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EXISTENCE OF SOLUTIONS FOR A CLASS OF NONLINEAR MULTI-TERM FRACTIONAL DIFFERENTIAL EQUATIONS WITH IMPULSIVE AND INTEGRAL BOUNDARY CONDITIONS

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Abstract

In this paper, we investigate the uniqueness and existence of solution for nonlinear multi-term fractional differential equations with impulsive and fractional integral boundary conditions by means of the standard fixed point theorems and nonlinear alternative theorem. To illustrate our results, we give two examples.

1. Introduction

Fractional differential equations appear widely in various fields of science and engineering such as physics, the polymer rheology, regular variation in thermodynamics and so on [1]-[5].

On the other hand, impulsive differential equations provide an exact description of the observed evolution processes and they are regarded as important mathematical tools for the better understanding of several real world problems in applied sciences [6]-[10].

In recent years, many scientists have studied the existence and uniqueness of the solution for fractional differential equations by using analytic and numerical methods.

In fact, the theory of impulsive differential equations is much richer than that of ordinary differential equations without impulse effects since a simple impulsive differential equation may enough exhibit several new phenomena such as rhythmical beating [11]-[14]. Thus impulsive fractional differential equations are widely studied recently [6, 15-22].

The investigation for impulsive multi-term fractional differential equation with the integral boundary conditions has not been appreciated well enough. The integral boundary conditions arise from many applications such as blood flow problems, chemical engineering, thermo-elasticity, underground water flow, and so on [23]-[25].

Additionally, it is well known that the integral boundary conditions include two-point, multi-point, and nonlocal boundary condition.

Guezane-Lakoud et al. [26] proved the existence of solutions for the following fractional differential equation with fractional integral condition

$$\begin{cases} {}^c D^{\alpha} x(t) = f(t, x(t), {}^c D^{\beta} x(t)), \ t \in [0, 1], \ 1 < \alpha \le 2, \ 0 < \beta < 1, \\ x(0) = 0, \ I^{\beta} x(1) = x'(1). \end{cases}$$

Fu et al. [27] considered the impulsive fractional differential equation

$$\begin{cases} {}^c D_0^{\alpha} x(t) = f(t, x(t)), \ t \in J = [0, 1], \ t \ne t_k, \ k = 1, 2, \cdots, m, \\ \Delta x(t_k) = I_k(x(t_k^-)), \ \Delta x'(t_k) = J_k(x(t_k^-)), \\ x(0) = 0, \ aI^{\gamma} x(1) + bx'(1) = c. \end{cases}$$

According above mentioned paper, in this article, we will investigate uniqueness and existence theorems of solution for the following nonlinear multi-term fractional differential equations with impulsive and fractional integral boundary conditions

$$\begin{cases} {}^c D_0^{\alpha} x(t) = f(t, x(t), {}^c D_0^{\beta} x(t)), \ t \in l = [0, 1], \ t \ne t_k, \ k = 1, 2, \cdots, m, \\ \Delta x(t_k) = I_k(x(t_k^-)), \ \Delta x'(t_k) = J_k(x(t_k^-)), \\ x(0) = 0, \ aI^{\gamma} x(1) + bx'(1) = c, \end{cases} \tag{1}$$

where ${}^c D_0^{\alpha} x(t)$ and ${}^c D_0^{\beta} x(t)$ are the Caputo fractional derivatives, $1 < \alpha \le 2$, $0 < \beta < 1$ and $1 < \gamma < 2$, $f : l \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a continuous, $I_k, J_k \in C(\mathbf{R}, \mathbf{R})$, $0 < t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $a, b, c \in \mathbf{R}$ are such that a $a \ne -b\Gamma(\gamma + 2)$, I^{γ} is the Riemann-Liouville fractional integral of order γ, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+)$ and $x(t_k^-)$ denote the right and the left limits of $x(t)$ at $t = t_k (k = 1, 2, \cdots, m)$, respectively and $\Delta x'(t_k)$ have a similar meaning for $x'(t)$.

The paper is organized as follows. First, we give some preliminary results and present two classical fixed point theorems. In Section 3, we present and prove our results which consist of uniqueness and existence theorems for solutions of the Equation (1). In Section 4, two examples are presented to illustrate our results. The conclusions are given in Section 5.

2. Preliminaries

Definition 1 ([28])**.** The Riemann-Liouville fractional integral of order $q \in \mathbf{R}^+$ of a function $f \in L_1[a, b]$ is given by

$$I_a^q f(t) := \frac{1}{\Gamma(q)} \int_a^t (t-\tau)^{q-1} f(\tau) d\tau,\ a \leq t \leq b.$$

Definition 2 ([28])**.** The Caputo's fractional derivative of order $\alpha \in \mathbf{R}^+$ of a function f is given by

$${}^c D_a^\alpha f(t) := I_a^{n-\alpha}[D^n f(t)] = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \\ \dfrac{d^n}{dt^n} f(t),\ \alpha = n \in \mathbf{N}, \end{cases}$$

where $n - 1 < \alpha \leq n$ and $f^{(n)} \in L_1[a, b]$.

Lemma 3 ([27])**.** *If* $\alpha > 0$, *then the differential equation*

$${}^c D_0^\alpha h(t) = 0$$

has the solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$ *and*

$$I_0^\alpha\, {}^c D_0^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbf{R},\ i = 0, 1, \cdots, n-1$ *and* $n = [\alpha] + 1$.

We introduce the following notations:

$l_0 = [0, t_1],\ l_1 = (t_1, t_2], \cdots, l_{m-1} = (t_{m-1}, t_m],\ l_m = (t_m, 1],\ l = [0, 1],$ $l' = l \setminus \{t_1, t_2, \cdots, t_m\},\ PC(l, \mathbf{R}) := \{y : l \to \mathbf{R} | y \in C(l_k, \mathbf{R}),\ y(0+0),\ y(1-0),$

$y(t_k + 0)$ *and* $y(t_k - 0)$ *exist and* $y(t_k - 0) = y(t_k)$, $k = \overline{1, p}\}$, $PC^1(l, \mathbf{R}) := \{y \in PC(l, \mathbf{R}) | y \in C^1(l_k, \mathbf{R}), y'(0+0), y'(1-0), y'(t_k + 0)$ *and* $y'(t_k - 0)$ *exist*, $k = \overline{1, p}\}$.

Lemma 4 ([27]). *For any* $y \in PC(l, \mathbf{R})$, *the impulsive boundary value problem*

$$\begin{cases} {}^cD_0^\alpha x(t) = y(t),\ t \in l = [0, 1],\ t \neq t_k,\ k = 1, 2, \cdots, m, \\ \Delta x(t_k) = I_k(x(t_k^-)),\ \Delta x'(t_k) = J_k(x(t_k^-)), \\ x(0) = 0,\ aI^\gamma x(1) + bx'(1) = c, \end{cases} \tag{2}$$

has a solution $x \in PC^1(l, \mathbf{R})$ *which is given by*

$$x(t) = \begin{cases} I_0^\alpha y(t) + \dfrac{(c - N(y) - M(x))t}{\frac{a}{\Gamma(\gamma+2)} + b} - \displaystyle\sum_{i=1}^{m} J_i(x(t_i^-))t,\ t \in l_0, \\ I_0^\alpha y(t) + I_1(x(t_1^-)) - t_1 J_1(x(t_1^-)) + \dfrac{(c - N(y) - M(x))t}{\frac{a}{\Gamma(\gamma+2)} + b} - \displaystyle\sum_{i=2}^{m} J_i(x(t_i^-))t,\ t \in l_1, \\ I_0^\alpha y(t) + \displaystyle\sum_{i=1}^{k} I_i(x(t_1^-)) - \sum_{i=1}^{k} t_i J_i(x(t_i^-)) + \dfrac{(c - N(y) - M(x))t}{\frac{a}{\Gamma(\gamma+2)} + b} - \displaystyle\sum_{i=k+1}^{m} J_i(x(t_i^-))t, \\ \qquad t \in l_k,\ k = 2, 3, \cdots, m-1, \\ I_0^\alpha y(t) + \displaystyle\sum_{i=1}^{m} I_i(x(t_i^-)) - \sum_{i=1}^{m} t_i J_i(x(t_i^-)) + \dfrac{(c - N(y) - M(x))t}{\frac{a}{\Gamma(\gamma+2)} + b},\ t \in l_m, \end{cases} \tag{3}$$

where

$$N(y) := a\int_0^1 \frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)} y(s)ds + b\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds,$$

$$M(x) := \frac{a\left(\displaystyle\sum_{i=1}^{m} I_i(x(t_i^-)) - \sum_{i=1}^{m} t_i J_i(x(t_i^-))\right)}{\Gamma(\gamma+1)}.$$

Theorem 5 (Nonlinear alternative of Leray-Schauder type [16]).

Let X *be a Banach space,* C *a nonempty convex subset of* X, U *a nonempty open subset of* C *with* $0 \in U$. *Suppose that* $P : \overline{U} \to C$ *is a continuous and compact map. Then either* (a) P *has a fixed point in* $\overline{U}$, *or* (b) *there exist an* $x \in \partial U$ *(the boundary of* U*) and* $\lambda \in (0, 1)$ *with* $x = \lambda P x$.

Theorem 6 (Schaefer fixed point theorem [16]).

Let X *be a normed space and* $P : X \to X$ *be a continuous mapping which is compact on each bounded subset* B *of* X. *Then either* (a) *the equation* $x = \lambda P x$ *has a solution for* $\lambda = 1$, *or* (b) *the set of all such solutions of* $x = \lambda P x$ *is unbounded for* $0 < \lambda < 1$.

3. Main Results

Let

$$\widetilde{X} = \{x | x \in PC(l, \mathbf{R}),\ {}^{c}D_0^{\beta} x \in PC(l, \mathbf{R})\},\ X = (\widetilde{X}, \|\cdot\|),$$

$$\|x(t)\| = \sup_{t \in l} |x(t)| + \sup_{t \in l} |{}^{c}D_0^{\beta} x(t)|.$$

Obviously, X is a complete space [29].

We define the operator $T : X \to X$ by

$$(Tx)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s), {}^{c}D_0^{\beta} x(s)) ds + \sum_{i=1}^{k} I_i(x(t_i^-)) - \sum_{i=1}^{k} t_i J_i(x(t_i^-))$$

$$+ \frac{(c - N(x) - M(x))t}{\frac{a}{\Gamma(\gamma+2)} + b} - \sum_{i=k+1}^{m} J_i(x(t_i^-))t,$$

$$t \in l_k,\ k = 0, 1, 2, \cdots, m. \tag{4}$$

Then the existence of the solutions for the Equation (1) is equivalent to the existence of the fixed point of the operator T.

Let denote $W(x, y) := f(s, x(s), {}^cD_0^\beta x(s)) - f(s, y(s), {}^cD_0^\beta y(s)), x, y \in X.$

Theorem 7. *Let* $f \in C(l \times \mathbf{R} \times \mathbf{R}, \mathbf{R})$ *and suppose that there exist constants* $k \in \mathbf{R}^+$, $L \in \mathbf{R}^+$ *and* $L^* \in \mathbf{R}^+$ *such that for any* $x, y \in X$, *the following inequality holds*:

$$|f(t, x(t), {}^cD_0^\beta x(t)) - f(t, y(t), {}^cD_0^\beta y(t))| \le k\{|x(t) - y(t)| + |{}^cD_0^\beta x(t) - {}^cD_0^\beta y(t)|\}, \tag{5}$$

and for any $u, v \in \mathbf{R}$,

$$|I_k(u) - I_k(v)| \le L|u - v|, \ |J_k(u) - J_k(v)| \le L^*|u - v|. \tag{6}$$

If

$$\max\left\{2\Big[k\Big(\frac{1}{\Gamma(\alpha+1)} + \frac{|a|}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(\alpha+\gamma+1)} + \frac{|b|}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(\alpha)}\right.$$

$$+ mL\Big(1 + \frac{|a|}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(\gamma+1)}\Big) + L^*\Big(m + \sum_{i=1}^m t_i + \frac{|a|\sum_{i=1}^m t_i}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(\gamma+1)}\Big)\Big],$$

$$2\Big[k\Big(\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{|a|}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(2-\beta)\Gamma(\alpha+\gamma+1)}$$

$$+ \frac{|b|}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(2-\beta)\Gamma(\alpha)}\Big) + \frac{|a|mL}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(2-\beta)\Gamma(\gamma+1)}$$

$$\left.+ L^*\Big(m + \frac{|a|\sum_{i=1}^m t_i}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(\gamma+1)\Gamma(2-\beta)}\Big)\Big]\right\} = d < 1,$$

then the Equation (1) *has a unique solution* $x \in X$ *on* l.

Proof. By Equations (5) and (6), we have

$$|N(x)-N(y)| \leq |a|\int_0^1 \frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\gamma+\alpha)}|W(x,\ y)|ds + |b|\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}|W(x,\ y)|ds$$

$$\leq \frac{|a|k}{\Gamma(\alpha+\gamma+1)}\|x-y\| + \frac{|b|k}{\Gamma(\alpha)}\|x-y\|, \tag{7}$$

$$|M(x)-M(y)| \leq \frac{|a|}{\Gamma(\gamma+1)}\sum_{i=1}^{m}|I_i(x(t_i^-)) - I_i(y(t_i^-))| + \frac{|a|}{\Gamma(\gamma+1)}\sum_{i=1}^{m} t_i |J_i(x(t_i^-))$$

$$- J_i(y(t_i^-))| \leq \frac{|a|mL}{\Gamma(\gamma+1)}\|x-y\| + \frac{|a|L^*}{\Gamma(\gamma+1)}\sum_{i=1}^{m} t_i\|x-y\|. \tag{8}$$

Then by Equations (7) and (8), we obtain the following estimation:

$$|Tx(t)-Ty(t)| \leq [k(\frac{1}{\Gamma(\alpha+1)} + \frac{|a|}{|\frac{a}{\Gamma(\gamma+2)}+b|\Gamma(\alpha+\gamma+1)} + \frac{|b|}{|\frac{a}{\Gamma(\gamma+2)}+b|\Gamma(\alpha)}$$

$$+ mL(1+\frac{|a|}{|\frac{a}{\Gamma(\gamma+2)}+b|\Gamma(\gamma+1)}) + L^*(m + \sum_{i=1}^{m} t_i + \frac{|b|\sum_{i=1}^{m} t_i}{|\frac{a}{\Gamma(\gamma+2)}+b|\Gamma(\gamma+1)}]\|x-y\|. \tag{9}$$

Next, let estimate $|{}^cD_0^\beta Tx(t) - {}^cD_0^\beta Ty(t)|$.

$${}^cD_0^\beta Tx(t) = I_0^{\alpha-\beta}f + \frac{c-N(x)-M(x)}{\frac{a}{\Gamma(\gamma+2)}+b}\frac{t^{1-\beta}}{\Gamma(2-\beta)} + \sum_{i=k+1}^{m} J_i(x(t_i^-))\frac{t^{1-\beta}}{\Gamma(2-\beta)},$$

$$t \in l_k,\ k = 0,\ 1,\ 2,\ \cdots,\ m.$$

$$|{}^{c}D_{0}^{\beta}Tx(t) - {}^{c}D_{0}^{\beta}Ty(t)|$$

$$\leq \frac{1}{\Gamma(\alpha-\beta)}\int_{0}^{t}(t-s)^{\alpha-\beta-1}|W(x,\ y)|ds + \frac{1}{|\Gamma(2-\beta)||\frac{a}{\Gamma(\gamma+2)}+b|}$$

$$\times [|a|\int_{0}^{1}\frac{(1-s)^{\alpha+\gamma-1}}{\Gamma(\gamma+\alpha)}|W(x,\ y)|ds + |b|\int_{0}^{1}\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}|W(x,\ y)|ds$$

$$+ \frac{|a|}{\Gamma(\gamma+1)}\sum_{i=1}^{m}|I_i(x(t_i^-)) - I_i(y(t_i^-))| + \frac{|a|}{\Gamma(\gamma+1)}\sum_{i=1}^{m}t_i|J_i(x(t_i^-))$$

$$- J_i(y(t_i^-))|] + \sum_{i=1}^{m}t_i|J_i(x(t_i^-)) - J_i(y(t_i^-))|$$

$$\leq [k(\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{|a|}{|\frac{a}{\Gamma(\gamma+2)}+b|\Gamma(2-\beta)\Gamma(\alpha+\gamma+1)}$$

$$+ \frac{|b|}{|\frac{a}{\Gamma(\gamma+2)}+b|\Gamma(2-\beta)\Gamma(\alpha)}) + \frac{|a|mL}{|\frac{a}{\Gamma(\gamma+2)}+b|\Gamma(2-\beta)\Gamma(\gamma+1)}$$

$$+ L^{*}(m + \frac{|a|\sum_{i=1}^{m}t_i}{|\frac{a}{\Gamma(\gamma+2)}+b|\Gamma(\gamma+1)\Gamma(2-\beta)})]\|x-y\|. \tag{10}$$

Hence by Equations (9) and (10), we have

$$\|Tx - Ty\| = \max_{t\in l}|Tx(t) - Ty(t)| + \max_{t\in l}|{}^{c}D_{0}^{\beta}Tx(t) - {}^{c}D_{0}^{\beta}Ty(t)| \leq d\|x-y\|,$$

and since $0 < d < 1$, T is a contractive on X. By Banach's fixed point theorem, the Equation (1) has a unique solution $x \in X$ on l and this completes the proof.

Lemma 8. *The operator* T *defined in Equation* (4) *is completely continuous on X.*

Proof. Since f, I_k, J_k are continuous, we know that T is continuous on X. Let $B \subset X$ be a bounded set. Then there exist positive numbers $k_1, k_2,$ and k_3 such that

$$\forall t \in l,\ x \in B,\ |f(t, x(t),\ {}^cD_0^\beta x(t))| \le k_1,\ |I_k(x(t_k^-))| \le k_2,\ |J_k(x(t_k^-))| \le k_3,$$

$$|(Tx)(t)| \le \frac{k_1}{\Gamma(\alpha+1)} + mk_2 + k_3\left(\sum_{i=1}^{m} t_i + m\right) + \frac{|c| + |N(x)| + |M(x)|}{\left|\frac{a}{\Gamma(\gamma+2)+b}\right|},$$

$$|N(x)| \le \frac{|a|k_1}{\Gamma(\alpha+\gamma+1)} + \frac{|b|k_1}{\Gamma(\alpha)}, \tag{11}$$

$$|M(x)| \le \frac{|a|\left(mk_2 + \sum_{i=1}^{m} t_i k_3\right)}{\Gamma(\gamma+2)}. \tag{12}$$

So for any $x \in B$ and $t \in l$, we have

$$|(Tx)(t)| \le \frac{k_1}{\Gamma(\alpha+1)} + \frac{k_1}{\left|\frac{a}{\Gamma(\gamma+2)} + b\right|}\left(\frac{|a|}{\Gamma(\alpha+\gamma+1)} + \frac{|b|}{\Gamma(a)}\right)$$

$$+ mk_2\left(1 + \frac{|a|}{\left|\frac{a}{\Gamma(\gamma+2)} + b\right|\Gamma(\gamma+2)}\right) + \frac{|c|}{\left|\frac{a}{\Gamma(\gamma+2)} + b\right|}$$

$$+ k_3\left(\left(\sum_{i=1}^{m} t_i + m\right) + \frac{|a|\sum_{i=1}^{m} t_i}{\left|\frac{a}{\Gamma(\gamma+2)} + b\right|\Gamma(\gamma+2)}\right),$$

which shows that $T(B)$ is uniformly bounded.

$$|(Tx)(t_2) - (Tx)(t_1)| \le \frac{k_1(t_2^\alpha - t_1^\alpha)}{\Gamma(\alpha+1)} + \left(\frac{|c| + |N(x)| + |M(x)|}{\left|\frac{a}{\Gamma(\gamma+2)+b}\right|} + mL^*\right)(t_2 - t_1), \tag{13}$$

$$|{}^cD_0^\beta Tx(t_2) - {}^cD_0^\beta Tx(t_1)|$$

$$\leq \left| \int_0^{t_2} \frac{(t_2 - s)^{\alpha-\beta-1}}{\Gamma(\alpha - \beta)} f(s, x(s), {}^cD_0^\beta x(s))ds - \int_0^{t_1} \frac{(t_1 - s)^{\alpha-\beta-1}}{\Gamma(\alpha - \beta)} f(s, x(s), {}^cD_0^\beta x(s))ds \right|$$

$$+ \frac{|c| + |N(x)| + |M(x)|}{|\frac{a}{\Gamma(\gamma+2)} + b|} \frac{(t_2^{1-\beta} - t_1^{1-\beta})}{\Gamma(2 - \beta)} + mL^* \frac{1}{\Gamma(2 - \beta)} (t_2^{1-\beta} - t_1^{1-\beta}) \tag{14}$$

$$\leq \frac{k_1(t_2^{\alpha-\beta} - t_1^{\alpha-\beta})}{\Gamma(\alpha - \beta + 1)} + \left(\frac{|c| + |N(x)| + |M(x)|}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(2 - \beta)} + mL^* \frac{1}{\Gamma(2 - \beta)} \right)(t_2^{1-\beta} - t_1^{1-\beta}).$$

Hence by Equations (11), (12), (13), and (14), we have

$$\|Tx(t_2) - Tx(t_1)\| \to 0(t_2 \to t_1).$$

That is to say, $T(B)$ is equi-continuous and the operator T is completely continuous. This completes the proof.

Theorem 9. *Assume that*

(1) *there exist* $h(t) \in L^\infty(l, \mathbf{R}^+)$ *and a* $\varphi : [0, +\infty) \to (0, +\infty)$ *continuous, nondecreasing function such that for any* $t \in l$ *and* $x \in X$,

$$|f(t, x, {}^cD_0^\beta x)| \leq h\varphi(\|{}^cD_0^\beta x\|).$$

(2) $f \in C(l \times \mathbf{R} \times \mathbf{R}, \mathbf{R})$.

Also let assume that there exist $\psi, \psi^* : [0, +\infty) \to (0, +\infty)$ *continuous, nondecreasing functions such that*

$$|J_k(x)| \leq \psi(\|x\|), |J_k(x)| \leq \psi^*(\|x\|).$$

Let $h^* := \sup\{|h(t)| : t \in l\}$, $U := \{x \in PC(l, \mathbf{R}) | \|x\| < M\}$.

Suppose that there exists $M > 0$ *such that*

$$\min\{\frac{M}{2(P\varphi(M)+Q\psi(M)+R\psi^*(M)+H)}, \frac{M}{2(P'\varphi(M)+Q'\psi(M)+R'\psi^*(M))}\} < 1, \tag{15}$$

where

$$P = \frac{h^*}{\Gamma(\alpha+1)} + \frac{h^*}{|\frac{a}{\Gamma(\gamma+2)}+b|}\left(\frac{|a|}{\Gamma(\alpha+\gamma+1)}+\frac{|b|}{\Gamma(\alpha)}\right),\ H = \frac{|c|}{|\frac{a}{\Gamma(\gamma+2)}+b|},$$

$$Q = m\left(1+\frac{|a|}{|\frac{a}{\Gamma(\gamma+2)}+b|\Gamma(\gamma+2)}\right),\ R = \left(\sum_{i=1}^{m} t_i + m + \frac{|a|\sum_{i=1}^{m} t_i}{|\frac{a}{\Gamma(\gamma+2)}+b|\Gamma(\gamma+2)}\right),$$

$$P' = \frac{h^*}{\Gamma(\alpha-\beta+1)} + \frac{h^*}{|\frac{a}{\Gamma(\gamma+2)}+b|\Gamma(2-\beta)}\left(\frac{|a|}{\Gamma(\alpha+\gamma+1)}+\frac{|b|}{\Gamma(\alpha)}\right),$$

$$Q' = m\left(1+\frac{|a|}{|\frac{a}{\Gamma(\gamma+2)}+b|\Gamma(2-\beta)\Gamma(\gamma+1)}\right),\ R' = \frac{|a|\sum_{i=1}^{m} t_i}{|\frac{a}{\Gamma(\gamma+2)}+b|\Gamma(2-\beta)} + \frac{m}{\Gamma(2-\beta)}.$$

Then the Equation (1) *has at least one solution* x *in* $\overline{U}$.

Proof. We will show that the operator T satisfies the assumptions of the nonlinear alternative of Leray-Schauder type.

First, we know that T is completely continuous by Lemma 8.

Let $x \in X$ be such that $x(t) = \lambda(Tx)(t)$ for some $\lambda \in (0, 1)$.

Similarly to the proof of Lemma 8,

$$|Tx(t)| \le h^*\varphi(\|x\|)\frac{1}{\Gamma(\alpha+1)} + m\psi(\|x\|) + \psi^*(\|x\|)\sum_{i=1}^{m} t_i + \frac{|c|}{|\frac{a}{\Gamma(\gamma+2)}+b|} + \frac{h^*\varphi(\|x\|)}{|\frac{a}{\Gamma(\gamma+2)}+b|}$$
$$\left(\frac{|a|}{\Gamma(\alpha+\gamma+1)}+\frac{|b|}{\Gamma(\alpha)}\right) + \frac{1}{|\frac{a}{\Gamma(\gamma+2)}+b|}\left(\frac{|a|m\psi(\|x\|)+|a|\psi^*(\|x\|)\sum_{i=1}^{m} t_i}{\Gamma(\gamma+1)}\right) + m\psi^*, \tag{16}$$

and

$$|{}^{c}D_0^{\beta}Tx(t)| \leq h^*\varphi(\|x\|)\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{|c|}{|\frac{a}{\Gamma(\gamma+2)}+b|\Gamma(2-\beta)} + \frac{1}{|\frac{a}{\Gamma(\gamma+2)}+b|\Gamma(2-\beta)}\cdot$$

$$\left[\frac{h^*\varphi(\|x\|)|a|}{\Gamma(\alpha+\gamma+1)} + \frac{h^*\varphi(\|x\|)|b|}{\Gamma(\alpha)} + \frac{|a|m\psi(\|x\|)+|a|\psi^*(\|x\|)\sum_{i=1}^{m}t_i}{\Gamma(\gamma+1)}\right] + \frac{m\psi^*(\|x\|)}{\Gamma(2-\beta)}. \tag{17}$$

By Equations (16) and (17), we have

$$\|x\| = \|Tx\| = \max_{t\in J}|Tx(t)| + \max_{t\in J}|{}^{c}D_0^{\beta}Tx(t)| \leq \max\{2(P\varphi(\|x\|)+Q\psi(\|x\|)+R\psi^*(\|x\|)+H),$$

$$2(P'\varphi(\|x\|)+Q'\psi(\|x\|)+R'\psi^*(\|x\|))\}.$$

That is,

$$\frac{\|x\|}{\max\{2(P\varphi(\|x\|)+Q\psi(\|x\|)+R\psi^*(\|x\|)+H),\ 2(P'\varphi(\|x\|)+Q'\psi(\|x\|)+R'\psi^*(\|x\|))\}} \leq 1.$$

Since Equation (15), we have

$$\frac{\|x\|}{\max\{2(P\varphi(\|x\|)+Q\psi(\|x\|)+R\psi^*(\|x\|)+H),\ 2(P'\varphi(\|x\|)+Q'\psi(\|x\|)+R'\psi^*(\|x\|))\}}$$

$$< \min\left\{\frac{M}{2(P\varphi(M)+Q\psi(M)+R\psi^*(M)+H)}, \frac{M}{2(P'\varphi(M)+Q'\psi(M)+R'\psi^*(M))}\right\}.$$

Hence $\|x\| \neq M$ and consequently, $T:\overline{U}\to X$ is completely continuous.

From the choice of the set U, there is no $x\in\partial U$ such that $x=\lambda Tx$ for some $0<\lambda<1$. Therefore by the nonlinear alternative of Leray-Schauder type Theorem 5, we know that there exists at least one fixed point in $\overline{U}$ and this completes the proof.

Theorem 10. *Let* $h \in L^{\infty}(J, \mathbf{R}^{+})$ *and* H_1 *and* H_2 *be positive constants.*

If the following inequalities hold for any $t \in l$ *and* $x \in X$, $k = 1, 2, \cdots, m$, *then Equation* (1) *has at least one solution in X.*

$$|f(t, x, {}^{c}D_{0}^{\beta}x(t))| \leq h(t),\ |I_k(x)| \leq H_1,\ |J_k(x)| \leq H_2.$$

Proof. Obviously, $T : X \to X$ is completely continuous.

Now we show that $V = \{v \in V : v = \lambda Tv,\ 0 < \lambda < 1\}$ is a bounded set.

Let $x \in V$ be such that $x = \lambda Tx$ for some $0 < \lambda < 1$. Similarly to the proof of Theorem 9, for any $t \in J$, we have

$$\|x\| = \|Tx\| = \max_{t \in J}|\lambda Tx(t)| + \max_{t \in J}|{}^{c}D^{\beta}Tx(t)|.$$

Then

$$|x(t)| \leq h^{*}\left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{|\frac{a}{\Gamma(\gamma+2)} + b|}\left(\frac{|a|}{\Gamma(\alpha+\gamma+1)} + \frac{|b|}{\Gamma(\alpha)}\right)\right]$$

$$+ mH_1\left(1 + \frac{|a|}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(\gamma+2)} + \frac{|c|}{|\frac{a}{\Gamma(\gamma+2)} + b|}\right)$$

$$+ H_2\left(\left(\sum_{i=1}^{m} t_i + m\right) + \frac{|a|\sum_{i=1}^{m} t_i}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(\gamma+2)}\right), \tag{18}$$

and

$$|{}^{c}D_{0}^{\beta}x(t)| \leq h^{*}\left[\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{1}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(2-\beta)}\left(\frac{|a|}{\Gamma(\alpha+\gamma+1)} + \frac{|b|}{\Gamma(\alpha)}\right)\right]$$

$$+ H_1\frac{m|a|}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(2-\beta)\Gamma(\gamma+1)}$$

$$+ H_2\left(\frac{\sum_{i=1}^{m} t_i}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(2-\beta)\Gamma(\gamma+1)} + \frac{m}{\Gamma(2-\beta)}\right)$$

$$+ \frac{|c|}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(2-\beta)}. \tag{19}$$

By Equations (18) and (19), we have

$$M := 2\max\left\{h^*\left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{|\frac{a}{\Gamma(\gamma+2)} + b|}\left(\frac{|a|}{\Gamma(\alpha+\gamma+1)} + \frac{|b|}{\Gamma(\alpha)}\right)\right] + mH_1\right.$$

$$\left(1 + \frac{|a|}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(\gamma+2)} + \frac{|c|}{|\frac{a}{\Gamma(\gamma+2)} + b|}\right) + H_2\left((\sum_{i=1}^{m} t_i + m) + \frac{|a|\sum_{i=1}^{m} t_i}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(\gamma+2)}\right),$$

$$h^*\left[\frac{1}{\Gamma(\alpha-\beta+1)} + \frac{1}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(2-\beta)}\left(\frac{|a|}{\Gamma(\alpha+\gamma+1)} + \frac{|b|}{\Gamma(\alpha)}\right)\right]$$

$$+ H_1\frac{m|a|}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(2-\beta)\Gamma(\gamma+1)} + H_2\left(\frac{\sum_{i=1}^{m} t_i}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(2-\beta)\Gamma(\gamma+1)} + \frac{m}{\Gamma(2-\beta)}\right)$$

$$\left. + \frac{|c|}{|\frac{a}{\Gamma(\gamma+2)} + b|\Gamma(2-\beta)}\right\}.$$

Hence there exists some $M > 0$ such that $\|x\| \leq M$ for all $x \in V$, i.e., V is a bounded set.

Thus, by Theorem 6, T has at least one fixed point in X. This completes the proof.

4. Examples

Example 11.

$$\begin{cases} {}^{c}D^{\frac{7}{4}}x(t) = f(t,\ x(t),\ {}^{c}D^{\frac{1}{2}}x(t))t \in [0, 1],\ t \neq \frac{1}{2}, \\ \Delta x(\frac{1}{2}) = \dfrac{\left|x(\frac{1}{2}^{-})\right|}{\left|10 + x(\frac{1}{2}^{-})\right|},\ \Delta x'(\frac{1}{2}) = \dfrac{\left|x(\frac{1}{2}^{-})\right|}{20 + \left|x(\frac{1}{2}^{-})\right|}, \\ x(0) = 0,\ I^{\frac{5}{3}}x(1) + \frac{1}{2}x'(1) = 2, \end{cases}$$

where $\alpha = \frac{7}{4}$, $\gamma = \frac{5}{3}$, $\beta = \frac{1}{2}$, $m = 1$, $a = 1$, $b = \frac{1}{2}$, $c = 2$ and

$$f(t,\ x(t),\ {}^{c}D^{\frac{1}{2}}x(t)) = \frac{1}{40(e^{t})(1 + 3|x(t)| + 5|x^{\frac{1}{2}}(t)|)}.$$

Since

$$|f(t,\ x(t),\ {}^{c}D^{\frac{1}{2}}x(t)) - f(t,\ y(t),\ {}^{c}D^{\frac{1}{2}}y(t))|$$

$$\leq \frac{|3|y| + 5|y^{\frac{1}{2}}| - 3|x| - 5|x^{\frac{1}{2}}||}{40(e^{t})(1 + 3|x(t)| + 5|x^{\frac{1}{2}}(t)|)(1 + 3|y(t)| + 5|y^{\frac{1}{2}}(t)|)}.$$

$$= \frac{1}{40}\left(5(|x| - |y|) + 5(|x^{\frac{1}{2}}| - |y^{\frac{1}{2}}|)\right) = \frac{1}{8}\left(|x - y| + |x^{\frac{1}{2}} - y^{\frac{1}{2}}|\right),$$

we can take $k = \frac{1}{8}$. Then since

$$d_1 = 2\left\{\frac{1}{8}\left(\frac{1}{\Gamma(\frac{11}{4})} + \frac{1}{(\frac{1}{\Gamma(\frac{11}{3})} + \frac{1}{2})\Gamma(\frac{53}{12})} + \frac{1}{(\frac{1}{\Gamma(\frac{11}{3})} + \frac{1}{2})\Gamma(\frac{7}{4})}\right)\right.$$

$$\left. + \frac{1}{10}\left(1 + \frac{1}{(\frac{1}{\Gamma(\frac{11}{3})} + \frac{1}{2})\Gamma(\frac{8}{3})}\right) + \frac{1}{20}\left(1 + \frac{1}{2} + \frac{1}{(\frac{1}{\Gamma(\frac{11}{3})} + \frac{1}{2})\Gamma(\frac{8}{3})}\right)\right\} = 0.94 < 1,$$

$$d_2 = 2\left\{\frac{1}{8}\left(\frac{1}{\Gamma(\frac{9}{4})} + \frac{1}{\Gamma(\frac{3}{2})(\frac{1}{\Gamma(\frac{11}{3})} + \frac{1}{2})\Gamma(\frac{53}{12})} + \frac{1}{\Gamma(\frac{9}{4})(\frac{1}{\Gamma(\frac{11}{3})} + \frac{1}{2})\Gamma(\frac{7}{4})}\right)\right.$$

$$\left. + \frac{\frac{1}{10}}{\Gamma(\frac{9}{4})(\frac{1}{\Gamma(\frac{11}{3})} + \frac{1}{2})\Gamma(\frac{8}{3})} + \frac{1}{20}\left(1 + \frac{1}{\Gamma(\frac{9}{4})(\frac{1}{\Gamma(\frac{11}{3})} + \frac{1}{2})\Gamma(\frac{8}{3})}\right)\right\} = 0.751 < 1,$$

and $d = \max\{d_1, d_2\}$, there exists a unique solution by Theorem 7.

Example 12.

$$\begin{cases} {}^cD^{\frac{3}{2}}x(t) = t^5 + e^{-|x^{\frac{1}{2}(t)}|} + \arctan(|x(t)|),\ t \in [0, 1],\ t \neq \frac{1}{3}, \\ \Delta x(\frac{1}{3}) = \dfrac{3\left|x(\frac{1}{3}^-)\right|}{1 + \left|x(\frac{1}{3}^-)\right|},\ \Delta x'(\frac{1}{3}) = 2\cos x(\frac{1}{3}^-) + 3, \\ x(0) = 0,\ I^{\frac{4}{3}}x(1) + \frac{1}{5}x'(1) = -3. \end{cases}$$

Here $|f(t, x(t), {}^cD^{\beta}x(t))| = |t^5 + e^{-|x^{\frac{1}{2}(t)}|} + \arctan(|x(t)|)| \leq 2 + \frac{\pi}{2}$, $t \in [0, 1]$, $x \in X$. Since $|I_k(x) \leq 3|$ and $|J_k(x) \leq 5|$ for $x \in X$, we can take $h(t) = 2 + \frac{\pi}{2}$, $H_1 = 3$ and $H_2 = 5$. Then by Theorem 10, the problem has at least one solution on $[0, 1]$.

5. Conclusion

By using the standard fixed point theorems and nonlinear alternative theorem, we obtained the uniqueness and existence of the solution for nonlinear multi-term fractional differential equations with impulsive and fractional integral boundary conditions. We generalized the results given in [6], [20], [21], [26], [27].

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