b_Q)\chi_{(2Q)^c}}$$

$$\times I_\alpha T^{j,2}(f)(x) = A_1(x) + A_2(x),$$

and

$$B_{b-b_Q}(x)=\sum_{j=1}^{m}T^{j,3}I_\alpha M_{(b-b_Q)\chi_{2Q}}T^{j,4}(f)(x)+\sum_{j=1}^{m}T^{j,3}I_\alpha M_{(b-b_Q)\chi_{(2Q)^c}}$$

$$\times\, T^{j,4}(f)(x)=B_1(x)+B_2(x).$$

Then

$$\frac{1}{|Q|}\int_Q\left|A_{b-b_Q}(f)(x)-C_0\right|dx\le\frac{1}{|Q|}\int_Q|A_1(x)|dx$$

$$+\frac{1}{|Q|}\int_Q|A_2(x)-C_0|dx=I_1+I_2,$$

and

$$\frac{1}{|Q|}\int_Q\left|B_{b-b_Q}(f)(x)-B_2(x_0)\right|dx\le\frac{1}{|Q|}\int_Q|B_1(x)|dx$$

$$+\frac{1}{|Q|}\int_Q|B_2(x)-B_2(x_0)|dx=I_3+I_4,$$

where Q is any a cube centered at x_0, $C_0=\sum_{j=1}^{m}\sum_{i=1}^{l}g_j^i\phi_i(x_0-x)$ and $g_j^i=\int_{R^n}B_i(x_0-y)M_{(b-b_Q)\chi_{(2Q)^c}}I_\alpha T^{j,2}(f)(y)dy.$

For I_1, by L^s-boundedness of T (see Lemma 1) and Hölder's inequality, we obtain

$$\frac{1}{|Q|}\int_Q|T^{j,1}M_{(b-b_Q)\chi_{2Q}}I_\alpha T^{j,2}(f)(x)|dx$$

$$\le\left(\frac{1}{|Q|}\int_{R^n}|T^{j,1}M_{(b-b_Q)\chi_{2Q}}I_\alpha T^{j,2}(f)(x)|^s dx\right)^{1/s}$$

$$\le C|Q|^{-1/s}\left(\int_{R^n}|M_{(b-b_Q)\chi_{2Q}}I_\alpha T^{j,2}(f)(x)|^s dx\right)^{1/s}$$

$$\leq C|Q|^{-1/s}\left(\int_{2Q}(|b(x)-b_Q||I_\alpha T^{j,2}(f)(x)|)^s dx\right)^{1/s}$$

$$\leq C|Q|^{-1/s}\|b\|_{Lip_\beta}|2Q|^{\beta/n}|2Q|^{1/s-\beta/n}\left(\frac{1}{|2Q|^{1-s\beta/n}}\int_{2Q}|I_\alpha T^{j,2}(f)(x)|^s dx\right)^{1/s}$$

$$\leq C\|b\|_{Lip_\beta}M_{\beta,s}(I_\alpha T^{j,2}(f))(\tilde{x}),$$

thus

$$I_1 \leq \sum_{j=1}^{m}\frac{1}{|Q|}\int_{R^n}|T^{j,1}M_{(b-b_Q)\chi_{2Q}}I_\alpha T^{j,2}(f)(x)|dx$$

$$\leq C\|b\|_{Lip_\beta}\sum_{j=1}^{m}M_{\beta,s}(I_\alpha T^{j,2}(f))(\tilde{x}).$$

For I_2, we get, for $x \in Q$,

$$|T^{j,1}M_{(b-b_Q)\chi_{(2Q)^c}}I_\alpha T^{j,2}(f)(x)-C_0|$$

$$=|T^{j,1}M_{(b-b_Q)\chi_{(2Q)^c}}I_\alpha T^{j,2}(f)(x)-\sum_{i=1}^{l}g_j^i\phi_i(x_0-x)|$$

$$\leq\left|\int_{R^n}(K(x-y)-\sum_{i=1}^{m}B_i(x_0-y)\phi_i(x_0-x))(b(y)-b_{2Q})\chi_{(2Q)^c}(y)I_\alpha T^{j,2}(f)(y)dy\right|$$

$$\leq\sum_{k=1}^{\infty}\int_{2^k d\leq|y-x_0|<2^{k+1}d}|K(x-y)-\sum_{i=1}^{l}B_i(x_0-y)\phi_i(x_0-x)||b(y)-b_{2Q}||$$

$$\times\ I_\alpha T^{j,2}(f)(y)|dy$$

$$\leq C\sum_{k=1}^{\infty}\int_{2^k d\leq|y-x_0|<2^{k+1}d}\frac{|x-x_0|^\delta}{|y-x_0|^{n+\delta}}|b(y)-b_{2Q}||I_\alpha T^{j,2}(f)(y)|dy$$

$$\leq C\sum_{k=1}^{\infty}\frac{d^{\delta}}{(2^k d)^{n+\delta}}(2^k d)^{\beta}\|b\|_{Lip_\beta}(2^k d)^{n-\beta}$$

$$\times\left(\frac{1}{|2^{k+1}Q|^{1-s\beta/n}}\int_{2^{k+1}Q}|I_\alpha T^{j,2}(f)(y)|^s dy\right)^{1/s}$$

$$\leq C\|b\|_{Lip_\beta}M_{\beta,s}(I_\alpha T^{j,2}(f))(\tilde{x})\sum_{k=1}^{\infty}2^{-k\delta}$$

$$\leq C\|b\|_{Lip_\beta}M_{\beta,s}(I_\alpha T^{j,2}(f))(\tilde{x}),$$

thus

$$I_2 \leq \frac{1}{|Q|}\int_Q\sum_{j=1}^{m}|T^{j,1}M_{(b-b_Q)\chi_{(2Q)^c}}T^{j,2}(f)(x)-C_0|dx$$

$$\leq C\|b\|_{Lip_\beta}\sum_{j=1}^{m}M_{\beta,s}(T^{j,2}(f))(\tilde{x}).$$

Similarly, by (L^s, L^r)-boundedness of I_α with $1 < s < n/\alpha$ and $1/r = 1/s - \alpha/n$ (see Lemma 4), we get

$$I_3 \leq \sum_{j=1}^{m}\left(\frac{1}{|Q|}\int_{R^n}|I_\alpha M_{(b-b_Q)\chi_{2Q}}T^{j,4}(f)(x)|^r dx\right)^{1/r}$$

$$\leq C\sum_{j=1}^{m}|Q|^{-1/r}\left(\int_{2Q}(|b(x)-b_Q||T^{j,4}(f)(x)|)^s dx\right)^{1/s}$$

$$\leq C\|b\|_{Lip_\beta}\sum_{j=1}^{m}|Q|^{-1/r}|2Q|^{\beta/n}|2Q|^{1/s-(\beta+\alpha)/n}$$

$$\times\left(\frac{1}{|2Q|^{1-s(\beta+\alpha)/n}}\int_{2Q}|T^{j,4}(f)(x)|^s dx\right)^{1/s}$$

$$\leq C\|b\|_{Lip_\beta}\sum_{j=1}^{m} M_{\beta+\alpha,s}(T^{j,4}(f))(\tilde{x}),$$

$$I_4 \leq \sum_{j=1}^{m}\frac{1}{|Q|}\int_Q\int_{(2Q)^c}|b(y)-b_{2Q}|\left|\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|x_0-y|^{n-\alpha}}\right|$$

$$\times |T^{j,4}(f)(y)|dydx$$

$$\leq C\sum_{j=1}^{m}\sum_{k=1}^{\infty}\|b\|_{Lip_\beta}|2^{k+1}Q|^{\beta/n}\int_{2^k d\leq|y-x_0|<2^{k+1}d}\frac{d}{|x_0-y|^{n-\alpha+1}}$$

$$\times |T^{j,4}(f)(y)|dy$$

$$\leq C\|b\|_{Lip_\beta}\sum_{j=1}^{m}\sum_{k=1}^{\infty}(2^k d)^{\beta} d(2^k d)^{-n+\alpha-1}(2^k d)^{n(1-1/s)}(2^k d)^{n/s-\beta-\alpha}$$

$$\times\left(\frac{1}{|2^{k+1}Q|^{1-s(\beta+\alpha)/n}}\int_{2^{k+1}Q}|T^{j,4}(f)(y)|^s dy\right)^{1/s}$$

$$\leq C\|b\|_{Lip_\beta}\sum_{j=1}^{m} M_{\beta+\alpha,s}(T^{j,4}(f))(\tilde{x})\sum_{j=1}^{\infty}2^{-j}$$

$$\leq C\|b\|_{Lip_\beta}\sum_{j=1}^{m} M_{\beta+\alpha,s}(T^{j,4}(f))(\tilde{x}).$$

These complete the proof of Theorem 1.

Proof of Theorem 2. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_Q, the following inequality holds:

$$\frac{1}{|Q|^{1+\beta/n}}\int_Q\left|T^b(f)(x)-C_Q\right|dx \leq C\|b\|_{Lip_\beta}\sum_{j=1}^{m}(M_s(I_\alpha T^{j,2}(f))(\tilde{x})$$

$$+ M_{\alpha,s}(T^{j,4}(f))(\tilde{x})).$$

Without loss of generality, we may assume $T^{j,1}$ are $T(j = 1, \ldots, m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Similar to the proof of Theorem 1, we have

$$\frac{1}{|Q|^{1+\beta/n}} \int_Q \left| T^b(f)(x) - C_0 - B_2(x_0) \right| dx$$

$$\leq \frac{1}{|Q|^{1+\beta/n}} \int_Q |A_1(x)| dx + \frac{1}{|Q|^{1+\beta/n}} \int_Q |A_2(x) - C_0| dx$$

$$+ \frac{1}{|Q|^{1+\beta/n}} \int_Q |B_1(x)| dx + \frac{1}{|Q|^{1+\beta/n}} \int_Q |B_2(x) - B_2(x_0)| dx$$

$$= I_5 + I_6 + I_7 + I_8.$$

By using the same argument as in the proof of Theorem 1, we get, for $1/r = 1/s - \alpha/n$,

$$I_5 \leq |Q|^{-\beta/n} \sum_{j=1}^{m} \left(\frac{1}{|Q|} \int_{R^n} |T^{j,1} M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{j,2}(f)(x)|^s dx \right)^{1/s}$$

$$\leq C|Q|^{-\beta/n} \sum_{j=1}^{m} |Q|^{-1/s} \left(\int_{2Q} (|b(x) - b_Q| |I_\alpha T^{j,2}(f)(x)|)^s dx \right)^{1/s}$$

$$\leq C|Q|^{-\beta/n} \sum_{j=1}^{m} |Q|^{-1/s} \|b\|_{Lip_\beta} |2Q|^{\beta/n} |Q|^{1/s}$$

$$\times \left(\frac{1}{|2Q|} \int_{2Q} |I_\alpha T^{j,2}(f)(x)|^s dx \right)^{1/s}$$

$$\leq C\|b\|_{Lip_\beta} \sum_{j=1}^{m} M_s(I_\alpha T^{j,2}(f))(\tilde{x}),$$

$$I_6 \le |Q|^{-\beta/n} \sum_{j=1}^{m} \frac{1}{|Q|} \int_Q \sum_{k=1}^{\infty} \int_{2^k d \le |y-x_0| < 2^{j+1} d} \left| K(x-y) - \sum_{i=1}^{m} B_i(x_0 - y) \right.$$

$$\left. \times\, \phi_i(x_0 - x) \right| \times |b(y) - b_{2Q}| \, |I_\alpha T^{j,2}(f)(y)| dy dx$$

$$\le C|Q|^{-\beta/n} \sum_{k=1}^{m} \int_{2^k d \le |y-x_0| < 2^{k+1} d} \frac{|x-x_0|^\delta}{|y-x_0|^{n+\delta}} |b(y) - b_{2Q}|$$

$$\times |I_\alpha T^{j,2}(f)(y)| dy$$

$$\le C|Q|^{-\beta/n} \sum_{k=1}^{\infty} \frac{d^\delta}{(2^k d)^{n+\delta}} (2^k d)^\beta \|b\|_{Lip_\beta} (2^k d)^n$$

$$\times \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |I_\alpha T^{j,2}(f)(y)|^s dy \right)^{1/s}$$

$$\le C\|b\|_{Lip_\beta} M_s(I_\alpha T^{j,2}(f))(\tilde{x}) \sum_{k=1}^{\infty} 2^{k(\beta-\delta)}$$

$$\le C\|b\|_{Lip_\beta} M_s(I_\alpha T^{j,2}(f))(\tilde{x}),$$

$$I_7 \le |Q|^{-\beta/n} \sum_{j=1}^{m} \left(\frac{1}{|Q|} \int_{R^n} |I_\alpha M_{(b-b_Q)\chi_{2Q}} T^{j,4}(f)(x)|^r dx \right)^{1/r}$$

$$\le C|Q|^{-\beta/n - 1/r} \sum_{j=1}^{m} \left(\int_{2Q} (|b(x) - b_Q| \, |T^{j,4}(f)(x)|)^s dx \right)^{1/s}$$

$$\le C\|b\|_{Lip_\beta} \sum_{j=1}^{m} |Q|^{-\beta/n - 1/r} |2Q|^{\beta/n} |2Q|^{1/s - \alpha/n}$$

$$\times \left(\frac{1}{|2Q|^{1-s\alpha/n}} \int_{2Q} |T^{j,4}(f)(x)|^s dx \right)^{1/s}$$

$$\leq C\|b\|_{Lip_\beta}\sum_{j=1}^{m} M_{\alpha,s}(T^{j,4}(f))(\tilde{x}),$$

$$I_8 \leq |Q|^{-\beta/n-1}\sum_{j=1}^{m}\int_Q\int_{(2Q)^c}|b(y)-b_{2Q}|\left|\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|x_0-y|^{n-\alpha}}\right|$$

$$\times |T^{j,4}(f)(y)|dydx$$

$$\leq C|Q|^{-\beta/n}\sum_{j=1}^{m}\sum_{k=1}^{\infty}\|b\|_{Lip_\beta}|2^{k+1}Q|^{\beta/n}\int_{2^k d\leq|y-x_0|<2^{k+1}d}\frac{d}{|x_0-y|^{n-\alpha+1}}$$

$$\times |T^{j,4}(f)(y)|dy$$

$$\leq C\|b\|_{Lip_\beta}\sum_{j=1}^{m}\sum_{k=1}^{\infty} d^{-\beta}(2^k d)^{\beta} d(2^k d)^{-n+\alpha-1}(2^k d)^{n(1-1/s)}(2^k d)^{n/s-\alpha}$$

$$\times\left(\frac{1}{|2^{k+1}Q|^{1-s\alpha/n}}\int_{2^{k+1}Q}|T^{j,4}(f)(y)|^s dy\right)^{1/s}$$

$$\leq C\|b\|_{Lip_\beta}\sum_{j=1}^{m} M_{\alpha,s}(T^{j,4}(f))(\tilde{x})\sum_{k=1}^{\infty}2^{k(\beta-1)}$$

$$\leq C\|b\|_{Lip_\beta}\sum_{j=1}^{m} M_{\alpha,s}(T^{j,4}(f))(\tilde{x}).$$

These complete the proof of Theorem 2.

Proof of Theorem 3. It suffices to prove for $f \in C_0^{\infty}(R^n)$ and some constant C_Q, the following inequality holds:

$$\frac{1}{|Q|}\int_Q\left|T^b(f)(x)-C_Q\right|dx \leq C\|b\|_{BMO}\sum_{j=1}^{m}(M_s(I_\alpha T^{j,2}(f))(\tilde{x})$$

$$+\ M_{\alpha,s}(T^{j,4}(f))(\tilde{x})),$$

where Q is any a cube centered at x_0, $C_0 = \sum_{j=1}^m \sum_{i=1}^l g_j^i \phi_i(x_0 - x)$ and $g_j^i = \int_{R^n} B_i(x_0 - y)(b(y) - b_{2Q})M_{(b-b_Q)\chi_{2Q}^c} T^{j,2}(f)(y)dy$. Without loss of generality, we may assume $T^{j,1}$ are $T(j = 1, \dots, m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Similar to the proof of Theorem 1, we have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - C_0 - B_2(x_0)\right| dx \\
&\leq \frac{1}{|Q|}\int_Q |A_1(x)| dx + \frac{1}{|Q|}\int_Q |A_2(x) - C_0| dx \\
&\quad + \frac{1}{|Q|}\int_Q |B_1(x)| dx + \frac{1}{|Q|}\int_Q |B_2(x) - B_2(x_0)| dx \\
&= I_9 + I_{10} + I_{11} + I_{12}.
\end{aligned}$$

For I_9, choose $1 < r < s$, by Hölder's inequality and the L^r-boundedness of T, we obtain

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |T^{j,1} M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{j,2}(f)(x)| dx \\
&\leq \left(\frac{1}{|Q|}\int_{R^n} |T^{j,1} M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{j,2}(f)(x)|^r dx\right)^{1/r} \\
&\leq C|Q|^{-1/r}\left(\int_{R^n} |M_{(b-b_Q)\chi_{2Q}} I_\alpha T^{j,2}(f)(x)|^r dx\right)^{1/r} \\
&\leq C|Q|^{-1/r}\left(\int_{2Q} |I_\alpha T^{j,2}(f)(x)|^s dx\right)^{1/s}\left(\int_{2Q} |b(x) - b_Q|^{sr/(s-r)} dx\right)^{(s-r)/sr} \\
&\leq C\|b\|_{BMO}\left(\frac{1}{|2Q|}\int_{2Q} |I_\alpha T^{j,2}(f)(x)|^s dx\right)^{1/s} \\
&\leq C\|b\|_{BMO} M_s(I_\alpha T^{j,2}(f))(\tilde{x}),
\end{aligned}$$

thus

$$I_9 \le \sum_{j=1}^{l} \frac{1}{|Q|} \int_Q |I_\alpha T^{j,1} M_{(b-b_Q)\chi_{2Q}} T^{j,2}(f)(x)| dx$$

$$\le C\|b\|_{BMO} \sum_{j=1}^{m} M_s(I_\alpha T^{j,2}(f))(\tilde{x}).$$

For I_{10}, by using the same argument as in the proof of Theorem 1, we get, for $x \in Q$,

$$|T^{j,1} M_{(b-b_Q)\chi_{(2Q)^c}} I_\alpha T^{j,2}(f)(x) - \sum_{i=1}^{l} g_j^i \phi_i(x_0 - x)|$$

$$\le \sum_{k=1}^{\infty} \int_{2^k d \le |y-x_0| < 2^{k+1} d} |K(x-y) - \sum_{i=1}^{l} B_i(x_0 - y)\phi_i(x_0 - x)| |b(y) - b_{2Q}|$$

$$\times |I_\alpha T^{j,2}(f)(y)| dy$$

$$\le C \sum_{k=1}^{\infty} \int_{2^k d \le |y-x_0| < 2^{k+1} d} \frac{|x-x_0|^\delta}{|y-x_0|^{n+\delta}} |b(y) - b_{2Q}| |I_\alpha T^{j,2}(f)(y)| dy$$

$$\le C \sum_{k=1}^{\infty} \frac{d^\delta}{(2^k d)^{n+\delta}} (2^k d)^n \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |b(y) - b_Q|^{s'} dy \right)^{1/s'}$$

$$\times \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |I_\alpha T^{j,2}(f)(y)|^s dy \right)^{1/s}$$

$$\le C\|b\|_{BMO} \sum_{k=1}^{\infty} k 2^{-k\delta} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |I_\alpha T^{j,2}(f)(y)|^s dy \right)^{1/s}$$

$$\le C\|b\|_{BMO} M_s(I_\alpha T^{j,2}(f))(\tilde{x}) \sum_{j=1}^{\infty} k 2^{-k\delta}$$

$$\le C\|b\|_{BMO} M_s(I_\alpha T^{j,2}(f))(\tilde{x}),$$

thus

$$I_{10} \le \frac{1}{|Q|}\int_Q \sum_{j=1}^{m} |T^{j,1} M_{(b-b_Q)\chi_{(2Q)^c}} I_\alpha T^{j,2}(f)(x) - C_0| dx$$

$$\le C\|b\|_{BMO} \sum_{j=1}^{m} M_s(I_\alpha T^{j,2}(f))(\tilde{x}).$$

For I_{11} and I_{12}, by using the same argument as in the proof of Theorem 1, we get, for $1 < t < s$ and $1/r = 1/t - \alpha/n$,

$$I_{11} \le \sum_{j=1}^{m}\left(\frac{1}{|Q|}\int_{R^n} |I_\alpha M_{(b-b_Q)\chi_{2Q}} T^{j,4}(f)(x)|^r dx\right)^{1/r}$$

$$\le C\sum_{j=1}^{m} |Q|^{-1/r}\left(\int_{2Q} (|b(x) - b_Q||T^{j,4}(f)(x)|)^t dx\right)^{1/t}$$

$$\le C\sum_{j=1}^{m} |Q|^{-1/r}\left(\int_{2Q} |T^{j,4}(f)(x)|^s dx\right)^{1/s}$$

$$\times \left(\int_{2Q} |b(x) - b_Q|^{st/(s-t)} dx\right)^{(s-t)/st}$$

$$\le C\sum_{j=1}^{m}\left(\frac{1}{|2Q|^{1-\alpha s/n}}\int_{2Q} |T^{j,4}(f)(x)|^s dx\right)^{1/s}$$

$$\times \left(\frac{1}{|2Q|}\int_{2Q} |b(x) - b_Q|^{st/(s-t)} dx\right)^{(s-t)/st}$$

$$\le C\|b\|_{BMO} \sum_{j=1}^{m} M_{\alpha,\delta}(T^{j,4}(f))(\tilde{x}),$$

$$I_{12} \le \sum_{j=1}^{m} \frac{1}{|Q|} \int_Q \int_{(2Q)^c} |b(y) - b_{2Q}| \left| \frac{1}{|x - y|^{n-\alpha}} - \frac{1}{|x_0 - y|^{n-\alpha}} \right|$$

$$\times |T^{j,4}(f)(y)| dydx$$

$$\le C \sum_{j=1}^{m} \sum_{k=1}^{\infty} \frac{d}{(2^k d)^{n-\alpha+1}} \int_{2^k d \le |y - x_0| < 2^{k+1} d} |b(y) - b_{2Q}| |T^{j,4}(f)(y)| dy$$

$$\le C \sum_{j=1}^{m} \sum_{k=1}^{\infty} d(2^k d)^{-n+\alpha-1} (2^k d)^n (2^k d)^{n/s'} (2^k d)^{n(1/s - \alpha/n)}$$

$$\times \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |b(y) - b_Q|^{s'} dy \right)^{1/s'}$$

$$\times \left(\frac{1}{|2^{k+1}Q|^{1-\alpha s/n}} \int_{2^{k+1}Q} |T^{j,4}(f)(y)|^s dy \right)^{1/s}$$

$$\le C \|b\|_{BMO} \sum_{j=1}^{m} M_{\alpha,s}(T^{j,4}(f))(\tilde{x}) \sum_{k=1}^{\infty} k 2^{-k}$$

$$\le C \|b\|_{BMO} \sum_{j=1}^{m} M_{\alpha,s}(T^{j,4}(f))(\tilde{x}).$$

This completes the proof of Theorem 3.

Proof of Theorem 4. Choose $1 < s < p$ in Theorem 1 and set $1/v = 1/p - \alpha/n$, we have, by Lemmas 2 and 4,

$$\|T^b(f)\|_{L^q} \le \|M(T^b(f))\|_{L^q} \le C \|M_\Phi^{\#}(T^b(f))\|_{L^q}$$

$$\le C \|b\|_{Lip_\beta} \sum_{j=1}^{m} (\|M_{\beta,s}(I_\alpha T^{j,2}(f))\|_{L^q} + \|M_{\beta+\alpha,s}(T^{j,4}(f))\|_{L^q})$$

$$\leq C\|b\|_{Lip_\beta}\sum_{j=1}^{m}(\|I_\alpha T^{j,2}(f)\|_{L^v}+\|T^{j,4}(f)\|_{L^p})$$

$$\leq C\|b\|_{Lip_\beta}\sum_{j=1}^{m}(\|T^{j,2}(f)\|_{L^p}+\|f\|_{L^p})$$

$$\leq C\|b\|_{Lip_\beta}\|f\|_{L^p}.$$

This completes the proof of Theorem 4.

Proof of Theorem 5. Choose $1 < s < p$ in Theorem 1 and set $1/v = 1/p - \alpha/n,$ we have, by Lemmas 5, 6, and 7,

$$\|T^b(f)\|_{L^{q,\varphi}} \leq \|M(T^b(f))\|_{L^{q,\varphi}} \leq C\|M_\Phi^{\#}(T^b(f))\|_{L^{q,\varphi}}$$

$$\leq C\|b\|_{Lip_\beta}\sum_{j=1}^{m}(\|M_{\beta,s}(I_\alpha T^{j,2}(f))\|_{L^{q,\varphi}}+\|M_{\beta+\alpha,s}(T^{j,4}(f))\|_{L^{q,\varphi}})$$

$$\leq C\|b\|_{Lip_\beta}\sum_{j=1}^{m}(\|I_\alpha T^{j,2}(f)\|_{L^{v,\beta,\varphi}}+\|T^{j,4}(f)\|_{L^{p,\alpha+\beta,\varphi}})$$

$$\leq C\|b\|_{Lip_\beta}\sum_{j=1}^{m}(\|T^{j,2}(f)\|_{L^{p,\alpha+\beta,\varphi}}+\|f\|_{L^{p,\alpha+\beta,\varphi}})$$

$$\leq C\|b\|_{Lip_\beta}\|f\|_{L^{p,\alpha+\beta,\varphi}}.$$

This completes the proof of Theorem 5.

Proof of Theorem 6. Choose $1 < s < p$ in Theorem 2, we have, by Lemmas 2 and 3,

$$\|T^b(f)\|_{\dot{F}_q^{\beta,\infty}} \le C\left\|\sup_{Q\ni\cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q \left|T^b(f)(x) - C_0\right| dx\right\|_{L^q}$$

$$\le C\|b\|_{Lip_\beta} \sum_{j=1}^m (\|M_s(I_\alpha T^{j,2}(f))\|_{L^q} + \|M_{\alpha,s}(T^{j,4}(f))\|_{L^q})$$

$$\le C\|b\|_{Lip_\beta} \sum_{j=1}^m (\|I_\alpha T^{j,2}(f)\|_{L^q} + \|T^{j,4}(f)\|_{L^p})$$

$$\le C\|b\|_{Lip_\beta} \sum_{j=1}^m (\|T^{j,2}(f)\|_{L^p} + \|f\|_{L^p})$$

$$\le C\|b\|_{Lip_\beta} \|f\|_{L^p}.$$

This completes the proof of Theorem 6.

Proof of Theorem 7. Choose $1 < s < p$ in Theorem 3, we have, by Lemmas 2 and 4,

$$\|T^b(f)\|_{L^q} \le \|M(T^b(f))\|_{L^q} \le C\|M_\Phi^\#(T^b(f))\|_{L^q}$$

$$\le C\|b\|_{BMO} \sum_{j=1}^m (\|M_s(I_\alpha T^{j,2}(f))\|_{L^q} + \|M_{\alpha,s}(T^{j,4}(f))\|_{L^q})$$

$$\le C\|b\|_{BMO} \sum_{j=1}^m (\|I_\alpha T^{j,2}(f)\|_{L^q} + \|T^{j,4}(f)\|_{L^p})$$

$$\le C\|b\|_{BMO} \sum_{j=1}^m (\|T^{j,2}(f)\|_{L^p} + \|f\|_{L^p})$$

$$\le C\|b\|_{BMO} \|f\|_{L^p}.$$

This completes the proof of Theorem 7.

Proof of Theorem 8. Choose $1 < s < p$ in Theorem 3, we have, by Lemmas 5, 6 and 7,

$$\|T^b(f)\|_{L^{q,\varphi}} \le \|M(T^b(f))\|_{L^{q,\varphi}} \le C\|M_\Phi^{\#}(T^b(f))\|_{L^{q,\varphi}}$$

$$\le C\|b\|_{BMO}\sum_{j=1}^{m}(\|M_s(I_\alpha T^{j,2}(f))\|_{L^{q,\varphi}} + \|M_{\alpha,s}(T^{j,4}(f))\|_{L^{q,\varphi}})$$

$$\le C\|b\|_{BMO}\sum_{j=1}^{m}(\|I_\alpha T^{j,2}(f)\|_{L^{q,\varphi}} + \|T^{j,4}(f)\|_{L^{p,\alpha,\varphi}})$$

$$\le C\|b\|_{BMO}\sum_{j=1}^{m}(\|T^{j,2}(f)\|_{L^{p,\alpha,\varphi}} + \|f\|_{L^{p,\alpha,\varphi}})$$

$$\le C\|b\|_{BMO}\|f\|_{L^{p,\alpha,\varphi}}.$$

This completes the proof of Theorem 8.

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