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SOME MONOTONIC PROPERTIES OF GENERALIZED SIGMOID FUNCTION

JIACAI TIAN, LI YIN and JUMEI ZHANG

School of Science
Binzhou University
P. R. China
e-mail: yinli7979@163.com

Abstract

The completely monotonicity, convexity and inequalities are obtained involving p-generalized sigmoid function, and the properties can also generalize to its m-order derivative.

1. Introduction

The sigmoid function, which is also known as the standard logistic function is defined as

$$S(x) = \frac{e^x}{1 + e^x} = \frac{1}{1 + e^{-x}}, \quad x \in (-\infty, +\infty), \tag{1}$$

$$= \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2}\right), \quad x \in (-\infty, +\infty). \tag{2}$$

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The sigmoid function plays important role in many scientific disciplines including biology, machine learning, probability and statistics, demography, ecology, population dynamics, and mathematical psychology (see [2, 12], and the references therein).

Specially, the function is widely applied in artificial neural networks, where it plays as an activation function at the output of each neuron (see [3, 4, 5, 8, 13]). As well, in the business field, the function been used to study performance growth in manufacturing and service management (see [11]). Another area of application is in the field of medicine, where the function is used to model the growth of tumors or to study pharmacokinetic reactions (see [12]). It is also used in forestry. For instance, in [6], a generalized form of the function is applied to predict the site index of unmanaged loblolly and slash pine plantations in East Texas. Moreover, it also can be used in computer graphics or image processing to enhance image contrast (see [7, 10]).

The above important roles of the function makes its properties a matter of interest and hence worth studying. In the recent work [9], Ezeafulukwe et al. studied some analytic properties of the function such as convexity and starlikeness in a unit disc. In [14], Nantomah study the properties such as inequalities, subadditivity, convexity and super-multiplicativity of the sigmoid function.

In this paper, we study the properties of p-generalized sigmoid function such as inequalities, subadditivity, convexity and super-multiplicativity.

2. Main Results

We define the p-generalized sigmoid function as

$$S_p(x) = \frac{e^{px}}{1 + e^{px}} = \frac{1}{1 + e^{-px}}, \quad x \in (-\infty, +\infty), \tag{3}$$

$$= \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{px}{2}\right), \quad x \in (-\infty, +\infty), \tag{4}$$

where $p > 0$. We can easily calculate the first and second derivatives of the generalized sigmoid function

$$S'_p(x) = \frac{pe^{px}}{(1 + e^{px})^2} = pS_p(x)(1 - S_p(x)), \tag{5}$$

$$S''_p(x) = \frac{p^2 e^{px}(1 - e^{px})}{(1 + e^{px})^3} = p^2 S_p(x)(1 - S_p(x))(1 - 2S_p(x)), \tag{6}$$

for all $x \in (-\infty, +\infty)$. From (5), we can get that $S_p(x)$ is increasing on $x \in (-\infty, +\infty)$. Moreover, the p-generalized sigmoid function have the following properties:

$$S_p(x) + S_p(-x) = 1, \tag{7}$$

$$S'_p(x) = pS_p(x)S_p(-x), \tag{8}$$

$$S'_p(x) = S'_p(-x), \tag{9}$$

$$\lim_{x \to +\infty} S_p(x) = 1, \tag{10}$$

$$\lim_{x \to 0} S_p(x) = \frac{1}{2}, \tag{11}$$

$$\lim_{x \to -\infty} S_p(x) = 0, \tag{12}$$

$$\lim_{x \to +\infty} S'_p(x) = 0, \tag{13}$$

$$\lim_{x \to 0} S'_p(x) = \frac{p}{4}, \tag{14}$$

$$\int S_p(x)dx = \frac{1}{p} \ln(1 + e^{px}) + C, \tag{15}$$

with C is a constant of integration. The derivative of (15) gives the p-generalized sigmoid function.

Lemma 2.1. *The function*

$$v(x) = \frac{e^{px}}{(1 + e^{px})^2} \tag{16}$$

is decreasing for all $x \in (0, +\infty)$ *and increasing for all* $x \in (-\infty, 0)$.

Proof. For

$$v'(x) = \frac{pe^{px}(1 - e^{px})}{(1 + e^{px})^3}, \tag{17}$$

which indicates that $v'(x) < 0$ for $x \in (0, +\infty)$. That is $v(x)$ decreasing for all $x \in (0, +\infty)$. And $v'(x) > 0$ for $x \in (-\infty, 0)$. That is $v(x)$ increasing for all $x \in (-\infty, 0)$. □

Theorem 2.2. *The p-generalized sigmoid function satisfies the inequality*

$$S_p(x + y) < S_p(x) + S_p(y), \text{ for all } x, y \in (-\infty, +\infty). \tag{18}$$

Namely, the function $S_p(x)$ *is subadditive on* $(-\infty, +\infty)$.

Proof. The case $x = y = 0$ is trivial. Hence we only prove the case $x, y \in (0, +\infty)$ and the case $x, y \in (-\infty, 0)$. Let

$$g(x, y) = S_p(x + y) - S_p(x) - S_p(y), \tag{19}$$

$$= \frac{e^{p(x+y)}}{1 + e^{p(x+y)}} - \frac{e^{px}}{1 + e^{px}} - \frac{e^{py}}{1 + e^{py}}. \tag{20}$$

For any fixed y, we have

$$\frac{\partial}{\partial x} g(x, y) = \frac{pe^{p(x+y)}}{(1 + e^{p(x+y)})^2} - \frac{pe^{px}}{(1 + e^{px})^2}. \tag{21}$$

For $v(x)$ is decreasing on $(0, +\infty)$, we get $g(x, y)$ is decreasing on $(0, +\infty)$. Then for $x \in (0, +\infty)$, we can have

$$g(x, y) < g(0, y) = \lim_{x \to 0} g(x, y) = -\frac{1}{2} < 0. \tag{22}$$

For $v(x)$ is increasing on $(-\infty, 0)$, we get $g(x, y)$ is increasing on $(-\infty, 0)$. Then for $(-\infty, 0)$, we can have

$$g(x, y) < g(0, y) = \lim_{x \to 0} g(x, y) = -\frac{1}{2} < 0. \tag{23}$$

That completes the proof. □

Theorem 2.3. *The function* $S_p(x)$ *satisfies the following inequalities*:

$$1 < \frac{S_p(x + \frac{1}{p})}{S_p(x)} < e, \quad x \in (-\infty, +\infty), \tag{24}$$

$$\frac{2e}{1+e} < \frac{S_p(x + \frac{1}{p})}{S_p(x)} < e, \quad x \in (-\infty, 0), \tag{25}$$

$$1 < \frac{S_p(x + \frac{1}{p})}{S_p(x)} < \frac{2e}{1+e}, \quad x \in (0, +\infty). \tag{26}$$

Proof. Since

$$\left(\frac{S_p'(x)}{S_p(x)}\right)' = -\frac{pe^{px}}{(1 + e^{px})^2} < 0, \text{ for all } x \in (-\infty, +\infty). \tag{27}$$

Hence, the function $\frac{S_p'(x)}{S_p(x)}$ is decreasing for all $x \in (-\infty, +\infty)$. Let $G(x) = \frac{S_p(x + \frac{1}{p})}{S_p(x)}$, $x \in (-\infty, +\infty)$ and $v(x) = \ln G(x)$. Then,

$$v'(x) = \frac{S_p'(x + \frac{1}{p})}{S_p(x + \frac{1}{p})} - \frac{S_p'(x)}{S_p(x)} < 0. \tag{28}$$

Thus $v(x)$ and $G(x)$ are decreasing. Hence, for $x \in (-\infty, +\infty)$, we get

$$1 = \lim_{x \to +\infty} G(x) < G(x) < \lim_{x \to -\infty} G(x) = e. \tag{29}$$

For $x \in (-\infty, 0)$, we get

$$\frac{2e}{1+e} = \lim_{x \to 0} G(x) < G(x) < \lim_{x \to -\infty} G(x) = e. \tag{30}$$

For $x \in (0, +\infty)$, we get

$$1 = \lim_{x \to +\infty} G(x) < G(x) < \lim_{x \to 0} G(x) = \frac{2e}{1+e}. \tag{31}$$

That completes the proof. □

From the definition of MN-convex, MN-concave and logarithmically concave and the Corollary 2.5 in [1], we can get the following.

Theorem 2.4. *The function* $S_p(x)$

(1) *is GG-convex on* $(0, 1)$;

(2) *is AH-concave on* $(0, \infty)$;

(3) *is logarithmically concave on* $(0, \infty)$.

Theorem 2.5. *The function* $S_p(x)$ *satisfies the following inequalities*:

$$S_p^2(xy) \leq S_p(x) S_p(y),\ x,\ y \in (0, 1], \tag{32}$$

$$S_p^2(xy) \geq S_p(x) S_p(y),\ x \in [1, \infty). \tag{33}$$

Equality holds if $x = y = 1$.

Proof. For $x, y \in (0, 1]$, we have $xy \le x$ and $xy \le y$. Since $S_p(x)$ is increasing, we get

$$0 < S_p(xy) \le S_p(x), \tag{34}$$

$$0 < S_p(xy) \le S_p(y). \tag{35}$$

Product (34) and (35), we get (32). Using the similar method, we can get (33). □

Theorem 2.6. *The function* $S_p(x)$ *is super-multiplicative on* $(1, \infty)$. *Namely*,

$$S_p(xy) > S_p(x)S_p(y),$$

for all $x, y \in (1, \infty)$.

Proof. For $0 < S_p(z) < 1$ for all $z \in (-\infty, +\infty)$, then $S_p^2(z) < S_p(z)$ for all $z \in (-\infty, +\infty)$. Hence $S_p(xy) > S_p^2(xy) > S_p(x)S_p(y)$. □

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