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THERE ARE INFINITELY MANY FIBONACCI COMPOSITES WITH PRIME SUBSCRIPTS

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Abstract

From the entire set of natural numbers successively deleting some residue classes mod a prime, we invent a recursive sieve method. This is a modulo algorithm on natural numbers and their sets. The recursively sifting process mechanically yields a sequence of sets, which converges to the set of the certain subscripts of Fibonacci composites. The corresponding cardinal sequence is strictly increasing. Then the well known theory, set valued analysis, allows us to prove that the set of the certain subscripts is an infinite set, namely, the set of Fibonacci composites with prime subscripts is infinite.

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1. Introduction

Fibonacci number F_x is defined by the recursive formula

$$F_1 = 1,$$

$$F_2 = 1,$$

$$F_{x+1} = F_x + F_{x-1}.$$

From the aspect of primality, like the Mersenne numbers, in the Fibonacci sequence there is a conjecture and an open problem.

There are infinitely many Fibonacci primes.

There are infinitely many Fibonacci composites with prime subscripts.

It is well known that the Fibonacci sequence is a divisibility sequence, so we consider Fibonacci composites with prime subscripts.

Today in analytic number theory, by the normal sieve method, like the twin prime conjecture, it is still extremely difficult if not hopeless to solve the above open problem.

But mathematical research often off the beaten path.

In this paper, we solve this open problem by a recursive sieve method.

In 1998, Drobot showed a theorem: if $p > 7$ is a prime such that $p \equiv 2, 4 \mod 5$, and $2p - 1$ is also prime, then $2p - 1 | F_p$ [6], [9].

For example:

p = 19, 37, 79, 97, 139, 157, 199, 229, 307, 337, 367, 379, 439, 499, 547, 577, ….

When p and $2p + 1$ both are prime, the prime p is said to be a Sophie Germain prime.

When p and $2p - 1$ both are prime, the prime p is said to be another Sophie Germain prime.

If we prove that there are infinitely many another Sophie Germain primes of the form $5k + 2$, $5k + 4$, then we prove that there are infinitely many Fibonacci composites with prime subscripts F_p.

In 2011, author used the recursive sieve method, which reveals some exotic structures for various sets of primes, to prove the Sophie Germain prime conjecture [3].

Recently, author solved a similar open problem: there are infinitely many Mersenne composites with prime exponents [5].

Here we extend the above structural result to solve the open problem about Fibonacci composites.

In order to be self-contained we repeat some contents in the paper [3, 4, 5].

2. A Recursive Sieve Method for Fibonacci Composites

For expressing a recursive sieve method by well formed formulas, we extend both basic operations addition and multiplication $+, \times$ into finite sets of natural numbers.

We use small letters a, x, t to denote natural numbers and capital letters A, X, T to denote sets of natural numbers except F_x.

For arbitrary both finite sets of natural numbers A, B, we write

$$A = \langle a_1, a_2, \ldots, a_i, \ldots, a_n \rangle,\ a_1 < a_2 < \cdots < a_i < \cdots < a_n,$$

$$B = \langle b_1, b_2, \ldots, b_j, \ldots, b_m \rangle,\ b_1 < b_2 < \cdots < b_j < \cdots < b_m.$$

We define

$$A + B = \langle a_1 + b_1, a_2 + b_1, \ldots, a_i + b_j, \ldots, a_{n-1} + b_m, a_n + b_m \rangle,$$

$$AB = \langle a_1 b_1, a_2 b_1, \ldots, a_i b_j, \ldots, a_{n-1} b_m, a_n b_m \rangle.$$

Example.

$$\langle 7, 9\rangle + \langle 0, 10, 20\rangle = \langle 7, 9, 17, 19, 27, 29\rangle,$$

$$\langle 10\rangle \langle 0, 1, 2\rangle = \langle 0, 10, 20\rangle.$$

For the empty set $\emptyset$, we define $\emptyset + B = \emptyset$ and $\emptyset B = \emptyset$.

We write $A \setminus B$ for the set difference of A and B.

Let

$$X \equiv A = \langle a_1, a_2, \dots, a_i, \dots, a_n \rangle \quad \bmod a$$

be several residue classes mod a.

If $\gcd(a, b) = 1$, we define the solution of the system of congruences

$$X \equiv A = \langle a_1, a_2, \dots, a_i, \dots, a_n \rangle \quad \bmod a,$$

$$X \equiv B = \langle b_1, b_2, \dots, b_j, \dots, b_m \rangle \quad \bmod b,$$

to be

$$X \equiv D = \langle d_{11}, d_{21}, \dots, d_{ij}, \dots, d_{n-1m}, d_{nm} \rangle \quad \bmod ab,$$

where $x \equiv d_{ij} \bmod ab$ is the solution of the system of congruences

$$x \equiv a_i \quad \bmod a,$$

$$x \equiv b_j \quad \bmod b.$$

The solution $X \equiv D \bmod ab$ is computable and unique by the Chinese remainder theorem.

For example, $X \equiv D = \langle 9, 17, 27, 29\rangle \bmod 30$ is the solution of the system of congruences

$$X \equiv \langle 7, 9\rangle \quad \bmod 10,$$

$$X \equiv \langle 0, 2\rangle \quad \bmod 3.$$

The reader, who is familiar with model theory, know that we found a model and formal system of the second order arithmetic [4]. Here we do not discuss the model and formal system.

From the entire set of natural numbers successively deleting some residue classes modulo a prime, and leave residue classes, we invented a recursive sieve method or modulo algorithm on natural numbers and their sets.

Now we introduce the recursive sieve method for another Sophie Germain primes of the form $5k + 2$, $5k + 4$.

Let p_i be i-th prime, $p_0 = 2$. For every prime p_i, let $B_i \bmod p_i$ be the solution of the congruence

$$x(2x - 1) \equiv 0 \mod p_i.$$

Example.

$$B_0 \equiv \langle 0 \rangle \mod 2,$$

$$B_1 \equiv \langle 0, 2 \rangle \mod 3,$$

$$B_2 \equiv \langle 0, 3 \rangle \mod 5,$$

$$B_3 \equiv \langle 0, 4 \rangle \mod 7,$$

$$B_4 \equiv \langle 0, 6 \rangle \mod 11,$$

$$B_i \equiv \langle 0, (p_i + 1) / 2 \rangle \mod p_i.$$

Let

$$m_0 = 5,$$

$$m_1 = 10,$$

$$m_2 = 30,$$

for all $i > 2$, let

$$m_{i+1} = \prod_{0}^{i} p_j.$$

From the residue class $x \equiv \langle 2, 4 \rangle \bmod 5$ we successively delete the residue classes $B_1 \bmod p_1, \ldots, B_i \bmod p_i$, leave the residue class $T_{i+1} \bmod m_{i+1}$. Then the left residue class $T_{i+1} \bmod m_{i+1}$ is the set of all numbers x of the form $5k + 2, 5k + 4$, such that $x(2x - 1)$ does not contain any prime $p_j \leq p_i$ as a factor $(x(2x - 1), m_{i+1}) = 1$.

Let $X \equiv D_i \bmod m_{i+1}$ be the solution of the system of congruences

$$X \equiv T_i \quad \bmod m_i,$$

$$X \equiv B_i \quad \bmod p_i.$$

Let T_{i+1} be the set of least nonnegative representatives of the left residue class $T_{i+1} \bmod m_{i+1}$.

We obtain a recursive formula for the set T_{i+1}, which describe the recursive sieve method or modulo algorithm for another Sophie Germain primes of the form $5k + 2, 5k + 4$.

$$T_0 = \langle 2, 4 \rangle,$$

$$T_{i+1} = (T_i + \langle m_i \rangle \langle 0, 1, 2, \ldots, p_i - 1 \rangle) \setminus D_i. \tag{2.1}$$

The number of elements of the set T_{i+1} is

$$|T_{i+1}| = 2\prod_{3}^{i} (p_j - 2). \tag{2.2}$$

We exhibit the first few terms of formula (2.1) and briefly prove that the algorithm is valid by mathematical induction.

The residue class $T_0 = \langle 2, 4\rangle \bmod 5$ is the set of all numbers x of the form $5k + 2, 5k + 4$. Now the set $X \equiv \langle 2, 4\rangle \bmod 5$ is equivalent to the set

$$X \equiv (\langle 2, 4\rangle + \langle 5\rangle\langle 0, 1\rangle = \langle 2, 4, 7, 9\rangle \quad \bmod 10,$$

from them we delete the solution of the system of congruences $D_1 = \langle 2, 4\rangle \bmod 10$, and leave

$$T_1 = (\langle 2, 4\rangle + \langle 5\rangle\langle 0, 1\rangle) \setminus \langle 2, 4\rangle = \langle 7, 9\rangle.$$

The residue class T_1 mod 10 is the set of all numbers x of the form $5k + 2, 5k + 4$ such that $(x(2x - 1), 10) = 1$. Now the set $X \equiv \langle 7, 9\rangle \bmod 10$ is equivalent to the set

$$X \equiv (\langle 7, 9\rangle + \langle 10\rangle\langle 0, 1, 2\rangle) = \langle 7, 9, 17, 19, 27, 29\rangle \quad \bmod 30,$$

from them we delete the solution of the system of congruences

$$D_2 = \langle 9, 17, 27, 29\rangle \quad \bmod 30,$$

and leave

$$T_2 = (\langle 7, 9\rangle + \langle 10\rangle\langle 0, 1, 2\rangle) \setminus \langle 9, 17, 27, 29\rangle = \langle 7, 19\rangle.$$

The residue class T_2 mod 30 is the set of all numbers x of the form $5k + 2, 5k + 4$ such that $(x(2x - 1), 30) = 1$.

We delete nothing by the prime 5 from T_2 mod 30. So that let $T_3 = T_2$. The set T_3 mod 30 is equivalent to the set

$$X \equiv \langle 7, 19, 37, 49, 67, 79, 97, 109, 127, 139, 157, 169, 187, 199\rangle \bmod 210.$$

From them we delete

$$D_3 \equiv \langle 7, 49, 67, 109\rangle \quad \bmod 210,$$

and leave

$$T_4 \equiv \langle 19, 37, 79, 97, 139, 127, 157, 169, 187, 199 \rangle \quad \bmod 210.$$

The residue class $T_4 \bmod 210$ is the set of all numbers x of the form $5k + 2, 5k + 4$ such that $(x(2x - 1), 210) = 1$. And so on.

Suppose that the residue class $T_i \bmod m_i$, for $i > 2$ is the set of all numbers x of the form $5k + 2, 5k + 4$ such that $(x(2x - 1), m_i) = 1$. We delete the residue class $B_i \bmod p_i$ from them. In other words, we delete the solution $X \equiv D_i \bmod m_{i+1}$ of the system of congruences

$$X \equiv T_i \quad \bmod m_i,$$

$$X \equiv B_i \quad \bmod p_i.$$

Now the residue class $T_i \bmod m_i$ is equivalent to the residue class

$$(T_i + \langle m_i \rangle \langle 0, 1, 2, \dots, p_i - 1 \rangle) \quad \bmod m_{i+1}.$$

From them we delete the solution $D_i \bmod m_{i+1}$, which is the set of all numbers x of the form $5k + 2, 5k + 4$ such that $x(2x - 1) \equiv 0 \bmod p_i$. It follows that the left residue class $T_{i+1} \bmod m_{i+1}$ is the set of all numbers x of the form $5k + 2, 5k + 4$, and $(x(2x - 1), m_{i+1}) = 1$. Our algorithm is valid. It is easy to compute $|T_{i+1}| = 2|T_i|(p_i - 2)$ for $i > 2$ by the above algorithm.

We may rigorously prove formulas (2.1) and (2.2) by mathematical induction, the proof is left to the reader.

In the next section we refine formula (2.1) and solve the open problem.

3. A Theorem About Fibonacci Composites

We call another Sophie Germain primes $p > 7$ of the form $5k + 2,\ 5k + 4$ S-primes.

Let T_e be the set of all S-primes

$$T_e = \{x : x \text{ is a } S\text{-prime}\}.$$

We shall determine an exotic structure for the set T_e based on the limit of a sequence of sets (T_i'),

$$\lim T_i' = T_e,\ \lim|T_i'| = \aleph_0.$$

Then we prove that the cardinality of the set T_e is infinite by well known theory of those structures,

$$|T_e| = \aleph_0.$$

Based on the recursive algorithm, formula (2.1), we successively delete all numbers x of the form $5k + 2,\ 5k + 4$ such that $x(2x - 1)$ contains the least prime factor p_i. We delete non S-primes or non S-primes together with a S-prime. The sifting condition or 'sieve' is

$$x(2x - 1) \equiv 0 \bmod p_i \wedge p_i \le x.$$

For $p_i > 7$ we modify the sifting condition to be

$$x(2x - 1) \equiv 0 \bmod p_i \wedge p_i < x. \tag{3.1}$$

According to this new sifting condition or 'sieve', we successively delete the set C_i of all numbers x, such that either x or $2x - 1$ is composite with the least prime factor p_i.

For $p_i \le 7$,

$$C_i = \{x : x \in X \equiv T_i \quad \bmod m_i \wedge x(2x - 1) \equiv 0 \quad \bmod p_i \wedge p_i \le x\}.$$

For $p_i > 7$,

$$C_i = \{x : x \in X \equiv T_i \mod m_i \wedge x(2x-1) \equiv 0 \mod p_i \wedge p_i < x\},$$

but remain the S-prime x if there is a $x > 7$ such that $p_i = x$ in $T_i \bmod m_i$.

Note: If $p = 3, 7,$ then $2p - 1 = 5, 13$ are Fibonacci primes F_5, F_7, both are deleted.

We delete all sets C_j with $0 \le j < i$ from the set N_0 of all natural numbers x of the form $5k + 2, 5k + 4,$ and leave the set

$$L_i = N_0 \setminus \bigcup_0^{i-1} C_j.$$

The set of all S-primes is

$$T_e = N_0 \setminus \bigcup_0^{\infty} C_i.$$

The recursive sieve (3.1) is a perfect tool, with this tool we delete all non S-primes and leave all S-primes. So that we only need to determine the number of all S-primes $|T_e|$. If we do so successfully, then the parity obstruction, a ghost in house of primes, has been automatically evaporated [8].

With the recursive sieve (3.1), each non S-prime is deleted exactly once, there is need neither the inclusion-exclusion principle nor the estimation of error terms, which cause all the difficulty in normal sieve theory.

Let A_i be the set of all S-primes x less than p_i.

$$A_i = \{x : x < p_i \wedge x \text{ is } S\text{-prime}\}.$$

From the recursive formula (2.1), we know that the left set L_i is the union of the set A_i of S-primes and the residue class T_i mod m_i,

$$L_i = A_i \bigcup T_i \quad \mod m_i. \tag{3.2}$$

Now we intercept the initial segment T_i' from the left set L_i, which is the union of the set A_i of S-primes and the set T_i of least nonnegative representatives. Then we obtain a new recursive formula

$$T_i' = A_i \bigcup T_i. \tag{3.3}$$

Except remaining all S-primes x less than p_i in the initial segment T_i', both sets T_i' and T_i are the same.

Formula (3.3) expresses the recursively sifting process according to the sifting condition (3.1), and provides a recursive definition of the initial segment T_i'. The initial segment T_i' is a well chosen notation. We shall consider some properties of the initial segment T_i', and reveal some structures of the sequence of the initial segments (T_i') to determine the set of all S-primes and its cardinality.

Let $|A_i|$ be the number of S-primes less than p_i. Then the number of elements of the initial segment T_i' is

$$|T_i'| = |A_i| + |T_i|. \tag{3.4}$$

From formula (2.2), we deduce that the cardinal sequence $(|T_i'|)$ is strictly increasing for all $i > 2$

$$|T_i'| < |T_{i+1}'|. \tag{3.5}$$

Based on order topology obviously we have

$$\lim |T_i'| = \aleph_0. \tag{3.6}$$

Intuitively we see that the initial segment T_i' approaches the set of all S-primes T_e, and the corresponding cardinality $|T_i'|$ approaches infinity as $i \to \infty$. Thus the set of all S-primes is limit computable and is an infinite set.

Next we give a formal proof based on set valued analysis.

3.1. A formal proof. Let A_i' be the subset of all S-primes in the initial segment T_i',

$$A_i' = \{x \in T_i' : x \text{ is } S\text{-prime}\}. \tag{3.7}$$

We consider the structures of both sequences of sets (T_i') and (A_i') to solve the open problem.

Lemma 3.1. *The sequence of the initial segments (T_i') and the sequence of its subsets (A_i') of S-primes both converge to the set of all S-primes T_e.*

First from set theory, next from order topology we prove this lemma.

Proof. For the convenience of the reader, we quote a definition of the set theoretic limit of a sequence of sets [2].

Let (F_n) be a sequence of sets, we define $\limsup_{n=\infty} F_n$ and $\liminf_{n=\infty} F_n$ as follows:

$$\limsup_{n=\infty} F_n = \bigcap_{n=0}^{\infty}\bigcup_{i=0}^{\infty} F_{n+i},$$

$$\liminf_{n=\infty} F_n = \bigcup_{n=0}^{\infty}\bigcap_{i=0}^{\infty} F_{n+i}.$$

It is easy to check that $\limsup_{n=\infty} F_n$ is the set of those elements x, which belongs to F_n for infinitely many n. Analogously, x belongs to

$\liminf_{n=\infty} F_n$ if and only if it belongs to F_n for almost all n, that is it belongs to all but a finite number of the F_n.

If

$$\limsup_{n=\infty} F_n = \liminf_{n=\infty} F_n,$$

we say that the sequence of sets (F_n) converges to the limit

$$\lim F_n = \limsup_{n=\infty} F_n = \liminf_{n=\infty} F_n.$$

We know that the sequence of left sets (L_i) is descending

$$L_1 \supset L_2 \supset \cdots \supset L_i \supset \cdots\cdots.$$

According to the definition of the set theoretic limit of a sequence of sets, we obtain that the sequence of left sets (L_i) converges to the set T_e

$$\lim L_i = \bigcap L_i = T_e. \tag{3.8}$$

The sequence of subsets (A_i') of S-primes is ascending

$$A_1' \subset A_2' \subset \cdots \subset A_i' \subset \cdots\cdots,$$

we obtain that the sequence of subsets (A_i') converges to the set T_e,

$$\lim A_i' = \bigcup A_i' = T_e. \tag{3.9}$$

The initial segment T_i' is located between two sets A_i' and L_i

$$A_i' \subset T_i' \subset L_i.$$

Thus the sequence of the initial segments (T_i') converges to the set T_e

$$\lim T_i' = T_e. \tag{3.10}$$

According to set theory, we have proved that both sequences of sets (T_i') and (A_i') converge to the set of all S-primes T_e

$$\lim T_i' = \lim A_i' = T_e. \tag{3.11}$$

Next we prove that according to order topology both sequences of sets (T_i') and (A_i') converge to the set of all S-primes T_e.

We quote a definition of the order topology [1].

Let X be a set with a linear order relation; assume X has more one element. Let $\mathbb{B}$ be the collection of all sets of the following types:

(1) All open intervals (a, b) in X.

(2) All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) in X.

(3) All intervals of the form $[a, b_0)$, where b_0 is the largest element (if any) in X.

The collection $\mathbb{B}$ is a bases of a topology on X, which is called the order topology.

According to the definition there is no order topology on the empty set or sets with a single element.

The recursively sifting process, formula (3.3), produces both sequences of sets together with the set theoretic limit point T_e.

$$\mathbf{X_1} : T_1', T_2', \dots, T_i', \dots\dots; T_e,$$

$$\mathbf{X_2} : A_1', A_2', \dots, A_i', \dots\dots; T_e.$$

We further consider the structures of sets $\mathbf{X_1}$ and $\mathbf{X_2}$ using the recursively sifting process (3.3) as an order relation

$$i < j \to T_i' < T_j', \forall i(T_i' < T_e),$$

$$i < j \to A_i' < A_j', \forall i(A_i' < T_e).$$

The set $\mathbf{X_1}$ has no repeated term. It is a well ordered set with the order type $\omega + 1$ using the recursively sifting process (3.3) as an order relation. Thus the set $\mathbf{X_1}$ may be endowed an order topology.

The set $\mathbf{X_2}$ may have some repeated terms. We have computed out the first few S-primes x. The set $\mathbf{X_2}$ contains more than one element, may be endowed an order topology using the recursively sifting process (3.3) as an order relation.

Obviously, for every neighbourhood $(c, T_e]$ of T_e there is a natural number i_0, for all $i > i_0$, we have $T_i' \in (c, T_e]$ and $A_i' \in (c, T_e]$, thus both sequences of sets (T_i') and (A_i') converge to the set of all S-primes T_e.

$$\lim A_i' = T_e,$$

$$\lim T_i' = T_e.$$

According to the order topology, we have again proved that both sequences of sets (T_i') and (A_i') converge to the set of all S-primes T_e. We also have

$$\lim T_i' = \lim A_i'. \tag{3.12}$$

The formula T_i' is a recursive asymptotic formula for the set of all S-primes T_e. □

In general, if $T_e = \emptyset$, the set $\boldsymbol{X_2} = \{\emptyset\}$ only has a single element, which has no order topology. In this case formula (3.12) is not valid and our method of proof may be useless [3].

Lemma 3.1 reveals an order topological structure and a set theoretic structure for the set of all S-primes on the recursive sequences of sets. By the well known theory of those structures, we easily prove that the cardinality of the set of all S-primes is infinite.

Theorem 3.2. *The set of all S-primes is an infinite set.*

We give two proofs.

Proof. We consider the cardinalities $|T_i'|$ and $|A_i'|$ of sets on two sides of the equality (3.12), and the order topological limits of cardinal sequences $(|T_i'|)$ and $(|A_i'|)$ as the sets T_i' and A_i' both tend to T_e.

From general topology we know, if the limits of both cardinal sequences $(|T_i'|)$ and $(|A_i'|)$ on two sides of the equality (3.12) exist, then both limits are equal; if lim $|A_i'|$ does not exist, then the condition for the existence of the limit lim $|T_i'|$ is not sufficient [7].

For S-primes, the set T_e is nonempty $T_e \neq \emptyset$, the formula (3,12) is valid, obviously the order topological limits lim $|A_i'|$ and lim $|T_i'|$ on two sides of the equality (3.12) exist, thus both limits are equal

$$\lim |A_i'| = \lim |T_i'|.$$

From formula (3.6) lim $|T_i'| = \aleph_0$, we have

$$\lim |A_i'| = \aleph_0. \tag{3.13}$$

Usually, let $\pi_2(n)$ be the counting function, the number of S-primes less than or equal to n. Normal sieve theory is unable to provide non-trivial lower bounds of $\pi_2(n)$ by the parity obstruction [8]. Let n be a natural number. Then the number sequence (m_i) is a subsequence of the number sequence (n), we have

$$\lim \pi_2(n) = \lim \pi_2(m_i).$$

By formula (3.7), the A_i' is the set of all S-primes less than m_i, and the $|A_i'|$ is the number of all S-primes less than m_i, thus $\pi_2(m_i) = |A_i'|$. We have

$$\lim \pi_2(m_i) = \lim |A_i'|.$$

From formula (3.13), we prove

$$\lim \pi_2(n) = \aleph_0. \tag{3.14}$$

□

We directly prove that the number of all S-primes is infinite with the counting function. Next we give another proof by the continuity of the cardinal function.

Proof. Let $f : \mathbf{X} \to \mathbf{Y}$ be the cardinal function $f(T) = |T|$ from the order topological space $\boldsymbol{X}$ to the order topological space $\mathbf{Y}$

$$\mathbf{X} : T_1', T_2', \ldots, T_i', \ldots\ldots; T_e,$$

$$\mathbf{Y} : |T_1'|, |T_2'|, \ldots, |T_i'|, \ldots\ldots; \aleph_0.$$

It is easy to check that for every open set $[|T_1'|, |d|), (|c|, |d|), (|c|, \aleph_0]$ in $\mathbf{Y}$ the preimage $[T_1', d), (c, d), (c, T_e]$ is also an open set in $\mathbf{X}$. So that the cardinal function $|T|$ is continuous at T_e with respect to the above order topology.

Both order topological spaces are first countable, hence the cardinal function $|T|$ is sequentially continuous. By a usual topological theorem [1] (Theorem 21.3, p. 130), the cardinal function $|T|$ preserves limits

$$|\lim T_i'| = \lim |T_i'|. \tag{3.15}$$

Order topological spaces are Hausdorff spaces. In Hausdorff spaces, the limit point of the sequence of sets (T_i') and the limit point of cardinal sequence $(|T_i'|)$ are unique.

We have proved Lemma 3.1, $\lim T_i' = T_e$, and formula (3.6), $\lim |T_i'| = \aleph_0$. Substitute, we obtain that the set of all S-primes is an infinite set,

$$|T_e| = \aleph_0. \tag{3.16}$$

Without any estimation or statistical data, without the Riemann hypothesis, by the recursive algorithm, we well understand the recursive structure, set theoretic structure and order topological structure for the set of all S-primes on sequences of sets. We obtain a formal proof of the open problem in pure mathematics. □

By Drobot's theorem we have solved the open problem about Fibonacci composites.

Theorem 3.3. *There are infinitely many Fibonacci composite numbers with prime subscripts.*

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