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POINTS ACCESSIBLE IN AVERAGE BY REARRANGEMENT OF SEQUENCES I

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Abstract

We investigate the set of limit points of averages of rearrangements of a given sequence. We study how the properties of the sequence determine the structure of that set and what type of sets we can expect as the set of such accessible points.

1. Introduction

When in [6] we started building the theory of means on infinite sets, at one point we faced the problem that how the average behaves for the rearrangements of an arbitrary bounded sequence. More precisely, if a bounded sequence (a_n) is given, then we wanted to determine the set of points of the limit of averages of the rearranged sequences. I.e. take all rearrangements (a'_n) of the sequence, choose those where the limit of the averages $\lim\limits_{n\to\infty}\frac{a'_1+\cdots+a'_n}{n}$ exists, and examine the set of all such limit points.

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Many authors studied the rearrangement of the underlying sequence of a series and investigated what effect it has for the sum of the series, see [1], [3], [4], [5], [8]. In their research the rearrangement was always associated to a series. In [9], Sarigöl investigates the permutations that preserves bounded variation of sequences.

It is well known that the accumulation points, hence limit point of a rearranged sequence are identical to such points of the original sequence. Hence it does not make sense to study. However if we take the average of the rearranged sequence, that is not so trivial.

In this paper, our main aim will be to investigate which set of points can be accessed in average by rearrangement of sequences. How the properties of the sequence determine the structure of that set. What type of sets we can expect as the set of such accessible points.

In the first part of the paper, we prove some generic results that will provide theorems for bounded sequences. Unbounded sequences behaves differently, their investigation is our goal in the remainder of the paper. For more details see Subsection 1.2.

1.1. Basic notions and notations

Throughout this paper function $\mathcal{A}()$ will denote the arithmetic mean of any number of variables. We will also use the notation $\mathcal{A}(a_i : 1 \le i \le n)$ for $\mathcal{A}(a_1, \cdots, a_n)$. If $H \subset \mathbb{R}$ is a finite set, then $\mathcal{A}(H)$ denotes the arithmetic mean of its distinct points.

Let us use the notation $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ and consider $\overline{\mathbb{R}}$ as a 2 point compactification of $\mathbb{R}$, i.e., a neighbourhood base of $+\infty$ is $\{(c, +\infty] : c \in \mathbb{R}\}$.

Definition 1.1. Let (a_n) be a sequence. We say that a_n tends to $\alpha \in \overline{\mathbb{R}}$ in average if

$$\lim_{n\to\infty} \mathcal{A}(a_1, \cdots, a_n) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} a_i}{n} = \alpha.$$

We denote it by $a_n \xrightarrow{\mathcal{A}} \alpha$. We also use the expression that α is the limit in average of (a_n).

With this notation if a series $\sum a_n$ is Cesaro summable with sum c then we may say that $s_n \xrightarrow{\mathcal{A}} c$, where $s_n = \sum_{i=1}^{n} a_i$.

Definition 1.2. Let (a_n) be a sequence, $\alpha \in \overline{\mathbb{R}}$. We say that α is accessible in average by rearrangement of (a_n) if there exists a rearrangement of a_n, i.e., a bijection $p : \mathbb{N} \to \mathbb{N}$ such that $a_{p(n)} \xrightarrow{\mathcal{A}} \alpha$.

The set of all such accessible points will be denoted by $AAR_{(a_n)}$.

Definition 1.3. If $(a_n), (b_n)$ are two sequences then let $(c_n) = (a_n)||(b_n)$ be the sequence defined by $c_{2n} = b_n, c_{2n-1} = a_n (n \in \mathbb{N})$.

The following theorem is well know in the theory of Cesaro summation or can be proved easily.

Theorem 1.4. *If* $a_n \to \alpha (\alpha \in \overline{\mathbb{R}})$, *then* $a_n \xrightarrow{\mathcal{A}} \alpha$. □

Corollary 1.5. *If* $a_n \to \alpha$ $(\alpha \in \overline{\mathbb{R}})$, *then* $AAR_{(a_n)} = \{\alpha\}$.

Proof. It is also well known that for every rearrangement (a_{p_n}) of (a_n), $a_{p_n} \to \alpha$. □

Proposition 1.6. *If* $a_n \xrightarrow{\mathcal{A}} a \in \mathbb{R}, b_n \xrightarrow{\mathcal{A}} b \in \mathbb{R}, c \in \mathbb{R}$, *then* $a_n + c \xrightarrow{\mathcal{A}} a + c, ca_n \xrightarrow{\mathcal{A}} ca, a_n + b_n \xrightarrow{\mathcal{A}} a + b, a_n\|b_n \xrightarrow{\mathcal{A}} \frac{a+b}{2}$. □

1.2. Brief summary of the main results

We just enumerate some of the most interesting results to give a taste of the topic.

Proposition. *If* $\alpha \in \overline{\mathbb{R}}$ *is an accumulation point of* (a_n), *then* $\alpha \in AAR_{(a_n)}$.

Proposition. *If* $a_n \xrightarrow{\mathcal{A}} c$, *then* $c \in [\underline{\lim}\, a_n, \overline{\lim}\, a_n]$.

Theorem. $AAR_{(a_n)}$ *is closed in* $\overline{\mathbb{R}}$.

Theorem. *Let* (a_n) *be a bounded sequence. Then* $AAR_{(a_n)} = [\underline{\lim}\, a_n, \overline{\lim}\, a_n]$.

Theorem. *Let* $(a_n) = (b_n)||(c_n)$, *where* $b_n \equiv 0$, $c_n \to +\infty$. *If*

$$\frac{c_n}{\sum_{i=1}^{n-1} c_i} \to 0,$$

then 1 *is accessible in average by rearrangement of* (a_n).

Theorem. *Let* $(a_n) = (b_n)||(c_n)$, *where* $b_n \equiv 0$, $c_n \to +\infty$ *and* (c_n) *is increasing. If* 1 *is accessible in average by rearrangement of* (a_n), *then*

$$\frac{c_n}{\sum_{i=1}^{n-1} c_i} \to 0.$$

Theorem. *Let* $(a_n) = (b_n)||(c_n)$, *where* $b_n \to a$, $c_n \to +\infty$. *If there is* $b \in \mathbb{R}$ *such that* $a < b$, $b \in AAR_{(a_n)}$, *then* $AAR_{(a_n)} = [a, +\infty]$.

Corollary. *Let* $k \in \mathbb{N}$, $(a_n) = (b_n)\|(c_n)$, *where* $b_n \equiv 0$, $c_n = n^k$. *Then* $AAR_{(a_n)} = [0, +\infty]$.

Corollary. *Let* $d > 1$, $(a_n) = (b_n)\|(c_n)$, *where* $b_n \equiv 0$, $c_n = d^n$. *Then* $AAR_{(a_n)} = \{0, +\infty\}$.

2. General Results

First we need some preparation.

Lemma 2.1. *Let* (b_n) *be a sequence,* $c \in \mathbb{R}$. *Assume* $b_n \xrightarrow{\mathcal{A}} b$. *Then* $\forall \epsilon > 0$ *we can merge* c *into* (b_n), *i.e., create a new sequence* (d_n) *with* $d_i = b_i (i < k)$, $d_k = c$, $d_i = b_{i-1} (i > k)$ *such that* $n \geq k$ *implies that* $b - \epsilon < \mathcal{A}(d_1, \ldots, d_n) < b + \epsilon$.

Proof. Choose $k \in \mathbb{N}$ such that $n \geq k - 1$ implies that

(1) $b - \frac{\epsilon}{3} < \mathcal{A}(b_1, \ldots, b_n) < b + \frac{\epsilon}{3}$,

(2) $\left|\frac{c}{n}\right| < \frac{\epsilon}{3}$,

(3) $b - \frac{2\epsilon}{3} < (b - \frac{\epsilon}{3})\frac{n-1}{n}$.

If $m \geq k$, then

$$\mathcal{A}(d_1, \ldots, d_m) = \frac{c + \sum_{i=1}^{m-1} b_i}{m} = \frac{c}{m} + \mathcal{A}(b_1, \ldots, b_{m-1})\frac{m-1}{m},$$

hence $b - \epsilon < \mathcal{A}(d_1, \ldots, d_m) < b + \frac{2\epsilon}{3}$. □

Lemma 2.2. *Let* (b_n) *be a sequence,* $c \in \mathbb{R}$*. Assume* $b_n \xrightarrow{\mathcal{A}} +\infty$*. Then* $\forall M > 0$ *we can create a new sequence* (d_n) *with* $d_i = b_i (i < k)$, $d_k = c$, $d_i = b_{i-1} (i > k)$ *such that* $n \geq k$ *implies that* $M < \mathcal{A}(d_1, \dots, d_n)$*. Similar holds for* $-\infty$*.*

Proof. Choose $k \in \mathbb{N}$ such that $n \geq k - 1$ implies that

(1) $M + 2 < \mathcal{A}(b_1, \dots, b_n)$,

(2) $|\frac{c}{n}| < 1$,

(3) $M + 1 < (M + 2)\frac{n-1}{n}$.

If $m \geq k$, then

$$\mathcal{A}(d_1, \dots, d_m) = \frac{c + \sum_{i=1}^{m-1} b_i}{m} = \frac{c}{m} + \mathcal{A}(b_1, \dots, b_{m-1})\frac{m-1}{m},$$

hence $M < \mathcal{A}(d_1, \dots, d_m)$. $\square$

Lemma 2.3. *Let* $(b_n), (c_n)$ *be two sequences. Assume* $b_n \xrightarrow{\mathcal{A}} b \in \overline{\mathbb{R}}$*. Then we can merge the two sequences into a new sequence* (d_n) *such that* $d_n \xrightarrow{\mathcal{A}} b$*.*

Proof. We define sequences $(b_n^{(l)})$ and associated constants k_l recursively. Let $(b_n^{(0)}) = (b_n)$, $k_0 = 1$. Let first $b \in \mathbb{R}$. If $\epsilon = \frac{1}{2}$ then by Lemma 2.1 we can merge c_1 into (b_n) such that $d'_i = b_i (i < k_1)$, $d'_{k_1} = c_1$, $d'_i = b_{i-1} (i > k_1)$ and $n > k_1$ implies that $b - \frac{1}{2} < \mathcal{A}(d'_1, \dots, d'_n) < b + \frac{1}{2}$. Let $(b_n^{(1)}) = (d'_n)$. If we have already defined $(b_n^{(i)})$ and k_i for $i \leq l$ then apply 2.1 for $(b_n^{(l)})$, c_l, $\epsilon = \frac{1}{2^{l+1}}$. Then we end up with sequence $(b_n^{(l+1)})$ and $k_{l+1} > k_l + 1$ such that $n > k_{l+1}$ implies that $b - \frac{1}{2^{l+1}} < \mathcal{A}(b_1^{(l+1)}, \dots, b_n^{(l+1)}) < b + \frac{1}{2^{l+1}}$.

Then let us define (d_n) by $d_j = b_j^{(l)}$, where $k_l \leq j < k_{l+1}$. Obviously (d_n) is a merge of the two original sequences and $d_n \xrightarrow{\mathcal{A}} b$.

Now if $b = \pm\infty$ then replace $\epsilon = \frac{1}{2^{l+1}}$ by $M = 2^{l+1}$ in the first part of the proof and apply 2.2 instead of 2.1. $\square$

Corollary 2.4. *Let (a_n) be a sequence. If there is a subsequence (a'_n) such that it can be rearranged to (a''_n) such that $a''_n \xrightarrow{\mathcal{A}} \alpha$, then there is a rearrangement of (a_n) which tends to α in average.* $\square$

Corollary 2.5. *If $\alpha \in \overline{\mathbb{R}}$ is an accumulation point of (a_n), then $\alpha \in AAR_{(a_n)}$.*

Proof. Let (b_n) be a subsequence of (a_n) such that $b_n \to \alpha$ and (c_n) be the rest. Then apply 2.3.

Proposition 2.6. *Let $(a_n), (b_n)$ be two sequences with $a_n \to a$, $b_n \to b$, $a < b$, $a, b \in \mathbb{R}$. Then for $\forall\alpha > 0, \forall\beta > 0, \alpha + \beta = 1$ the two sequences can be merged into a new sequence (d_n) such that $d_n \xrightarrow{\mathcal{A}} \alpha a + \beta b$.*

Proof. Let $\alpha \le \beta$ (the opposite case can be handled similarly). Let $\gamma = 1 + \frac{\beta}{\alpha} = \frac{1}{\alpha}$. Obviously, $\mathbb{N} \cap [1, \infty) = \bigcup_{n=1}^{\infty} \mathbb{N} \cap [(n-1)\gamma, n\gamma)$. Because of $\gamma \ge 2$ the length of each such interval is at least 2 hence contains at least 2 integers. Set $J_n = \mathbb{N} \cap [(n-1)\gamma, n\gamma)$.

If $i \in \mathbb{N}$ is given then $i \in [(n-1)\gamma, n\gamma)$ for some $n \in \mathbb{N}$. For every first index i of $[(n-1)\gamma, n\gamma)$ let d_i come from the sequence (a_n), for all other indexes from (b_n) using the not-yet-used elements from the sequences and from the original order. In this way we have defined (d_n) as a merge of $(a_n), (b_n)$.

Let $\epsilon > 0$ be given. Then there is $N \in \mathbb{N}$ such that $n > N$ implies that an $a_n \in (a - \frac{\epsilon}{4\alpha}, a + \frac{\epsilon}{4\alpha}), b_n \in (b - \frac{\epsilon}{4\beta}, b + \frac{\epsilon}{4\beta})$. Let $M \in \mathbb{N}$ such that $\{a_n : n \le N\} \cup \{b_n : n \le N\} \subseteq \{d_m : m \le M\}$. If $m > M$, then set $I_1 = \{1, \dots, M\}$,

$$I_2 = \{i \in \mathbb{N} : i > M, \exists l \in \mathbb{N} \text{ such that } i \in J_l \subset (M, m]\},$$

$I_3 = \{1, \dots, m\} - (I_1 \cup I_2)$. If $m > M$, then

$$\mathcal{A}(d_1, \dots, d_m) = \frac{\sum_{i \in I_1} d_i}{m} + \frac{\sum_{i \in I_2} d_i}{m} + \frac{\sum_{i \in I_3} d_i}{m}.$$

Clearly the first and third items can be arbitrarily small if $m \to \infty$ because the number of elements in the first sum is M, while it is at most 2γ in the third. Let us estimate the middle term now.

$$\frac{k}{m} \frac{(a - \frac{\epsilon}{4\alpha})r + (b - \frac{\epsilon}{4\beta})(k - r)}{k} < \frac{k}{m} \frac{\sum_{i \in I_2} d_i}{k} < \frac{k}{m} \frac{(a + \frac{\epsilon}{4\alpha})r + (b + \frac{\epsilon}{4\beta})(k - r)}{k},$$

where k denotes the number of elements in the sum and r is the number of J_l intervals which are subset of $(M, m]$.

The obvious estimation gives that $m - M - 2\gamma < k \leq m - M$ and $\frac{m - M - 2\gamma}{\gamma} < r \leq \frac{m - M}{\gamma}$. Therefore,

$$\frac{m - M - 2\gamma}{\gamma(m - M)} \leq \frac{r}{k} \leq \frac{m - M}{\gamma(m - M - 2\gamma)}.$$

When $m \to \infty$ then $\frac{k}{m} \to 1$ and $\frac{r}{k} \to \frac{1}{\gamma} = \alpha$, $\frac{k - r}{k} \to \beta$. Hence we get

$$\alpha a + \beta b - \frac{2\epsilon}{3} < \frac{\sum_{i \in I_2} d_i}{m} < \alpha a + \beta b + \frac{2\epsilon}{3},$$

if m is large enough. Finally

$$\alpha a + \beta b - \epsilon < \mathcal{A}(d_1, \dots, d_m) < \alpha a + \beta b + \epsilon,$$

if m is large enough. □

Proposition 2.7. *Proposition* 2.6 *is valid too if* $\alpha = 0$ *or* $\beta = 0$.

Proof. Apply Lemma 2.3. □

Proposition 2.8. *If* $a_n \xrightarrow{\mathcal{A}} c$, *then* $c \in [\underline{\lim}\, a_n, \overline{\lim}\, a_n]$.

Proof. Let $m = \underline{\lim}\, a_n$, $M = \overline{\lim}\, a_n$. If any of m, M is infinite the we do not have to check that side. Hence assume that $M \in \mathbb{R}$ (m can be handled similarly). First let $c \in \mathbb{R}$. Assume indirectly that $c > M$. Then there is N such that $n > N$ implies that $a_n < \frac{M + c}{2}$. Then

$$\mathcal{A}(a_1, \dots, a_n) = \frac{\sum_{i=1}^{N} a_i + \sum_{i=N+1}^{n} a_i}{n} < \frac{\sum_{i=1}^{N} a_i}{n} + \frac{M + c}{2} \cdot \frac{n - N}{n}.$$

The latter can be smaller than $\frac{M + 2c}{3}$ if n is large enough which is a contradiction.

The case $c = +\infty$ can be handled similarly: just apply $M + 1$ instead of $\frac{M + c}{2}$. $\square$

Theorem 2.9. *$AAR_{(a_n)}$ is closed in $\overline{\mathbb{R}}$.*

Proof. Let us note first that if sup $AAR_{(a_n)} = +\infty$, then $+\infty \in AAR_{(a_n)}$ by 2.8 and 2.5. And $-\infty$ can be handled similarly.

Let (b_n) be a sequence such that $\forall n\ b_n \in AAR_{(a_n)}$ and $b_n \to b \in \mathbb{R}$. We have to show that $b \in AAR_{(a_n)}$. For that it is enough to give a subsequence of (a_n) which tends to b in average (see 2.3).

We can assume that (b_n) is increasing, moreover $b - b_i < \frac{1}{3i}$. The other case when (b_n) is decreasing is similar.

We know that for each $i \in \mathbb{N}$ there is a rearrangement $p_i : \mathbb{N} \to \mathbb{N}$ such that $a_{p_i(n)} \xrightarrow{\mathcal{A}} b_i$. Let $N_i \in \mathbb{N}$ such that $n > N_i$ implies that $|\mathcal{A}(a_{p_i(1)}, \dots, a_{p_i(n)}) - b_i| < \frac{1}{3i}$.

We define a new rearrangement (d_n) of (a_n) recursively. We will add some elements of (a_n) to (d_n) in each step. Without mentioning we will assume that we just add new elements, i.e., that are not among the previously selected ones.

Step 1. Take $n_1 \geq N_1$ elements from $(a_{p_1(n)})$ such that

$$\{a_{p_2(1)}, \dots, a_{p_2(N_2)}\} \subset \{a_{p_1(1)}, \dots, a_{p_1(n_1)}\}, \tag{1}$$

and

$$|\mathcal{A}(a_{p_1(i)} : 1 \leq i \leq n_1,\ a_{p_1(i)} \notin \{a_{p_2(1)}, \dots, a_{p_2(N_2)}\}) - b_1| < \frac{2}{3}. \tag{2}$$

This can be done. (1) is obvious because p_1 is a bijection. To show (2) let

$$v_1 = \mathcal{A}(a_{p_1(i)} : 1 \leq i \leq n_1),$$

$$w_2 = \mathcal{A}(a_{p_2(i)} : 1 \leq i \leq N_2),$$

$$v_1' = \mathcal{A}(a_{p_1(i)} : 1 \leq i \leq n_1,\ a_{p_1(i)} \notin \{a_{p_2(1)}, \ldots, a_{p_2(N_2)}\}).$$

Then clearly

$$v_1 = \frac{(n_1 - N_2)v_1' + N_2 w_2}{n_1},$$

which gives that

$$v_1' = \frac{n_1 v_1 - N_2 w_2}{n_1 - N_2}.$$

From that we get that

$$|v_1' - v_1| = \frac{N_2|v_1 - w_2|}{n_1 - N_2} \leq N_2 \frac{|v_1 - b_1| + |b_1 - b_2| + |b_2 - w_2|}{n_1 - N_2} \leq \frac{N_2}{n_1 - N_2},$$

and

$$|v_1' - b_1| \leq |v_1' - v_1| + |v_1 - b_1| = \frac{N_2}{n_1 - N_2} + \frac{1}{3} < \frac{2}{3},$$

if n_1 is chosen big enough.

Then add those elements $a_{p_1(1)}, \ldots, a_{p_1(n_1)}$ to (d_n) as $(d_1, \ldots, d_{n_1})$.

Step k. Now (d_n) is already defined till index m_{k-1}, i.e., $(d_1, \ldots, d_{m_{k-1}})$.

Take $n_k \geq N_k$ elements from $(a_{p_k(n)})$ such that

$$\{a_{p_{k+1}(1)}, \ldots, a_{p_{k+1}(N_{k+1})}\} \subset \{d_1, \ldots, d_{m_{k-1}};\ a_{p_k(1)}, \ldots, a_{p_k(n_k)}\}, \tag{3}$$

and

$$|v_k - b_k| < \frac{1}{3k}, \tag{4}$$

and

$$|v'_k - b_k| < \frac{2}{3k}, \tag{5}$$

where

$$v_k = \mathcal{A}(d_1, \dots, d_{m_{k-1}}; a_{p_k(1)}, \dots, a_{p_k(n_k)} : a_{p_k(i)} \neq d_l (1 \le i \le n_k, 1 \le l \le m_{k-1}),$$

$$v'_k = \mathcal{A}(d_1, \dots, d_{m_{k-1}}; a_{p_k(1)}, \dots, a_{p_k(n_k)} : d_l \neq a_{p_k(i)} \neq a_{p_{k+1}(j)} \neq d_l$$

$$(1 \le i \le n_k, 1 \le j \le N_{k+1}, 1 \le l \le m_{k-1})).$$

This can be done. (3) is obvious because p_k is a bijection and (4) is evident too. To show (5) let

$$w_{k+1} = \mathcal{A}(a_{p_{k+1}(i)} : 1 \le i \le N_{k+1}).$$

Let n'_k be the number of distinct elements in $(d_1, \dots, d_{m_{k-1}}; a_{p_k(1)}, \dots, a_{p_k(n_k)})$. Then clearly

$$v_k = \frac{(n'_k - N_{k+1})v'_k + N_{k+1}w_{k+1}}{n'_k},$$

which gives that

$$v'_k = \frac{n'_k v_k - N_{k+1}w_{k+1}}{n'_k - N_{k+1}}.$$

From that we get that

$$|v'_k - v_k| = \frac{N_{k+1}|v_k - w_{k+1}|}{n'_k - N_{k+1}} \le N_{k+1}\frac{|v_k - b_k| + |b_k - b_{k+1}| + |b_{k+1} - w_{k+1}|}{n'_k - N_{k+1}}$$

$$\le \frac{N_{k+1}}{n'_k - N_{k+1}},$$

and

$$|v_k' - b_k| \leq |v_k' - v_k| + |v_k - b_k| = \frac{N_{k+1}}{n_k' - N_{k+1}} + \frac{1}{3k} < \frac{2}{3k}$$

if n_k is chosen big enough.

Then add those elements $a_{p_k(1)}, \ldots, a_{p_k(n_k)}$ to (d_n).

In that way we have constructed (d_n). We show that $d_n \xrightarrow{\mathcal{A}} b$. Let $\epsilon = \frac{1}{k}$. First we show that $b - \mathcal{A}(d_1, \ldots, d_{m_k}) < \frac{1}{k}$. It is clear that $b - \mathcal{A}$ $(a_{p_k(1)}, \ldots, a_{p_k(n_k)}) < \frac{1}{3k}$ and $(d_1, \ldots, d_{m_k})$ contains $(a_{p_k(1)}, \ldots, a_{p_k(n_k)})$. But in Step k we had $|v_k - b_k| < \frac{1}{3k}$. We remark that $v_k = \mathcal{A}$ $(d_1, \ldots, d_{m_k})$. Hence $|b - v_k| \leq |b - b_k| + |b_k - v_k| < \frac{1}{k}$.

Let $m_k < p \leq m_{k+1}$. By construction d_p is elements from $(a_{p_{k+1}(n)})$. Let $v = \mathcal{A}(d_1, \ldots, d_p)$ and let $v' = \mathcal{A}$(elements of $(a_{p_{k+1}(n)})$ among $d_1, \ldots, d_p$). Obviously,

$$v = \frac{(N_{k+1} + p - m_k)v' + (m_k - N_{k+1})v_k'}{p},$$

i.e., v is a weighted average of v' and v_k' therefore $v \in (v', v_k')$.

But

$$|v' - b| < |v' - b_{k+1}| + |b_{k+1} - b| < \frac{1}{3(k+1)} + \frac{1}{3(k+1)} = \frac{2}{3(k+1)} < \frac{1}{k},$$

and

$$|v_k' - b| < |v_k' - b_k| + |b_k - b| < \frac{2}{3k} + \frac{1}{3k} = \frac{1}{k},$$

which gives that $|v - b| < \frac{1}{k}$. We got that if $m_k \leq p \leq m_{k+1}$ then

$$b - \mathcal{A}(d_1, \ldots, d_p) < \frac{1}{k},$$

which proves the claim. □

3. On Bounded Sequences

Theorem 3.1. *Let (a_n) be a bounded sequence. Then $AAR_{(a_n)} = [\underline{\lim}\, a_n, \overline{\lim}\, a_n]$.*

Proof. Let $m = \underline{\lim}\, a_n$, $M = \overline{\lim}\, a_n$. Clearly if (a'_n) is a rearrangement of (a_n) then $m = \underline{\lim}\, a'_n$, $M = \overline{\lim}\, a'_n$. Hence by 2.8 $AAR_{(a_n)} \subset [m, M]$.

Now let l be choosen such that $m \leq l \leq M$. We can devide (a_n) into three distinct sequences: $b_n \to m$, $c_n \to M$ and (d_n) is the rest i.e., $\{b_n, c_n, d_n : n \in \mathbb{N}\} = \{a_n : n \in \mathbb{N}\}$ and $b_n \neq c_k \neq d_l \neq b_n (\forall n, k, l)$. It can happen that either (d_n) or $(c_n), (d_n)$ are empty. By Proposition 2.6 we can merge $(b_n), (c_n)$ into a new sequence (e_n) such that $e_n \xrightarrow{\mathcal{A}} l$. By Lemma 2.3 we can add (d_n) as well in a way that the limit does not change. □

Theorem 3.2. *Let $m = \underline{\lim}\, a_n$, $M = \overline{\lim}\, a_n$. If $m < M$, $m, M \in \mathbb{R}$, then we can create a rearrangement (e_n) such that $\lim_{n \to \infty} \frac{\sum_{i=1}^{n} e_i}{n}$ does not exist.*

Proof. Let us devide H into three distinct sequences as in the proof of Theorem 3.1 (let us use the same notations). Let $p = m + \frac{M - m}{3}$, $q = M - \frac{M - m}{3}$.

Now we define (e_n). Let the first element be d_1. Then take elements from (b_n) such that $\mathcal{A}(d_1, b_1, \ldots, b_{n_1}) < p$. Next element will be d_2. Then take elements from (c_n) such that $\mathcal{A}(d_1, b_1, \ldots, b_{n_1}, d_2, c_1, \ldots, c_{n_2}) > q$. Next element is d_3. Then take elements from (b_n) such that

$$\mathcal{A}(d_1, b_1, \ldots, b_{n_1}, d_2, c_1, \ldots, c_{n_2}, d_3, b_{n_1+1}, \ldots, b_{n_3}) < p,$$

and so on. Obviously we exhaust all elements from (a_n) and $\frac{\sum_{i=1}^{n} e_i}{n}$ will not converge.

4. On Unbounded Sequences

Lemma 4.1. *Let* (c_n) *be an increasing sequence such that* $c_n \to +\infty$, $c_n > 0$. *Let* (c'_n) *be any of its rearrangements. Then*

$$\overline{\lim_{n\to\infty}} \frac{c_n}{\sum_{i=1}^{n-1} c_i} \le \overline{\lim_{n\to\infty}} \frac{c'_n}{\sum_{i=1}^{n-1} c'_i}. \tag{6}$$

Proof. Take a subsequence (c_{n_k}) of (c_n) such that

$$\lim_{k\to\infty} \frac{c_{n_k}}{\sum_{i=1}^{n_k-1} c_i} = \overline{\lim_{n\to\infty}} \frac{c_n}{\sum_{i=1}^{n-1} c_i}.$$

We can assume that $\forall k$ if $m < n_k$, then $c_m < c_{n_k}$ because if there is $m < n_k$ such that $c_m = c_{n_k}$ then

$$\frac{c_{n_k}}{\sum_{i=1}^{n_k-1} c_i} < \frac{c_m}{\sum_{i=1}^{m-1} c_i},$$

hence put c_m into the subsequence instead of c_{n_k}.

Now find c_{n_k} in (c'_n), say $c_{n_k} = c'_{m_k}$. Let

$$l_k = \min\{n \in \mathbb{N} : n \le m_k,\, c'_n \ge c'_{m_k}\}.$$

Clearly if $n < l_k$ then $c'_n < c'_{m_k} = c_{n_k}$. This gives that

$$\frac{c_{n_k}}{\sum_{i=1}^{n_k-1} c_i} \le \frac{c'_{l_k}}{\sum_{i=1}^{l_k-1} c'_i},$$

because $c_{n_k} = c'_{m_k} \le c'_{l_k}$ and $\{c'_i : 1 \le i \le l_k - 1\} \subset \{c_i : 1 \le i \le n_k - 1\}$ since $\{c_i : 1 \le i \le n_k - 1\}$ contains all elements that are strictly smaller than c_{n_k}. That yields (6). □

Corollary 4.2. *Let* (c_n) *be a sequence such that* $c_n \to +\infty$, $c_n > 0$ *and*

$$\frac{c_n}{\sum_{i=1}^{n-1} c_i} \to 0.$$

If we rearrange it to an increasing sequence (c'_n), *then*

$$\frac{c'_n}{\sum_{i=1}^{n-1} c'_i} \to 0.$$ □

Theorem 4.3. *Let* $(a_n) = (b_n)\|(c_n)$, *where* $b_n \equiv 0$, $c_n \to +\infty$. *If*

$$\frac{c_n}{\sum_{i=1}^{n-1} c_i} \to 0,$$

then 1 *is accessible in average by rearrangement of* (a_n).

Proof. We can assume that c_n is increasing (by 4.2) and $c_n > 1$. Then let $d_{m_n} = c_n$, where (m_n) is a strictly increasing sequence determined by the followings: $d_k = 0$ if $\forall n\ k \neq m_n$ and $|\{d_i : d_i = 0, i < m_n\}| = \lfloor c_1 + \cdots + c_n \rfloor - n$. We show that $d_l \xrightarrow{\mathcal{A}} 1$. Obviously

$$\mathcal{A}(d_1, \ldots, d_{m_n}) = \frac{\sum_{i=1}^{n} c_i}{\left\lfloor \sum_{i=1}^{n} c_i \right\rfloor} \to 1,$$

$$\mathcal{A}(d_1, \ldots, d_{m_n - 1}) = \frac{\sum_{i=1}^{n-1} c_i}{\left\lfloor \sum_{i=1}^{n} c_i \right\rfloor - 1}.$$

With evident estimation

$$\frac{\sum_{i=1}^{n-1} c_i}{\left\lfloor \sum_{i=1}^{n-1} c_i \right\rfloor + c_n} \leq \frac{\sum_{i=1}^{n-1} c_i}{\left\lfloor \sum_{i=1}^{n} c_i \right\rfloor - 1} \leq \frac{\sum_{i=1}^{n-1} c_i}{\left\lfloor \sum_{i=1}^{n-1} c_i \right\rfloor + c_n - 2}.$$

If we take the reciprocal and apply the condition then we get that $\lim_{n\to\infty} \mathcal{A}(d_1, \ldots, d_{m_n - 1}) = 1$. To finish to proof we have to remark that if $m_{n-1} < l < m_n - 1$, then

$$\mathcal{A}(d_1, \ldots, d_{m_n - 1}) > \mathcal{A}(d_1, \ldots, d_l) > \mathcal{A}(d_1, \ldots, d_{m_n - 1}).$$ □

Theorem 4.4. *Let* $(a_n) = (b_n)\|(c_n)$, *where* $b_n \equiv 0$, $c_n \to +\infty$ *and* (c_n) *is increasing. If* 1 *is accessible in average by rearrangement of* (a_n), *then*

$$\frac{c_n}{\sum_{i=1}^{n-1} c_i} \to 0.$$

Proof. Let (d_n) be a rearrangement such that $d_n \xrightarrow{\mathcal{A}} 1$. This rearrangement defines a rearrangement of (c_n), namely take the elements from (c_n) exactly in the same order as they come in (d_n). Let us denote that rearranged sequence with (c'_n) and $c'_n = d_{m_n}$.

Let $\epsilon > 0$. Then there is $N \in \mathbb{N}$ such that $m \geq N$ implies that

$$1 - \epsilon < \frac{\sum_{i=1}^{m} d_i}{m} < 1 + \epsilon.$$

Let n be chosen such that $m_{n-1} > N$. Let $m = m_{n-1}$. We know that $\sum_{i=1}^{m_{n-1}} d_i = \sum_{i=1}^{n-1} c'_i$ which gives that $1 - \epsilon < \frac{s_n}{m} < 1 + \epsilon$, where $s_n = \sum_{i=1}^{n-1} c'_i$. Suppose that in (d_n) there are k_n zeros between c'_{n-1} and c'_n. It gives that

$$1 - \epsilon < \frac{s_n}{m + k_n} < 1 + \epsilon, \tag{7}$$

$$1 - \epsilon < \frac{s_n + c'_n}{m + k_n + 1} < 1 + \epsilon. \tag{8}$$

From (8) we get that

$$1 - \epsilon < \frac{1 + \frac{c'_n}{s_n}}{\frac{m + k_n}{s_n} + \frac{1}{s_n}} < 1 + \epsilon.$$

By multiplying with the denominator and using (7) we get that

$$\frac{1-\epsilon}{1+\epsilon}+\frac{1-\epsilon}{s_n}<(1-\epsilon)\left(\frac{m+k_n}{s_n}+\frac{1}{s_n}\right)<1+\frac{c'_n}{s_n}<(1+\epsilon)\left(\frac{m+k_n}{s_n}+\frac{1}{s_n}\right)$$

$$<\frac{1+\epsilon}{1-\epsilon}+\frac{1+\epsilon}{s_n},$$

and clearly both sides tend to 1 when $\epsilon \to 0$. Which finally gives that $\frac{c'_n}{s_n} \to 0$. Now 4.2 yields the statement. □

Theorem 4.5. *Let* $(a_n) = (b_n)\|(c_n)$, *where* $b_n \equiv 0$, $c_n \to +\infty$. *If* $1 \in AAR_{(a_n)}$, *then* $AAR_{(a_n)} = [0, +\infty]$.

Proof. We have to verify that if $l \in \mathbb{R}^+$, then $l \in AAR_{(a_n)}$.

Let (d_n) be a rearrangement such that $d_n \xrightarrow{\mathcal{A}} 1$.

First we show that if $l \in \mathbb{N}$, then $l \in AAR_{(a_n)}$.

Let k_n denotes the number of zeros in the first n terms of (d_n). We state that there is $N \in \mathbb{N}$ such that $n > N$ implies that $k_n > (1 - \frac{1}{l})n$. Assume the contrary: $\forall N\ \exists n > N$ such that $k_n \le (1 - \frac{1}{l})n$ which gives that there are at least $n' = \lceil \frac{1}{l} n \rceil$ elements (say $z_1, \dots, z_{n'}$) that are non zero. Then

$$\frac{\sum_{i=1}^{n} d_i}{n} = \frac{\sum_{i=1}^{n'} z_i}{n} = \frac{n'}{n}\frac{\sum_{i=1}^{n'} z_i}{n'} \ge \frac{1}{l}\frac{\sum_{i=1}^{n'} z_i}{n'}.$$

But the average of the non zero elements tends to infinite that gives a contradiction.

Now we construct a new rearrangement (d'_n) of (d_n). Till index N leave out the first $\lfloor(1-\frac{1}{l})N\rfloor$ many zeros. It can be done by the previous statement. Then we go on by recursion. Suppose we are done for $n > N$ and already left out $\lfloor(1-\frac{1}{l})n\rfloor$ many zeros. Now we are dealing with $n+1$. If $\lfloor(1-\frac{1}{l})n\rfloor = \lfloor(1-\frac{1}{l})(n+1)\rfloor$ then we do nothing. Otherwise leave out 1 more zeros. Again the previous statement guarantees that it is possible.

Let show that $d'_n \xrightarrow{\mathcal{A}} l$. Let $n > N - \lfloor(1-\frac{1}{l})N\rfloor$. Set $k_n = n - \lfloor(1-\frac{1}{l})n\rfloor$ that is the number of remainder elements after managing d_n. Observe that $\sum_{i=1}^{k_n} d'_i = \sum_{i=1}^{n} d_i$. By $n-(1-\frac{1}{l})n \le k_n \le n-(1-\frac{1}{l})n+1$, we get that

$$\frac{\frac{\sum_{i=1}^{n} d_i}{n}}{\frac{1}{l}+\frac{1}{n}} = \frac{\sum_{i=1}^{n} d_i}{n-(1-\frac{1}{l})(n+1)} \le \frac{\sum_{i=1}^{k_n} d'_i}{k_n} \le \frac{\sum_{i=1}^{n} d_i}{n-(1-\frac{1}{l})n} = \frac{\frac{\sum_{i=1}^{n} d_i}{n}}{\frac{1}{l}},$$

and both sides tend to l which proves the claim.

Now we show that if $l \in \mathbb{N}$, $0 < l' < l$ and $l \in AAR_{(a_n)}$, then $l' \in AAR_{(a_n)}$. Let (d_n) be a rearrangement such that $d_n \xrightarrow{\mathcal{A}} l$. Let $L = \frac{l}{l'} - 1$. Now let us put $\lfloor 2L\rfloor$ many zeros between d_1 and $d_2 (k \in \mathbb{N})$ and put $(\lfloor k \cdot L\rfloor$-previously added number of zeros) many zeros between d_{k-1} and d_k $(k \in \mathbb{N})$. Let us denote this new sequence by (d'_k). Let us denote

$d'_{n_k} = d_k$. Observe that $\mathcal{A}(d'_1, \dots, d'_{n_k}) \geq \mathcal{A}(d'_1, \dots, d'_n) \geq \mathcal{A}(d'_1, \dots, d'_{n_{k+1}-1})$ if $n_k < n < n_{k+1}$. Clearly $\mathcal{A}(d'_1, \dots, d'_{n_k}) = \dfrac{\sum_{i=1}^{k} d_i}{k + \lfloor k \cdot L \rfloor}$. By obvious estimation

$$\frac{l'}{l} \frac{\sum_{i=1}^{k} d_i}{k} = \frac{\sum_{i=1}^{k} d_i}{k + k \cdot L} \leq \frac{\sum_{i=1}^{k} d_i}{k + \lfloor k \cdot L \rfloor} \leq \frac{\sum_{i=1}^{k} d_i}{k + k \cdot L - 1} = \frac{1}{\frac{l}{l'} - \frac{1}{k}} \frac{\sum_{i=1}^{k} d_i}{k}.$$

Let us estimate $\mathcal{A}(d'_1, \dots, d'_{n_k-1}) = \dfrac{\sum_{i=1}^{k} d_i}{k + \lfloor (k+1) \cdot L \rfloor}$.

$$\frac{1}{\frac{l}{l'} - \frac{L}{k}} \frac{\sum_{i=1}^{k} d_i}{k} = \frac{\sum_{i=1}^{k} d_i}{k + k \cdot L + L} \leq \frac{\sum_{i=1}^{k} d_i}{k + \lfloor (k+1) \cdot L \rfloor} \leq \frac{\sum_{i=1}^{k} d_i}{k + k \cdot L + L - 1}$$

$$= \frac{1}{\frac{l}{l'} - \frac{1-L}{k}} \frac{\sum_{i=1}^{k} d_i}{k}.$$

Hence both $\mathcal{A}(d'_1, \dots, d'_{n_k}) \to l'$ and $\mathcal{A}(d'_1, \dots, d'_{n_k-1}) \to l'$ which give that $\mathcal{A}(d'_1, \dots, d'_n) \to l'$. □

Theorem 4.6. *Let* $(a_n) = (b_n) \| (c_n)$, *where* $b_n \to 0$, $c_n \to +\infty$. *If* $1 \in AAR_{(a_n)}$, *then* $AAR_{(a_n)} = [0, +\infty]$.

Proof. We have to verify that if $l \in \mathbb{R}^+$, then $l \in AAR_{(a_n)}$.

Let (d_n) be the rearranged sequence whose average tends to 1. Let k_n denote the number of elements from (b_n) among the first n terms of (d_n). Then

$$\frac{\sum_{j=1}^{n} d_j}{n} = \frac{k_n}{n} \frac{\sum_{j=1}^{k_n} b_{i_j}}{k_n} + \frac{\sum_{j=1}^{n-k_n} c_{l_j}}{n}.$$

Clearly $\frac{k_n}{n}$ is bounded, $\frac{\sum_{j=1}^{k_n} b_{i_j}}{k_n} \to 0$ therefore $\frac{\sum_{j=1}^{n-k_n} c_{l_j}}{n} \to 1$. Let us replace all b_n with 0 in (d_n) and denote the new sequence by (d'_n). Then

$$\frac{\sum_{j=1}^{n} d'_j}{n} = \frac{\sum_{j=1}^{n-k_n} c_{l_j}}{n}$$

hence $\mathcal{A}(d'_1, \dots, d'_n) \to 1$, i.e., $1 \in AAR_{(d'_n)}$.

By 4.5 (d'_n) can be rearranged to (d''_n) such that $\mathcal{A}(d''_1, \dots, d''_n) \to l$. Let k'_n denote the number of zeros among the first n terms of (d''_n). Let us replace all zeros with distinct elements from (b_n) in (d''_n) and denote the new sequence by (d'''_n). Then

$$\frac{\sum_{j=1}^{n} d''_j}{n} = \frac{\sum_{j=1}^{n-k'_n} c_{m_j}}{n} \to l,$$

$$\frac{\sum_{j=1}^{n} d'''_j}{n} = \frac{k'_n}{n} \frac{\sum_{j=1}^{k'_n} b_{n_j}}{k'_n} + \frac{\sum_{j=1}^{n-k'_n} c_{m_j}}{n}.$$

But the first term $\frac{k'_n}{n} \frac{\sum_{j=1}^{k'_n} b_{n_j}}{k'_n} \to 0$ hence $\mathcal{A}(d'''_1, \dots, d'''_n) \to l$. □

Theorem 4.7. *Let* $(a_n) = (b_n)\|(c_n)$, *where* $b_n \to a$, $c_n \to +\infty$. *If there is* $b \in \mathbb{R}$ *such that* $a < b$, $b \in AAR_{(a_n)}$, *then* $AAR_{(a_n)} = [a, +\infty]$.

Proof. Let $l \geq a$. Let $b'_n = \dfrac{b_n - a}{b - a}$, $c'_n = \dfrac{c_n - a}{b - a}$, $(a'_n) = (b'_n)\|(c'_n)$. Clearly $b'_n \to 0$, $c'_n \to +\infty$ hence by 4.6 (a'_n) can be rearranged to (d_n) such that $\mathcal{A}(d_1, \dots, d_n) \to \dfrac{l - a}{b - a}$. Let $d'_n = (b - a)d_n + a$. Then clearly $\mathcal{A}(d'_1, \dots, d'_n) \to l$ and (d'_n) is a rearrangement of (a_n). □

Proposition 4.8. *Let* $k \in \mathbb{N}$, $(a_n) = (b_n)\|(c_n)$, *where* $b_n \equiv 0$, $c_n = n^k$. *Then* $AAR_{(a_n)} = [0, +\infty]$.

Proof. By 4.6 and 4.3 it is enough to show that

$$\frac{c_n}{\sum_{i=1}^{n-1} c_i} \to 0.$$

It is known that $\sum_{i=1}^{n-1} i^k = p(n-1)$, where $p(x)$ is a polynomial of degree $k+1$, i.e., $p(n-1) = d_{k+1}(n-1)^{k+1} + q(n-1)$, where $q(x)$ is a polynomial of degree k and $d_{k+1} > 0$. Hence

$$\frac{c_n}{\sum_{i=1}^{n-1} c_i} = \frac{n^k}{d_{k+1}(n-1)^{k+1} + q(x)} = \frac{1}{d_{k+1}(n-1) \cdot (1 - \frac{1}{n})^k + \frac{q(x)}{n^k}} \to 0.$$

□

Proposition 4.9. *Let* $d > 1$, $(a_n) = (b_n)\|(c_n)$, *where* $b_n \equiv 0$, $c_n = d^n$. *Then* $AAR_{(a_n)} = \{0, +\infty\}$.

Proof. By 4.7 and 4.4 it is enough to show that $1 \notin AAR_{(a_n)}$, i.e.,

$$\frac{c_n}{\sum_{i=1}^{n-1} c_i} \not\to 0.$$

But

$$\frac{d^n}{\sum_{i=1}^{n-1} d^i} = \frac{d^n(d-1)}{d^n - 1} = \frac{d-1}{1 - \frac{1}{d^n}} \to d - 1 \neq 0. \qquad \square$$

Example 4.10. Let $(a_n) = (b_n)\|(c_n)$, where $b_n \equiv 0$, $c_n \to +\infty$ and $1 \in AAR_{(a_n)}$. Let (c'_n) be given such that $c'_n < c_n$, $c'_n \to +\infty$, $(a'_n) = (b_n)\|(c'_n)$. These conditions do not imply that $1 \in AAR_{(a'_n)}$.

Proof. Let $c_n = n^2$. By 4.8, $1 \in AAR_{(a_n)}$.

We define (c'_n) by recursion. Let $c'_1 = 1$. If (c'_n) is defined till n then let

$$c'_{n+1} = \begin{cases} c'_n, & \text{if } \sum_{i=1}^{n} c'_i \geq (n+1)^2, \\ \sum_{i=1}^{n} c'_i, & \text{otherwise.} \end{cases}$$

Properties of (c'_n):

(1) (c'_n) is increasing.

(2) $c'_n < c_n (n > 1)$. It can be seen by induction starting from $n = 2$, $c'_2 = 1$.

(3) $c'_n \to +\infty$. Assume the contrary. Then there is $N \in \mathbb{N}$ such that $n \geq N$ implies that $c'_N = c'_n$. Then for such n we get that $\sum_{i=1}^{n} c'_i \leq n \cdot c'_N$ using the monotonicity of (c'_n) too. But there is an n such that $n \cdot c'_N < (n+1)^2$ which is a contradiction.

(4) $1 \notin AAR_{(a'_n)}$. To show that it is enough to prove that

$$\frac{c'_n}{\sum_{i=1}^{n-1} c'_i} \not\to 0$$

by 4.4. There are infinitely many n where $c'_n \neq c'_{n-1}$. For such n

$$\frac{c'_n}{\sum_{i=1}^{n-1} c'_i} = \frac{\sum_{i=1}^{n-1} c'_i}{\sum_{i=1}^{n-1} c'_i} = 1,$$

therefore

$$\overline{\lim} \frac{c'_n}{\sum_{i=1}^{n-1} c'_i} \geq 1. \qquad \square$$

Proposition 4.11. *Let* $(a_n) = (b_n)\|(c_n)$, *where* $b_n \equiv 0$, $c_n = n$. *Let* (c'_n) *be given such that* $c'_n < c_n$, $c'_n \to +\infty$, $(a'_n) = (b_n)\|(c'_n)$. *Then* $1 \in AAR_{(a'_n)}$.

Proof. We can assume that $c'_n > 0$.

Let $\epsilon > 0$. Choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \frac{\epsilon}{2}$. Then there is $N \in \mathbb{N}$ such that $n > N$ implies that $c'_n \geq k$. Then

$$\frac{c'_n}{\sum_{i=1}^{n-1} c'_i} \leq \frac{n}{\sum_{i=N+1}^{n-1} k} = \frac{n}{(n-N-1)k} = \frac{1}{(1-\frac{N-1}{n})k} < \epsilon$$

if n is big enough which gives the statement by 4.3. □

Now we give some equivalent forms of the condition in 4.3.

Proposition 4.12. *Let* (c_n) *be a sequence such that* $\forall n\ c_n > 0$. *Then the followings hold*:

$$\frac{c_n}{\sum_{i=1}^{n-1} c_i} \to 0 \Leftrightarrow \frac{c_n}{\sum_{i=1}^{n} c_i} \to 0 \Leftrightarrow \frac{\sum_{i=1}^{n-1} c_i}{c_n} \to +\infty \Leftrightarrow \frac{\sum_{i=1}^{n} c_i}{c_n} \to +\infty.$$ □

Proposition 4.13. *Let* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *be an increasing function that is integrable over each finite interval and* $\lim_{+\infty} f = +\infty$. *Let* (c_n) *be defined by* $c_n = f(n)$. *Then*

$$\frac{c_n}{\sum_{i=1}^{n-1} c_i} \to 0 \Leftrightarrow \frac{f(n)}{\int_1^n f} \to 0.$$

Proof. By obvious estimation we get that

$$\frac{f(n)}{\sum_{i=1}^{n} f(i)} \leq \frac{f(n)}{\sum_{i=2}^{n} f(i)} \leq \frac{f(n)}{\int_1^n f} \leq \frac{f(n)}{\sum_{i=1}^{n-1} f(i)},$$

which gives the statement. □

Proposition 4.14. *Let* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *be an increasing function that has a primitive function F and* $\lim\limits_{+\infty} f = +\infty$. *Let* (c_n) *be defined by* $c_n = f(n)$. *Then*

$$\frac{c_n}{\sum_{i=1}^{n-1} c_i} \to 0 \Leftrightarrow \frac{f(n)}{F(n)} \to 0. \qquad \square$$

References

[1] F. Bagemihl and P. Erdös, Rearrangements of C_1-summable series, Acta Mathematica 92(1) (1954), 35-53.

DOI: https://doi.org/10.1007/BF02392700

[2] P. S. Bullen, Handbook of Means and their Inequalities, 560 Kluwer Academic Publisher, Dordrecht, The Netherlands, 2003.

[3] Y. Dybskiy and K. Slutsky, Riemann Rearrangement Theorem for Some Types of Convergence, Preprint, arXiv:math/0612840.

[4] R. Filipów and P. Szuca, Rearrangement of conditionally convergent series on a small set, Journal of Mathematical Analysis and Applications 362(1) (2010), 64-71.

DOI: https://doi.org/10.1016/j.jmaa.2009.07.029

[5] G. G. Lorentz and K. I. Zeller, Series rearrangements and analytic sets, Acta Mathematica 100(3-4) (1958), 149-169.

DOI: https://doi.org/10.1007/BF02559537

[6] A. Losonczi, Means of Infinite Sets I, Preprint, arXiv:1704.07115.

[7] A. Losonczi, Measures by means, means by measures, preprint, arXiv:1706.03658.

[8] P. A. B. Pleasants, Rearrangements that preserve convergence, Journal of the London Mathematical Society 15(1) (1977), 134-142.

DOI: https://doi.org/10.1112/jlms/s2-15.1.134

[9] M. A. Sarigöl, Rearrangements of bounded variation sequences, Proceedings of the Indian Academy of Sciences (Mathematical Sciences) 104(2) (1994), 373-376.

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