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VOLUME INEQUALITIES FOR GENERAL L_p ZONOIDS OF EVEN ISOTROPIC MEASURES

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Abstract

The volume inequalities for general L_p zonoids of even isotropic measures and for their duals are strengthened by Ball et al.. Motivated by their way, a stronger version of the Brascamp-Lieb inequality for a family of functions is proved, which can approximate arbitrary well some Gaussians when equality holds. Its application gives the L_p Loomis-Whitney inequality for even isotropic measures associated with the support function of L_p projection bodies with complete equality conditions. Moreover, we establish a dual version of the Loomis-Whitney inequality for isotropic measures with complete equality conditions, in which we give the sharp lower bound for the volumes of hyperplane sections. This extends Ball's Loomis-Whitney inequality and dual Ball's Loomis-Whitney inequality to the L_p space, respectively.

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1. Introduction

Suppose that S^{n-1} is the Euclidean unit sphere. By John's theorem [38] (see also Ball [1]), B^n is the ellipsoid of maximal volume in an origin symmetric convex body K if and only if $B^n \subseteq K$ and there exist $\pm u_1, \cdots,$ $\pm u_k \in S^{n-1} \bigcap \partial K$ and $c_1, \cdots, c_k > 0\,(k \geq n)$ such that

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = \mathrm{Id}_n. \tag{1.1}$$

Here $\otimes$ is the tensor product of vectors in $\mathbb{R}^n$, Id_n is the $n \times n$ identity matrix and ∂K is the boundary of K.

According to Giannopoulos and Papadimitrakis [31] as well as Lutwak et al. [52], one call an even Borel measure μ on the unit sphere S^{n-1} isotropic if

$$\int_{S^{n-1}} u \otimes u d\mu(u) = \mathrm{Id}_n. \tag{1.2}$$

In this case, equating traces of both sides we obtain that

$$\mu(S^{n-1}) = n. \tag{1.3}$$

The support function h_K of a convex compact set K in $\mathbb{R}^n$ at $v \in \mathbb{R}^n$ is defined by

$$h_K(v) = \max\{\langle v, x\rangle : x \in K\},$$

where $\langle \cdot, \cdot \rangle$ for the Euclidean scalar product and $\| \cdot \|$ is the induced norm in $\mathbb{R}^n$.

Denote by B_p^n the unit ball of l_p^n-space, that is,

$$B_p^n = \left\{ x \in \mathbb{R}^n : \left(\sum_{i=1}^{n} |\langle x, e_i \rangle|^p \right)^{\frac{1}{p}} \leq 1 \right\}, 1 \leq p < \infty,$$

and

$$B_\infty^n = \{x \in \mathbb{R}^n : |\langle x, e_i \rangle| \leq 1, \text{ for all } i = 1, \cdots, n\}, \ p = \infty,$$

where $\{e_1, \cdots, e_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$.

Schneider and Weil [68] introduced the notion of L_p zonoids which is an kernel ingredient in the L_p Brunn-Minkowski theory. Suppose $p \geq 1$ and μ is an even Borel measure on S^{n-1} such that its support, $\operatorname{supp} \mu$, is not contained in a subsphere of S^{n-1}. The L_p zonoid $Z_p := Z_p(\mu)$ related to μ is the origin-symmetric convex body whose representation is

$$h_{Z_p(\mu)}(v)^p = \int_{S^{n-1}} |\langle u, v \rangle|^p \, d\mu(u), \quad v \in S^{n-1},$$

if $p = 1$, this just is the classical zonoid. We let

$$Z_\infty(\mu) = \lim_{p \to \infty} Z_p(\mu) = \sup_{v \in \operatorname{supp}\mu} |\langle u, v \rangle|,$$

and for $1 \leq p \leq \infty$, let $Z_p^*(\mu)$ be the polar of $Z_p(\mu)$, namely,

$$Z_p^*(\mu) = \left\{ x \in \mathbb{R}^n : \int_{S^{n-1}} |\langle x, u \rangle|^p \, d\mu(u) \leq 1 \right\} \quad \text{for } p \in [1, \infty),$$

$$Z_\infty^*(\mu) = \{x \in \mathbb{R}^n : \langle x, u \rangle \leq 1 \text{ for } u \in \operatorname{supp}\mu\} \quad \text{for } p = \infty.$$

Then $Z_2(\mu) = B^n$ for any even isotropic measure μ. Obviously,

$$Z_p(\nu_n) = (B_p^n)^* = B_{p^*}^n. \tag{1.4}$$

For some isotropic measure μ on S^{n-1}, any n-dimensional subspace of L_p is isometric to $\|\cdot\|_{Z_p^*(\mu)}$ (see, e.g., [44, 51, 53]), where

$$\|x\|_{Z_p^*(\mu)} = \left(\int_{S^{n-1}} |\langle x, u\rangle|^p \, d\mu(u)\right)^{\frac{1}{p}}, \quad x \in \mathbb{R}^n.$$

Note that the Minkowski functional of a body K

$$\|x\|_K = \min\{\lambda \geq 0 : x \in \lambda K\}, \quad x \in \mathbb{R}^n.$$

We call a cross measure ν on S^{n-1} if there is an orthonormal basis $u_1, \cdots, u_n$ of $\mathbb{R}^n$ such that

$$\text{supp}\nu = \{\pm u_1, \cdots, \pm u_n\} = O\{\pm e_1, \cdots, \pm e_n\},$$

for some $O \in O(n)$. Since $\nu(\{u_i\}) = \nu(\{-u_i\}) = 1/2$ for $i = 1, \cdots, n$, ν is even and isotropic. If we fix a cross measure ν_n on S^{n-1}, note that $p \in [1, \infty]$ and $\Gamma(\cdot)$ is Euler's Gamma function, then

$$V(Z_p(\nu_n)) = \begin{cases} \dfrac{\Gamma(1+\frac{n}{2})\Gamma(1+\frac{p}{2})}{\Gamma(1+\frac{1}{2})\Gamma(1+\frac{n+p}{2})}, & \text{if } p \geq 1, \\ \dfrac{2^n}{n!}, & \text{if } p = \infty. \end{cases} \tag{1.5}$$

In addition,

$$V(Z_p^*(\nu_n)) = \begin{cases} \dfrac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})}, & \text{if } p \geq 1, \\ 2^n, & \text{if } p = \infty. \end{cases} \tag{1.6}$$

$Z_\infty^*(\mu)$ plays a crucial role in the reverse isoperimetric inequality.

Theorem 1.1. *If* μ *is an even isotropic measure on* S^{n-1}, *and* $p \in [1, \infty]$, *then*

$$V(Z_p(\mu)) \geq V(Z_p(\nu_n)),$$

$$V(Z_p^*(\mu)) \leq V(Z_p^*(\nu_n)).$$

Assuming $p \neq 2$, *equality holds if and only if* μ *is a cross measure.*

Theorem 1.1 is the work of Ball [1] and Barthe [8] if μ is discrete, which was extended to arbitrary even isotropic measures μ by Lutwak et al. [51]. The measures on S^{n-1} with an isotropic linear image are characterized by Böröczky et al. [16]. It is well known that isotropic measures on $\mathbb{R}^n$ play a central role in the KLS conjecture by Kannan et al. [40], see also, e.g., Barthe and Cordero-Erausquin [9], Guedon and Milman [32] and Klartag [41]. In particular, the following issues are obtained by Li et al. [48] in a recent work: The L_p cosine transform on Grassmann manifolds induces finite dimensional Banach norms whose unit balls are origin-symmetric convex bodies in $\mathbb{R}^n$, and they further established the reverse isoperimetric type volume inequalities for these bodies, which extend the results from the sphere to Grassmann manifolds.

A natural notion of distance between two isotropic measures μ and ν is the Wasserstein distance $\delta_W(\mu, \nu)$ which is also called the Kantorovich-Monge-Rubinstein distance. In order to give gits definition, let $\angle(v, w)$ be the angle between non-zero vectors v and w; that is, the geodesic distance of the unit vectors $\|v\|^{-1}v$ and $\|w\|^{-1}w$ on the unit sphere. Suppose that $\mathrm{Lip}_1(S^{n-1})$ is the family of Lipschitz functions with Lipschitz constant at most 1; namely, $f : S^{n-1 \to \mathbb{R}}$ is in $\mathrm{Lip}_1(S^{n-1})$ if

$\|f(x) - f(y)\| \leq \angle(x, y)$ for $x, y \in S^{n-1}$. Then the Wasserstein distance of μ and ν is defined by

$$\delta_W(\mu, \nu) = \max\left\{\int_{S^{n-1}} fd\mu - fd\nu : f \in \mathrm{Lip}_1(S^{n-1})\right\}.$$

In fact, in this paper, we need the Wasserstein distance of an isotropic measure μ from the closest cross measure. Therefore, in the case of two isotropic measures μ and ν, we define

$$\delta_{WO}(\mu, \nu) = \min\{\delta_W(\mu, \Phi_*\nu) : \Phi \in O(n)\},$$

where $\Phi_*\nu$ denotes the push forward of ν by $\Phi : S^{n-1} \to S^{n-1}$.

Recently, a stability version of Theorem 1.1 was established by Böröczky et al. [17] as follows:

Theorem 1.2 (see [17]). *Let* μ *be an even isotropic measure on* S^{n-1}, $n \geq 2$, *and let* $p \in [1, \infty]$ *with* $p \neq 2$. *If* $\delta_{WO}(\mu, \nu_n) \geq \varepsilon > 0$ *for* $\varepsilon \in [0, 1)$, *then*

$$V(Z_p(\mu)) \geq (1 + \gamma\varepsilon^3)V(Z_p(\nu_n)),$$

$$V(Z_p^*(\mu)) \leq (1 - \gamma\varepsilon^3)V(Z_p^*(\nu_n)),$$

where $\gamma = n^{-cn^3}\min\{|p-2|^2, 1\}$ *for an absolute constant* $c > 0$.

The notion of the generalized l_p^n-ball $B_{p,\alpha}^n := B_{p,\alpha}^n(\nu_n)$ formed by ν_n is defined by (see [46])

$$B_{p,\alpha}^n = \left\{x \in \mathbb{R}^n : \left(\sum_{i=1}^n |\langle x, u_i\rangle|^p \alpha(u_i)\right)^{\frac{1}{p}} \leq 1\right\}, \quad 1 \leq p < \infty, \tag{1.7}$$

and

$$B_{\infty,\alpha}^n = \{x \in \mathbb{R}^n : |\langle x, u_i\rangle| \leq 1, \text{ for all } i = 1, \cdots, n\}, \quad p = \infty,$$

where $\alpha(u_i) > 0$, $i = 1, \cdots, n$, and ν_n is a cross measure on S^{n-1} such that

$$\operatorname{supp}\nu_n = \{\pm u_1, \cdots, \pm u_n\} = O\{\pm e_1, \cdots, \pm e_n\},$$

for some $O \in O(n)$.

Suppose that μ is an even isotropic measure on S^{n-1} and $\alpha : S^{n-1} \to (0, +\infty)$ is an even positive continuous function. Much recently, Li and Huang [46] defined the "general" L_p zonoid $Z_{p,\alpha} := Z_{p,\alpha}(\mu)$ with parametric variables to be the origin-symmetric convex body whose support function is given, for each $x \in \mathbb{R}^n$, by

$$h_{Z_{p,\alpha}(\mu)}(x) = \left(\int_{S^{n-1}} |\langle x, u\rangle|^p \alpha(u) d\mu(u)\right)^{\frac{1}{p}}, \; p \in [1, \infty), \tag{1.8}$$

and

$$h_{Z_{\infty,\alpha}(\mu)}(x) = \lim_{p\to\infty} h_{Z_{p,\alpha}(\mu)}(x) = \sup_{u\in\operatorname{supp}\mu} |\langle x, u\rangle|, \; p = \infty.$$

If ν is a cross measure such that $\operatorname{supp}\nu = \{\pm u_1, \cdots, \pm u_n\}$, we have

$$h_{Z_{p,\alpha}(\nu)}(x) = \left(\sum_{i=1}^{n} |\langle x, u_i\rangle|^p \alpha(u_i)\right)^{\frac{1}{p}} = h_{(B^n_{p,\alpha})^*}(x) = h_{B^n_{p^*,1/\alpha}}(x), \tag{1.9}$$

for each $x \in \mathbb{R}^n$. For $1 \le p \le \infty$, let $Z^*_{p,\alpha}(\mu)$ be the polar of $Z_{p,\alpha}(\mu)$; i.e.,

$$Z^*_{p,\alpha}(\mu) = \left\{x \in \mathbb{R}^n : \int_{S^{n-1}} |\langle x, u\rangle|^p \alpha(u) d\mu(u) \le 1\right\} \text{ for } p \in [1, \infty),$$

$$Z^*_{\infty,\alpha}(\mu) = \{x \in \mathbb{R}^n : |\langle x, u\rangle| \le 1 \text{ for } u \in \operatorname{supp}\mu\} \text{ for } p = \infty.$$

For $p \in [1, \infty]$ and each $x \in \mathbb{R}^n$, the origin-symmetric star body $Z^*_{p,\alpha}(\mu)$ in $\mathbb{R}^n$ is defined by

$$\|x\|_{Z^*_{p,\alpha}(\mu)} = \left(\int_{S^{n-1}} |\langle x, u\rangle|^p \alpha(u) d\mu(u)\right)^{\frac{1}{p}}, 1 \leq p \leq \infty, \tag{1.10}$$

and

$$\|x\|_{Z^*_{\infty,\alpha}} = \sup_{u \in \mathrm{supp}\mu} |\langle x, u\rangle|, \ p = \infty. \tag{1.11}$$

Note that (1.10) and (1.11) tell us that any n-dimensional subspace of L_p is isometric to $\|\cdot\|_{Z^*_{p,\alpha}(\mu)}$ for some isotropic measure μ on S^{n-1}.

In [46], we know that

$$h_{Z_{p,\alpha}(\nu_n)}(x) = \left(\sum_{i=1}^{n} |\langle x, u_i\rangle|^p \alpha(u_i)\right)^{\frac{1}{p}} = h_{(B^n_{p,\alpha})^*}(x) = h_{B^n_{p^*, 1/\alpha}}(x), \tag{1.12}$$

and

$$B^n_{p,\alpha} = OA^{-1}B^n_p, \tag{1.13}$$

where $O \in O(n)$ and $A = \mathrm{diag}\{\alpha(u_1)^{1/p}, \cdots, \alpha(u_n)^{1/p}\}$. The volume of the generalized l^n_p-ball $B^n_{p,\alpha}$ and polar body $(B^n_{p,\alpha})^*$ in [46] are given by

$$V(B^n_{p,\alpha}) = V(B^n_p)\left(\prod_{i=1}^{n} \alpha(u_i)\right)^{-\frac{1}{p}} = V(Z^*_p(\nu_n))\left(\prod_{i=1}^{n} \alpha(u_i)\right)^{-\frac{1}{p}}, \tag{1.14}$$

and

$$V\left((B^n_{p,\alpha})^*\right) = V(B^n_{p^*})\left(\prod_{i=1}^{n} \alpha(u_i)\right)^{\frac{1}{p}} = V(Z_p(\nu_n))\left(\prod_{i=1}^{n} \alpha(u_i)\right)^{\frac{1}{p}}. \tag{1.15}$$

If the unit vectors $(u_i)_{i=1}^k (k \geq n)$ and positive numbers $(c_i)_{i=1}^k$ are satisfied John's condition

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = \mathrm{Id}_n,$$

then the notion of the generalized k-th l_p^n-unit ball $B_{p,\alpha}^{n,k} := B_{p,\alpha}^{n,k}(\nu)$ formed by ν is defined by

$$B_{p,\alpha}^{n,k} = \left\{ x \in \mathbb{R}^n : \left(\sum_{i=1}^{k} c_i |\langle x, u_i \rangle|^p \alpha(u_i) \right)^{\frac{1}{p}} \leq 1 \right\}, \quad 1 \leq p < \infty. \tag{1.16}$$

The main purpose of this article is to generalize the above Theorems 1.1 and 1.2 to the more general situations of "general" L_p zonoid $Z_{p,\alpha}(\mu)$ with parametric variables $\alpha(u)$. The following is our main results.

Theorem 1.3. *If* μ *is an even isotropic measure on* S^{n-1} *and* $p \in [1, \infty]$, *and let* $\alpha : S^{n-1} \to (0, +\infty)$ *be an even positive continuous function. Let* $k \geq n$, *if there are unit vectors* $(u_i)_{i=1}^k$ *and positive numbers* $(c_i)_{i=1}^k$ *satisfying John's condition*

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = \mathrm{Id}_n, \tag{1.17}$$

then

$$V(Z_{p,\alpha}(\mu)) \geq V(Z_p(\nu_n)) \left(\prod_{i=1}^{k} \alpha(u_i)^{c_i} \right)^{\frac{1}{p}}, \tag{1.18}$$

$$V(Z_{p,\alpha}^*(\mu)) \leq V(Z_p^*(\nu_n)) \left(\prod_{i=1}^{k} \alpha(u_i)^{c_i} \right)^{-\frac{1}{p}}. \tag{1.19}$$

If equality holds in (1.18) *and* (1.19), *then there is an origin symmetric regular crosspolytope in* $\mathbb{R}^n$ *such that* $u_1, \cdots, u_k$ *lie among its vertices. Conversely, equality holds in* (1.18) *and* (1.19) *if* $k = n$ *and* $u_1, \cdots, u_n$ *form an orthonormal basis of* $\mathbb{R}^n$.

When $k = n$ and taking $u_i = e_i$ with $c_i = 1$ for all $i = 1, \cdots, n,$ we see the following fact.

Corolloary 1.4. *If* μ *is an even isotropic measure on* S^{n-1} *and* $p \in [1, \infty]$, *then*

$$V(Z_{p,\alpha}(\mu)) \geq V((B^n_{p,\alpha})^*), \tag{1.20}$$

$$V(Z^*_{p,\alpha}(\mu)) \leq V(B^n_{p,\alpha}). \tag{1.21}$$

Assuming $p \neq 2$, *equality holds in* (1.20) *and* (1.21) *if and only if* μ *is a cross measure.*

Theorem 1.5. *Let* μ *be an even isotropic measure on* S^{n-1}, $n \geq 2$, *and let* $\alpha : S^{n-1} \to (0, +\infty)$ *be an even positive bounded continuous function, and* $p \in [1, \infty]$ *with* $p \neq 2$. *If* $\delta_{WO}(\mu, \nu_n) \geq \varepsilon > 0$ *for* $\varepsilon \in [0, 1)$, *and there are unit vectors* $(u_i)_{i=1}^k$ *as well as positive numbers* $(c_i)_{i=1}^k$ *satisfying John's condition* (1.17), *then*

$$V(Z_{p,\alpha}(\mu)) \geq (1 + \gamma\varepsilon^3)V(Z_p(\nu_n))\left(\prod_{i=1}^{k}\alpha(u_i)^{c_i}\right)^{\frac{1}{p}}, \tag{1.22}$$

$$V(Z^*_{p,\alpha}(\mu)) \leq (1 - \gamma\varepsilon^3)V(Z^*_p(\nu_n))\left(\prod_{i=1}^{n}\alpha(u_i)^{c_i}\right)^{-\frac{1}{p}}, \tag{1.23}$$

where $\gamma = n^{-cn^3}\min\{|p - 2|^2, 1\}$ *for an absolute constant* $c > 0$.

Letting $k = n$ and taking $u_i = e_i$ with $c_i = 1$ for all $i = 1, \cdots, n$, the following is a direct result.

Corolloary 1.6. *Let* μ *be an even isotropic measure on* S^{n-1}, $n \geq 2$, *and let* $\alpha : S^{n-1} \to (0, +\infty)$ *be an even positive bounded continuous function, and* $p \in [1, \infty]$ *with* $p \neq 2$. *If* $\delta_{WO}(\mu, \nu_n) \geq \varepsilon > 0$ *for* $\varepsilon \in [0, 1)$, *then*

$$V(Z_{p,\alpha}(\mu)) \geq (1 + \gamma\varepsilon^3)V((B^n_{p,\alpha}(\nu_n))^*), \tag{1.24}$$

$$V(Z^*_{p,\alpha}(\mu)) \leq (1 - \gamma\varepsilon^3)V(B^n_{p,\alpha}(\nu_n)), \tag{1.25}$$

where $\gamma = n^{-cn^3} \min\{|p - 2|^2, 1\}$ *for an absolute constant* $c > 0$.

The ideas and techniques of Ball [1], Barthe [6], Lutwak et al. [51], Böröczky et al. [16] and especially Böröczky et al. [17] play a critical role throughout this paper. It would be impossible to overstate our reliance on their work.

In order to obtain a generalized inequality of Theorem 1.1 and a stability version of Theorem 1.5, we need some basic concepts and facts, and also need some analytic inequalities such as the estimates of the derivatives of the corresponding transportation maps established in Section 4, a basic algebraic inequality provided in Section 4 of [15] and Aczel's inequality [43].

The rest of this paper is organized as follows: In Section 2, the background materials are provided. The Proof of Theorem 1.3 are completed in Section 3. In Sections 5 and 6, we deal with Theorem 1.5. Section 7 is dedicated to prove the L_p Loomis-Whitney and reverse L_p Loomis-Whitney inequalities.

2. Background Materials

2.1. Some definitions and notations

For quick later reference we recall some background materials. The excellent references are the books by Gardner [27] and Schneider [67].

Let $\mathcal{K}_o^n$ denote the space of convex bodies in $\mathbb{R}^n$ equipped with the Hausdorff metric. For $K \in \mathcal{K}_o^n$ and $A \in GL(n)$, we write $AK = \{Ax : x \in K\}$ for the image of K under A. If $\lambda > 0$, then $\lambda K = \{\lambda x : x \in K\}$ is the dilation of K by a factor of λ. The polar body K^* of K is defined by

$$K^* = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } y \in K\}.$$

It follows from the definition of the polar K^* of K that for $A \in GL(n)$, $(AK)^* = A^{-t}K^*$, where A^{-t} is the inverse and transpose of A.

The Minkowski functional $\|x\|_K$ of $K \in \mathcal{K}_o^n$ (or K is a star body with respect to the origin) is defined by

$$\|x\|_K = \min\{t > 0 : x \in tK\},$$

for $x \in \mathbb{R}^n$. It is easy to verify that $\|\cdot\|_K = h_{K^*}(\cdot)$.

We need some facts from the L_p Brunn-Minkowski theory of convex bodies. Firey [25] introduced the concept of L_p combinations of convex bodies in the early 1960s, which emerge the new theory. These L_p Minkowski-Firey combinations were shown to lead to an embryonic L_p Brunn-Minkowski theory in the works of Lutwak [54, 55]. This theory has witnessed a rapid growth. The detailed bibliography on the topic we refer the reader to Chapter 9 of [67] and the references therein.

For $p \geq 1$, $K, L \in \mathcal{K}_o^n$, and $\varepsilon > 0$, the L_p Minkowski-Firey combination $K +_p \varepsilon \cdot L$ is the convex body whose support function is given by

$$h^p_{K+_p\varepsilon\cdot L}(\cdot) = h^p_K(\cdot) + \varepsilon h^p_L(\cdot).$$

The L_p mixed volume $V_p(K, L)$ of $K, L \in \mathcal{K}_o^n$ was defined in [54] by

$$V_p(K, L) = \frac{p}{n} \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

In particular, $V_p(K, K) = V(K)$. The L_p Minkowski inequality [54] states that for $K, L \in \mathcal{K}_o^n$,

$$V_p(K, L)^n \geq V(K)^{n-p} V(L)^p, \tag{2.1}$$

with equality if and only if K and L are dilates when $p > 1$ and if and only if K and L are homothetic when $p = 1$.

It was shown in [54] that there is a positive Borel measure, $S_p(K, \cdot)$, on S^{n-1} so that

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h^p_L(u) dS_p(K, u), \tag{2.2}$$

for $K, L \in \mathcal{K}_o^n$, where $dS_p(K, \cdot) = h_K^{1-p} dS_K(\cdot)$ is the L_p surface area measure of K and dS_K is the classical surface area measure of K.

An important notion from the L_p Brunn-Minkowski theory is the L_p projection body $\prod_p K$ introduced by Lutwak et al. [59]. The L_p projection body $\prod_p K (p \geq 1)$ of $K \in \mathcal{K}_o^n$ is the origin-symmetric convex body defined by (also see [46])

$$h_{\prod_p K}(v) = \left(\frac{1}{V(B^n_{p*})^{\frac{p}{n}}} \int_{S^{n-1}} |\langle v,\, u \rangle|^p \, dS_p(K,\, u) \right)^{\frac{1}{p}},\ v \in S^{n-1}, \tag{2.3}$$

where $dS_p(K, \cdot)$ is the L_p surface area measure of K and B^n_{p*} is the unit ball of the space l^n_{p*}. Here p^* is the Höder conjugate of p; i.e., $1/p + 1/p^* = 1$. The case $p = 1$ is the classical projection body $\prod K$. The normalization above is chosen so that for $p = 1$, we have

$$h_{\prod K}(v) = \mathrm{vol}_{n-1}(K|v^{\perp}) = \frac{1}{2}\int_{S^{n-1}} |\langle v,\, u\rangle| dS_K(u), \quad v \in S^{n-1}, \tag{2.4}$$

where $dS_K(\cdot)$ is the surface area measure of K.

A compact set $K \subset \mathbb{R}^n$ is a star-shaped set (with respect to the origin) if the intersection of every straight line through the origin with K is a line segment. Let $K \subset \mathbb{R}^n$ be a compact star-shaped set (with respect to the origin); the radial function $\rho_K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is defined by

$$\rho_K(x) = \max\{\lambda \geq 0 : \lambda x \in K\}.$$

If ρ_K is positive and continuous, then we call K a star body (with respect to the origin). Let $\mathcal{S}^n_o$ be the class of star bodies (with respect to the origin) in $\mathbb{R}^n$. Two star bodies K and L are said to be dilates (of each other) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$. It is easy to see that for $K \in \mathcal{S}^n_o$, $\rho_K^{-1}(\cdot) = \|\cdot\|_K$.

In the following, we also require some basic facts of the dual Brunn-Minkowski theory due to Lutwak [56]. Some further details were provided in [57, 58]. The theory was developed very fast by many authors [14, 28, 29, 30, 33, 34, 35, 36, 50, 54, 55, 59, 60, 62, 63, 64, 69, 71].

For $p \in \mathbb{R}$, the dual mixed volume $\widetilde{V}_p(K, L)$ of $K, L \in \mathcal{S}_o^n$ was defined in [56] by

$$\widetilde{V}_p(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n-p}(u) \rho_L^p(u) du, \tag{2.5}$$

where the integration is with respect to spherical Lebesgue measure. In particular, $\widetilde{V}_p(K, K) = V(K)$. A basic inequality for the dual mixed volumes $\widetilde{V}_p$ is the dual Minkowski inequality, which states that, for $K, L \in \mathcal{S}_o^n$,

$$\widetilde{V}_p(K, L)^n \leq V(K)^{n-p} V(L)^p, \quad 0 < p < n, \tag{2.6}$$

$$\widetilde{V}_p(K, L)^n \geq V(K)^{n-p} V(L)^p, \quad p < 0 \text{ or } p > n. \tag{2.7}$$

Equality holds in each of the inequalities if and only if K and L are dilates.

Suppose $p > 0$ and $K \in \mathcal{S}_o^n$. The polar L_p-centroid body, $\Gamma_p^* K$, of K is the body whose Minkowski functional is given, for $x \in \mathbb{R}^n$, by

$$\|x\|_{\Gamma_p^* K}(u) = \left\{ \frac{1}{V(K)} \int_K |\langle x, y \rangle|^p dx \right\}^{\frac{1}{p}}. \tag{2.8}$$

Lutwak and Zhang [61] introduced a normalized definition for $p \geq 1$. When $p = 1$, the body ΓK is the classical centroid body, which was defined and investigated by Petty [65]. For more information about the L_p-centroid body, see, e.g., [21, 22, 34, 59, 63].

By the polar coordinate formula, for $u \in S^{n-1}$, we have

$$\|x\|_{\Gamma_p^* K}(u) = \left\{ \frac{1}{V(K)} \int_K |\langle u, y \rangle|^p dx \right\}^{\frac{1}{p}}$$

$$= \left(\frac{1}{(n+p)V(K)} \int_{S^{n-1}} |\langle u, v \rangle|^p \rho_K^{n+p}(v) d(v) \right)^{\frac{1}{p}}. \tag{2.9}$$

2.2. An auxiliary analytic stability result

The following two estimates is due to Ball [2]. For a simpler proof of (i), see [8].

Lemma 2.1. *The following two assertions are true*:

(i) *For any* $t_1, \cdots, t_k > 0$, *we have*

$$\det\left(\sum_{i=1}^{k} t_i c_i u_i \otimes u_i\right) \geq \prod_{i=1}^{k} t_i^{c_i}. \tag{2.10}$$

(ii) *If* $z = \sum_{i=1}^{k} c_i \theta_i u_i$, *for* $\theta_1, \cdots, \theta_k \in \mathbb{R}$, *then*

$$\|z\|^2 \leq \sum_{i=1}^{k} c_i \theta_i^2. \tag{2.11}$$

Lemma 2.2. *Let* $k \geq n+1$, $t_1, \cdots, t_k > 0$, *and let* $v_1, \cdots, v_k \in \mathbb{R}^n$ *satisfy* $\sum_{i=1}^{k} v_i \otimes v_i = \mathrm{Id}_n$. *Then*

$$\det\left(\sum_{i=1}^{k} t_i v_i \otimes v_i\right) \geq \theta^* \prod_{i=1}^{k} t_i^{\langle v_i, v_i \rangle},$$

where

$$\theta^* = 1 + \frac{1}{2} \sum_{1 \leq i_1 < \cdots < i_n \leq k} \det[v_{i_1}, \cdots, v_{i_n}]^2 \left(\frac{\sqrt{t_{i_1} \cdots t_{i_n}}}{t_0} - 1\right)^2,$$

$$t_0 = \sqrt{\sum_{1 \leq i_1 < \cdots < i_n \leq k} t_{i_1} \cdots t_{i_n} \det[v_{i_1}, \cdots, v_{i_n}]^2}.$$

The following observation from [15] will be estimated the θ^* from below.

Lemma 2.3 (see [15]). *If* $a,\ b,\ x > 0,$ *then*

$$(xa-1)^2+(xb-1)^2 \geq \frac{1}{2}\left(\frac{a^2-b^2}{a^2+b^2}\right)^2.$$

We need the following Aczel inequality.

Lemma 2.4 (see [43], p. 200). *Let* $a=(a_1, \cdots, a_k),\ b=(b_1, \cdots, b_k) \in \mathbb{R}^k,$ *and satisfy*

$$a_1^2-\sum_{i=2}^k a_i^2 > 0 \quad \textit{or} \quad b_1^2-\sum_{i=2}^k b_i^2 > 0.$$

Then

$$\left(a_1^2-\sum_{i=2}^k a_i^2\right)\left(b_1^2-\sum_{i=2}^k b_i^2\right) \leq \left(a_1 b_1-\sum_{i=2}^k a_i b_i\right)^2, \tag{2.12}$$

with equality if and only if $\dfrac{a_1}{b_1}=\cdots=\dfrac{a_k}{b_k}.$

3. Volume Inequalities for General L_p Zonoids of Even Isotropic Measures

The rank one geometric Brascamp-Lieb inequality (3.1) identified by Ball [2] is an essential case of the rank one Brascamp-Lieb inequality by Brascamp and Lieb [20]. The reverse form (3.2) is due to Barthe [7] and [8]. If $u_1, \cdots, u_k \in S^{n-1}$ are distinct unit vectors and $c_1, \cdots, c_k > 0$ satisfy

$$\sum_{i=1}^k c_i u_i \otimes u_i = \mathrm{Id}_n,$$

and $f_1, \cdots, f_k$ are non-negative measurable functions on $\mathbb{R}$, then

$$\int_{\mathbb{R}^n} \prod_{i=1}^{k} f_i(\langle x, u_i \rangle)^{c_i} dx \leq \prod_{i=1}^{k} \left(\int_{\mathbb{R}} f_i \right)^{c_i}, \tag{3.1}$$

and

$$\int_{\mathbb{R}^n}^{*} \sup_{x=\sum_{i=1}^{k} c_i \theta_i u_i} \prod_{i=1}^{k} f_i(\theta_i)^{c_i} dx \geq \prod_{i=1}^{k} \left(\int_{\mathbb{R}} f_i \right)^{c_i}. \tag{3.2}$$

In (3.2), the supremum extends over all $\theta_1, \cdots, \theta_k \in \mathbb{R}$. Note that the integrand is not a measurable function. Therefore, we need to consider the outer integral. If $k = n$, then $u_1, \cdots, u_n$ form an orthonormal basis and therefore $\theta_1, \cdots, \theta_k$ are uniquely determined for a given $x \in \mathbb{R}^n$.

Together with Barthe [8], if equality holds in (3.1) or in (3.2) and none of the functions f_i is identically zero or a scaled version of a Gaussian, then there is an origin symmetric regular crosspolytope in $\mathbb{R}^n$ such that $u_1, \cdots, u_k$ lie among its vertices. Conversely, equality holds in (3.1) and (3.2) if each f_i is a scalled version of the same centered Gaussian, or if $k = n$ and $u_1, \cdots, u_n$ form an orthonormal basis.

The rank one Brascamp-Lieb inequality has a deeper discussion by Carlen and Cordero-Erausquin [24]. The higher rank case is reproved by Lieb [45] and further explored by Barthe [8] (including a discussion of the equality case). That is again carefully analyzed by Bennett et al. [11]. In particular, what we need see is the work of Barthe et al [10] for an enlightening review of the relevant literature and an approach via Markov semigroups in a quite general framework.

According to the mass transportation by Ball [3], Barthe [7, 8] provided concise proofs of (3.1) and (3.2). The main ideas of his method were sketched, because it will be the starting point of subsequent refinements.

The following observation due to Ball [1] will often be used. If K is an origin symmetric convex body in $\mathbb{R}^n$ with associated norm $\|\cdot\|_K$ and if $p \in [1, \infty)$, then

$$V(K) = \frac{1}{\Gamma\left(1 + \frac{n}{p}\right)} \int_{\mathbb{R}^n} e^{-\|x\|_K^p} dx. \tag{3.3}$$

We note that for $p \geq 1$ and $\lambda > 0$, we have

$$\int_{\mathbb{R}} e^{-\lambda|t|^p} dt = 2\lambda^{-\frac{1}{p}} \Gamma\left(1 + \frac{1}{p}\right). \tag{3.4}$$

Suppose that each f_i is a positive continuous probability density both for (3.1) and (3.2), and $g(t) = e^{-\pi t^2}$ is the Gaussian density Here, for $i = 1, \cdots, k,$ we consider the transportation map $T_i : \mathbb{R} \to \mathbb{R}$ satisfying

$$\int_{-\infty}^{t} f_i(s) ds = \int_{-\infty}^{\alpha(u_i) T_i(t)} g(s) ds. \tag{3.5}$$

Obviously, T_i is bijective, differentiable and

$$f_i(t) = g(\alpha(u_i) T_i(t)) \cdot T_i'(t) \alpha(u_i), \quad t \in \mathbb{R}. \tag{3.6}$$

By these transportation maps, we associate the smooth transformation $\Theta : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\Theta(x) = \sum_{i=1}^{n} c_i \alpha(u_i) T_i(\langle u_i, x\rangle) u_i, \quad x \in \mathbb{R}^n,$$

satisfing

$$d\Theta(x) = \sum_{i=1}^{n} c_i \alpha(u_i) T_i'(\langle u_i, x\rangle) u_i \otimes u_i.$$

In this case, $d\Theta(x)$ is positive definite and $\Theta : \mathbb{R}^n \to \mathbb{R}^n$ is injective (see [7, 8]).

In the following, we start with a generalization of Theorem 1.1.

Proof of Theorem 1.3 in the case $Z^*_{p,\alpha}(\mu)$: Set $k \geq n$. For $i = 1, \cdots, k$ and any $u_i \in S^{n-1}$. Consider the following probability densities on $\mathbb{R}$ given by

$$f_i(t) = \begin{cases} \dfrac{\alpha(u_i)^{1/p}}{2\Gamma\left(1+\frac{1}{p}\right)} e^{-\alpha(u_i)|t|^p}, & \text{if } p \in [1, \infty); \\ \dfrac{1}{2} 1_{[-1,1]}, & \text{if } p = \infty, \end{cases} \tag{3.7}$$

where

$$1_{[-1,1]}(t) = \begin{cases} 1, & \text{if } t \in [-1, 1], \\ 0, & \text{otherwise.} \end{cases}$$

Let μ is discrete, $\operatorname{supp}\mu = \{u_1, \cdots, u_k\}$ and $\mu(\{u_i\}) = c_i > 0$, $i = 1, \cdots, k$. Since μ is isotropic, we get $\mu(S^{n-1}) = \sum_{i=1}^k c_i = n$. From (3.3), (1.10) and (3.6), it follows that (i) of Lemma 2.1 has $t_i = T_i'(\langle u_i, x\rangle)\alpha(u_i)$, the definition of Θ and (ii) of Lemma 2.1 have $\theta_i = \alpha(u_i)T_i(\langle u_i, x\rangle)$, and finally has the transformation formula. Therefore, if $p \in [1, \infty)$, we obtain

$$V(Z^*_{p,\alpha}(\mu)) = \frac{1}{\Gamma\left(1+\frac{n}{p}\right)} \int_{\mathbb{R}^n} e^{-\|x\|^p_{Z^*_{p,\alpha}}} dx$$

$$= \frac{1}{\Gamma\left(1+\frac{n}{p}\right)} \int_{\mathbb{R}^n} \exp\left(-\sum_{i=1}^k c_i \alpha(u_i)|\langle x, u_i\rangle|^p\right) dx$$

$$= \frac{2^n \Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma\left(1+\frac{n}{p}\right)} \left(\prod_{i=1}^k \alpha(u_i)^{c_i}\right)^{-\frac{1}{p}} \int_{\mathbb{R}^n} \prod_{i=1}^k f_i(\langle u_i, x\rangle)^{c_i} dx \tag{3.8}$$

$$= V(Z_p^*(\nu_n))\left(\prod_{i=1}^{k}\alpha(u_i)^{c_i}\right)^{-\frac{1}{p}}\int_{\mathbb{R}^n}\left(\prod_{i=1}^{k} g(\alpha(u_i)T_i(\langle u_i, x\rangle))^{c_i}\right)\left(\prod_{i=1}^{k} T_i'(\langle u_i, x\rangle\alpha(u_i))^{c_i}\right)dx$$

$$\leq V(Z_p^*(\nu_n))\left(\prod_{i=1}^{k}\alpha(u_i)^{c_i}\right)^{-\frac{1}{p}}$$

$$\times \int_{\mathbb{R}^n}\left(\prod_{i=1}^{k} e^{-\pi c_i(\alpha(u_i)T_i(\langle u_i, x\rangle))^2}\right)\det\left(\sum_{i=1}^{k} c_i\alpha(u_i)T_i'(\langle u_i, x\rangle)u_i \otimes u_i\right)dx$$

$$\leq V(Z_p^*(\nu_n))\left(\prod_{i=1}^{k}\alpha(u_i)^{c_i}\right)^{-\frac{1}{p}}\int_{\mathbb{R}^n} e^{-\pi\|\Theta(x)\|^2} \det(d\Theta(x))dx$$

$$\leq V(Z_p^*(\nu_n))\left(\prod_{i=1}^{k}\alpha(u_i)^{c_i}\right)^{-\frac{1}{p}}\int_{\mathbb{R}^n} e^{-\pi\|y\|^2}dy$$

$$= V(Z_p^*(\nu_n))\left(\prod_{i=1}^{k}\alpha(u_i)^{c_i}\right)^{-\frac{1}{p}}. \tag{3.9}$$

Using the Brascamp-Lieb inequality (3.1) directly and Equation (3.4), we have

$V(Z_{p,\alpha}^*(\mu))$

$$= \frac{1}{\Gamma\left(1+\frac{n}{p}\right)}\int_{\mathbb{R}^n} e^{-\|x\|^p_{Z_{p,\alpha}^*}}dx = \frac{1}{\Gamma\left(1+\frac{n}{p}\right)}\int_{\mathbb{R}^n}\exp\left(-\sum_{i=1}^{k} c_i\alpha(u_i)|\langle x, u_i\rangle|^p\right)dx$$

$$= \frac{2^n\Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma\left(1+\frac{n}{p}\right)}\left(\prod_{i=1}^{k}\alpha(u_i)^{c_i}\right)^{-\frac{1}{p}}\int_{\mathbb{R}^n}\prod_{i=1}^{k} f_i(\langle u_i, x\rangle)^{c_i}\,dx$$

$$\le \frac{2^n \Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma\left(1+\frac{n}{p}\right)} \left(\prod_{i=1}^{k} \alpha(u_i)^{c_i}\right)^{-\frac{1}{p}} \prod_{i=1}^{k} \left(\int_{\mathbb{R}} f_i\right)^{c_i}$$

$$= V(Z_p^*(\nu_n)) \left(\prod_{i=1}^{k} \alpha(u_i)^{c_i}\right)^{-\frac{1}{p}}.$$

On the other hand, if $p = \infty$ and $f_i = \frac{1}{2} 1_{[-1,1]}$, we give

$$V(Z_{\infty,\alpha}^*(\mu)) = 2^n \int_{\mathbb{R}^n} \prod_{i=1}^{k} f_i(\langle x, u_i \rangle)^{c_i} dx \le 2^n \prod_{i=1}^{k} \left(\int_{\mathbb{R}} f_i\right)^{c_i} = 2^n.$$

Equality in (3.9) forces equality in the Brascamp-Lieb inequality (3.1). Thus $k = 2n$ and $u_1, \cdots, u_k$ produce the vertices of a regular crosspolytope in $\mathbb{R}^n$. Conversely, equality holds in (3.9) if $k = n$ and $u_1, \cdots, u_n$ form an orthonormal basis of $\mathbb{R}^n$.

Now suppose that μ is an arbitrary isotropic measure on S^{n-1}. Similar to [6], we can construct a sequence μ_k, $k \in \mathbb{N}$, of discrete isotropic measures such that μ converges weakly to μ as $k \to \infty$. This obtain that $\lim_{k\to\infty} h_{Z_{p,\alpha}(\mu_k)}(v) = h_{Z_{p,\alpha}(\mu)}(v)$ for every $v \in S^{n-1}$. From the fact that the point wise convergence of support functions implies the convergence of the respective convex bodies in the Hausdorff metric (see, e.g., [67], Chapter 1), it follows that the continuity of volume and polarity on convex bodies containing the origin in their interiors finishes the proof.

□

Let each $\widetilde{f}_i$ be a positive continuous probability density both for (3.1) and (3.2), and also let

$$g(t) = \exp\left[-\pi\left(\prod_{i=1}^{k}\alpha(u_i)^{c_i}\right)^{\frac{2(1-p)}{p}} t^2\right]$$

be the Gaussian density. On the reverse Brascamp-Lieb inequality (3.2), the transportation map $S_i : \mathbb{R} \to \mathbb{R}$ satisfies

$$\int_{-\infty}^{\alpha(u_i)S_i(t)} \widetilde{f}_i(s)ds = \int_{-\infty}^{t} g(s)ds, \tag{3.10}$$

$$g(t) = \widetilde{f}_i(\alpha(u_i)S_i(t)) \cdot S_i'(t)\alpha(u_i), \quad t \in \mathbb{R}. \tag{3.11}$$

Moreover,

$$d\Psi(x) = \sum_{i=1}^{n} c_i \alpha(u_i) S_i'(\langle u_i, x\rangle) u_i \otimes u_i$$

holds for the smooth transformation $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\Psi(x) = \sum_{i=1}^{n} c_i \alpha(u_i) S_i(\langle u_i, x\rangle) u_i, \quad x \in \mathbb{R}^n.$$

In this case, $d\Psi(x)$ is positive definite and $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is injective (see [7, 8]).

Proof of Theorem 1.3 in the case $Z_{p,\alpha}(\mu)$: Without loss of generality, since μ is discrete, for the lower bound on the volume of the L_p zonotopes and $p \in [1, \infty]$ we choose $p^* \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{p^*} = 1$. If $p \in [1, \infty)$, then define an (auxiliary) origin symmetric convex body by

$$M_{p,\alpha}(\mu) = \left\{\sum_{i=1}^{k} c_i \theta_i u_i : \sum_{i=1}^{k} c_i \alpha(u_i)^{1-p} |\theta_i|^p \le 1\right\}.$$

We use the reference to μ when it do not cause any misunderstanding. In particular,

$$\|x\|_{M_{p,\alpha}} = \left(\inf_{x=\sum_{i=1}^k c_i\theta_i u_i} \sum_{i=1}^{k} c_i \alpha(u_i)^{1-p} |\theta_i|^p \right)^{\frac{1}{p}}, \quad x \in \mathbb{R}^n. \tag{3.12}$$

Define

$$M_{\infty,\alpha}(\mu) = \left\{ \sum_{i=1}^{k} c_i\theta_i u_i : |\theta_i/\alpha(u_i)| \le 1 \text{ for } i = 1, \cdots, k \right\}.$$

Then if $p \in [1, \infty]$, then

$$M_{p,\alpha}(\mu) \subseteq Z_{p^*,\alpha}(\mu). \tag{3.13}$$

In fact, let $x \in M_{p,\alpha}(\mu)$. Thus $x = \sum_{i=1}^k c_i\theta_i u_i$ with $\sum_{i=1}^k c_i\alpha(u_i)^{1-p}$ $|\theta_i|^p \le 1$ if $p \in [1, \infty)$ and $|\theta_i/\alpha(u_i)| \le 1$ for $i = 1, \cdots, k$ if $p = \infty$. Let $p \in (1, \infty)$. From Hölder's inequality, we obtain that

$$\langle x, v\rangle = \sum_{i=1}^{k} c_i\theta_i\langle u_i, v\rangle \le \left(\sum_{i=1}^{k} c_i\alpha(u_i)^{1-p}|\theta_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{k} c_i\alpha(u_i)|\langle u_i, v\rangle|^{p^*} \right)^{\frac{1}{p^*}} \le h_{Z_{p^*,\alpha}}(v).$$

If $p = 1$, then

$$\langle x, v\rangle = \sum_{i=1}^{k} c_i\theta_i\langle u_i, v\rangle \le \max_{i=1,\cdots,k} |\langle u_i, v\rangle| = h_{Z_{\infty,\alpha}}.$$

Furthermore, if $p = \infty$, then

$$\langle x, v\rangle = \sum_{i=1}^{k} c_i\theta_i\langle u_i, v\rangle \le \sum_{i=1}^{k} c_i\alpha(u_i)|\langle u_i, v\rangle| = h_{Z_{1,\alpha}}(v).$$

For $i = 1, \cdots, k$ and any $u_i \in S^{n-1}$, the following probability densities on $\mathbb{R}$ will be considered

$$\widetilde{f}_i(t) = \begin{cases} \dfrac{1}{2\Gamma\left(1+\frac{1}{p}\right)} e^{-\alpha(u_i)^{1-p}|t|^p}, & \text{if } p \in [1, \infty); \\ \dfrac{1}{2} 1_{[-\alpha(u_i), \alpha(u_i)]}, & \text{if } p = \infty, \end{cases} \tag{3.14}$$

where

$$1_{[-\alpha(u_i), \alpha(u_i)]}(t) = \begin{cases} 1, & \text{if } t \in [-\alpha(u_i), \alpha(u_i)], \\ 0, & \text{otherwise.} \end{cases}$$

Next if $p \in [1, \infty)$, then from (3.13), the volume formula (3.3), the norm (3.12) of $M_{p,\alpha}$ and the reverse Brascamp-Lieb inequality (3.2), we have that

$$V(Z_{p^*,\alpha}(\mu)) \geq V(M_{p,\alpha}(\mu)) = \frac{1}{\Gamma\left(1+\frac{n}{p}\right)} \int_{\mathbb{R}^n} \exp\left(-\|x\|_{M_{p,\alpha}}^p\right) dx$$

$$= \frac{2^n \Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma\left(1+\frac{n}{p}\right)} \int_{\mathbb{R}^n}^{*} \sup_{x=\sum_{i=1}^k c_i\theta_i u_i} \prod_{i=1}^k \widetilde{f}_i(\theta_i)^{c_i} dx \tag{3.15}$$

$$\geq V(Z_p^*(\nu_n)) \int_{\mathbb{R}^n}^{*} \left(\sup_{\Psi(y)=\sum_{i=1}^k c_i\theta_i u_i} \prod_{i=1}^k \widetilde{f}_i(\theta_i)^{c_i} \right) \det(d\Psi(y)) dy$$

$$\geq V(Z_p^*(\nu_n)) \int_{\mathbb{R}^n} \left(\prod_{i=1}^k \widetilde{f}_i(\alpha(u_i) S_i(\langle u_i, y\rangle))^{c_i} \right) \det\left(\sum_{i=1}^k c_i \alpha(u_i) S_i'(\langle u_i, y\rangle) u_i \otimes u_i \right) dy$$

$$\geq V(Z_p^*(\nu_n)) \int_{\mathbb{R}^n} \left(\prod_{i=1}^k \widetilde{f}_i(\alpha(u_i) S_i(\langle u_i, y\rangle))^{c_i} \right) \left(\prod_{i=1}^k (\alpha(u_i) S_i'(\langle u_i, y\rangle))^{c_i} \right) dy$$

$$= V(Z_p^*(\nu_n))\int_{\mathbb{R}^n}\left(\prod_{i=1}^k g(\langle u_i,\ y\rangle)^{c_i}\right)dy$$

$$= V(Z_p^*(\nu_n))\left(\prod_{i=1}^k \alpha(u_i)^{c_i}\right)^{\frac{p-1}{p}}\int_{\mathbb{R}^n} e^{-\pi\|y\|^2}dy$$

$$= V(Z_p^*(\nu_n))\left(\prod_{i=1}^k \alpha(u_i)^{c_i}\right)^{\frac{1}{p^*}} = V(Z_{p^*}(\nu_n))\left(\prod_{i=1}^k \alpha(u_i)^{c_i}\right)^{\frac{1}{p^*}}. \tag{3.16}$$

By the reverse Brascamp-Lieb inequality (3.2), we also have

$$V(Z_{p^*,\alpha}(\mu)) \ge V(M_{p,\alpha}(\mu)) = \frac{1}{\Gamma\left(1+\frac{n}{p}\right)}\int_{\mathbb{R}^n}\exp\left(-\|x\|_{M_{p,\alpha}}^p\right)dx$$

$$= \frac{2^n\Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma\left(1+\frac{n}{p}\right)}\int_{\mathbb{R}^n}^* \sup_{x=\sum_{i=1}^k c_i\theta_i u_i}\prod_{i=1}^k \widetilde{f}_i(\theta_i)^{c_i}\,dx$$

$$\ge \frac{2^n\Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma\left(1+\frac{n}{p}\right)}\prod_{i=1}^k\left(\int_{\mathbb{R}}\widetilde{f}_i\right)^{c_i} = \frac{2^n\Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma\left(1+\frac{n}{p}\right)}\left(\prod_{i=1}^k \alpha(u_i)^{c_i}\right)^{\frac{p-1}{p}}$$

$$= V(Z_p^*(\nu_n))\left(\prod_{i=1}^k \alpha(u_i)^{c_i}\right)^{\frac{1}{p^*}} = V(Z_{p^*}(\nu_n))\left(\prod_{i=1}^k \alpha(u_i)^{c_i}\right)^{\frac{1}{p^*}}.$$

Finally, if $p = \infty$, then $\widetilde{f}_i = \frac{1}{2}\mathbf{1}_{[-\alpha(u_i),\,\alpha(u_i)]}$ and

$$V(Z_{1,\alpha}(\mu)) \ge V(M_{\infty,\alpha}(\mu)) = 2^n\int_{\mathbb{R}^n}^* \sup_{x=\sum_{i=1}^k c_i\theta_i u_i}\prod_{i=1}^k f_i(\theta_i)^{c_i}\,dx$$

$$\ge 2^n\prod_{i=1}^k\left(\int_{\mathbb{R}} f_i\right)^{c_i} = 2^n.$$

Equality in (3.16) forces equality in the reverse Brascamp-Lieb inequality. Thus $k = 2n$ and $u_1, \cdots, u_k$ form the vertices of a regular crosspolytope in $\mathbb{R}^n$. Conversely, equality holds in (3.16) if $k = n$ and $u_1, \cdots, u_n$ form an orthonormal basis of $\mathbb{R}^n$. □

4. The Transportation Maps

Since $p \in [1, \infty]$ and the map $\alpha : S^{n-1} \to (0, \infty)$ is a continuous positive function, the density functions is considered

$$\varrho_p(s) = \begin{cases} \dfrac{\alpha(u)}{2\Gamma\left(1 + \dfrac{1}{p}\right)} e^{-\alpha(u)|s|^p}, & \text{if } p \in [1, \infty), \\ \dfrac{1}{2} \mathbf{1}_{[-1,1]}(s), & \text{if } p = \infty. \end{cases}$$

$$\tilde{\varrho}_p(s) = \begin{cases} \dfrac{1}{2\Gamma\left(1 + \dfrac{1}{p}\right)} e^{-\alpha(u)^{1-p}|s|^p}, & \text{if } p \in [1, \infty), \\ \dfrac{1}{2} \mathbf{1}_{[-1,1]}(s), & \text{if } p = \infty. \end{cases}$$

For each $u \in S^{n-1}$, the strictly increasing function $\varphi_{p,u}, \psi_{p,u} : \mathbb{R} \to \mathbb{R}$ for $p \in [1, \infty)\varphi_{\infty,u} : (-1, 1) \to \mathbb{R}$ and $\psi_{\infty,u} : \mathbb{R} \to (-1, 1)$ is defined by

$$\int_{-\infty}^{t} \varrho_p(s) ds = \int_{-\infty}^{\alpha(u)\varphi_{p,u}(t)} \tilde{\varrho}_2(s) ds, \tag{4.1}$$

$$\int_{-\infty}^{\alpha(u)\psi_{p,u}(t)} \tilde{\varrho}_p(s) ds = \int_{-\infty}^{t} \varrho_2(s) ds. \tag{4.2}$$

Hence,

$$\varrho_p(t) = \tilde{\varrho}_2(\alpha(u)\varphi_{p,u}(t))\varphi'_{p,u}(t)\alpha(u), \tag{4.3}$$

$$\varrho_2(t) = \tilde{\varrho}_p(\alpha(u)\psi_{p,u}(t))\psi'_{p,u}(t)\alpha(u). \tag{4.4}$$

Using

$$s - s^2 \leq \log(1+s) \leq s \text{ if } s \geq -\frac{1}{2},$$

and the following properties of the Γ function:

(i) $\log \Gamma(t)$ is strictly convex for $t > 0$;

(ii) $\Gamma(1) = \Gamma(2) = 1$;

(iii) $\Gamma\left(1 + \frac{1}{2.3}\right) < \Gamma\left(1 + \frac{1}{2}\right) = \sqrt{\pi} / 2$;

(iv) Γ has a unique minimum on $(0, \infty)$ at $x_{\min} = 1.4616 \cdots$ with $\Gamma(x_{\min}) = 0885603 \cdots$. In particular, $\Gamma(t) > 0.8856$ for $t > 0$, Γ is strictly decreasing on $[0, x_{\min}]$ and strictly increasing on $[1.5, \infty)$.

From (i)-(iv), the density functions involved satisfy

$$\frac{\alpha(u)}{2e^{\alpha(u)}} \leq \varrho_p(s) < \frac{\alpha(u)}{2 \times 0.8856} \text{ for } p \in [1, \infty] \text{ and } s \in [0, 1], \tag{4.5}$$

$$\frac{1}{2e^{\alpha(u)1-p}} \leq \tilde{\varrho}_p(s) < \frac{1}{2 \times 0.8856} \text{ for } p \in [1, \infty] \text{ and } s \in [0, 1]. \tag{4.6}$$

Since $e / 0.8856 < 3.1$, we have

$$\varphi_{p,u}(s) \in \left[0, e^{\alpha(u)^{-1}-1}\right) \text{ for } s \in \left[0, \frac{1}{3.1}\right]. \tag{4.7}$$

Indeed, let

$$\varphi_{p,u}\left(\frac{1}{3.1}\right) \geq e^{\alpha(u)^{-1}-1} = \varphi_{p,u}(t), \ t \in \left(0, \frac{1}{3.1}\right]$$

Then from (4.1), (4.5) and (4.6), it follows that

$$\frac{3.1^{-1}\alpha(u)}{2 \times 0.8856} > \int_0^t \varrho_p(s)ds = \int_0^{\alpha(u)\varphi_{p,u}(t)} \tilde{\varrho}_2(s)ds \geq \frac{\alpha(u)}{2e},$$

this is a contradiction. Thus together (4.3) with (4.7) get that

$$\frac{1}{3.1e^{\alpha(u)-1}} < \varphi'_{p,u}(t) < 3.1e^{\alpha(u)e^{2(\alpha(u)^{-1}-1)}-1} \text{ for } p \in [1, \infty] \text{ and } s \in \left[0, \frac{1}{3.1}\right]. \tag{4.8}$$

Similarly, since $e / 0.8856 < 3.1$, we have

$$\psi_{p,u}(s) \in \left[0, e^{\alpha(u)^{1-p}-1}\right) \text{ for } s \in \left[0, \frac{1}{3.1}\right]. \tag{4.9}$$

This implies

$$\psi_{p,u}\left(\frac{1}{3.1}\right) \geq e^{\alpha(u)^{1-p}-1} = \psi_{p,u}(t),\ t \in \left(0, \frac{1}{3.1}\right],$$

we have

$$\frac{3.1^{-1}\alpha(u)}{2 \times 0.8856} > \int_0^t \varrho_2(s)ds = \int_0^{\alpha(u)\psi_{p,u}(t)} \widetilde{\varrho}_p(s)ds \geq \frac{\alpha(u)}{2e},$$

this is a contradiction. Then, (4.4) and (4.9) yield that

$$\frac{1}{3.1e^{\alpha(u)-1}} < \psi'_{p,u}(s) < 3.1e^{\alpha(u)e^{p(\alpha(u)^{1-p}-1)}-1} \text{ for } p \in [1, \infty] \text{ and } s \in \left[0, \frac{1}{3.1}\right]. \tag{4.10}$$

The following simple estimate will play a crucial role in the proofs of Lemmas 4.2 and 4.3.

Lemma 4.1 (see [17]). *For* $p \in (1, 3)\setminus\{2\}$ *and* $a > 0$, *let* $f(t) = at - pt^{p-1}$ *for* $t \in [0, 1]$.

(a) *If* $p \in (1, 2)$, $f(\tau) \leq 0$ *for some* $\tau \in (0, 1]$ *and* $t \in (0, \tau/2]$, *then*

$$f(t) < -\frac{p(p-1)(2-p)}{2^{4-p}} \cdot t^{p-1}.$$

(b) *If* $p \in (2, 3)$, $f(\tau) \geq 0$ *for some* $\tau \in (0, 1]$ *and* $t \in (0, \tau/2]$, *then*

$$f(t) > \frac{p(p-1)(p-2)}{2^{4-p}} \cdot t^{p-1}.$$

For ease of notations, let $\varphi_p = \varphi_{p,u}$ and $\psi_p = \psi_{p,u}$.

Lemma 4.2. *Let* $p \in [1, \infty]/\{2\}$ *and* $t \in (0, \frac{1}{8})$, *and let* $\alpha : S^{n-1} \to (0, \infty)$ *be a continuous bounded function. Then*

$$\varphi_p''(t) < -\frac{(2-p)\alpha(u)e^{1-\alpha(u)}}{48} \cdot t \text{ if } p \in [1, 2), \tag{4.11}$$

$$\varphi_p''(t) > \frac{(p-2)\alpha(u)}{5} \cdot t^{1.3} \text{ if } p \in (2, 3], \tag{4.12}$$

$$\varphi_p''(t) > 0.2\alpha(u) \cdot t^{1.3} \text{ if } p \in (3, \infty]. \tag{4.13}$$

Proof. First, we let $\varphi = \varphi_p$. Thus $\varphi(0) = 0$ as φ is odd. From the fact that φ is strictly increasing, we have $\varphi(t) > 0$ if $t > 0$.

Let $p \in [1, \infty]/\{2\}$. For $t > 0$, differentiating (4.1) gives the formula

$$\frac{\alpha(u)e^{-\alpha(u)t^p}}{\Gamma\left(1 + \frac{1}{p}\right)} = \frac{e^{-\alpha(u)\varphi(t)^2}\varphi'(t)\alpha(u)}{\Gamma\left(1 + \frac{1}{2}\right)}, \tag{4.14}$$

i.e.,

$$\varphi'(t) = \frac{\Gamma\left(1 + \frac{1}{2}\right)}{\Gamma\left(1 + \frac{1}{p}\right)} e^{\alpha(u)(\varphi(t)^2 - t^p)}. \tag{4.15}$$

Differentiating again, we have

$$\varphi''(t) = \alpha(u)\left(2\varphi(t)\varphi'(t) - pt^{p-1}\right)\varphi'(t). \tag{4.16}$$

The following argument is to use the value

$$t_p = (2/p)^{\frac{1}{p-2}} \text{ for } p \in [1, \infty)\setminus\{2\}.$$

The function $p \mapsto t_p$ is continuously extended to $p = 2$ by $t_2 = e^{-1/2}$, and then this function is increasing on $[1, \infty)$. In particular, $t_p \geq 1/2$ for $p \in [1, \infty)$.

Moreover, we use the following fact to obtain that for given $t \in (0, 1/e)$, $p \mapsto pt^{p-1}$ is a decreasing function of $p \geq 1$.

First, we prove that for $p \in [1, 2)$ and $t \in (0, \frac{1}{4})$. Thus we obtain $\varphi''(t) < -\frac{(2-p)\alpha(u)e^{1-\alpha(u)}}{48} \cdot t,$ which proves (4.11).

In this case, $\varphi'(0) < 1$ by (4.15), (i), (ii) and (iv). Since φ' is continuous, there exists a largest $s_p \in (0, \infty]$ such that $\varphi'(t) < 1$ if $0 < t < s_p$. Thus, if $t \in (0, s_p)$, then $\varphi(t) < t$, and in turn (4.16) yields that

$$\varphi''(t) = \alpha(u)\left(2\varphi(t)\varphi'(t) - pt^{p-1}\right)\varphi'(t) < \alpha(u)\left(2t - pt^{p-1}\right)\varphi'(t).$$

For $p \in [1, 2)$ and $t \in [0, t_p]$, we have $2t - pt^{p-1} \leq 0$. In particular, $\varphi'(t)$ is monotone decreasing on $(0, \min\{s_p, t_p\})$, which in turn implies that $s_p \geq t_p$. We obtain from (4.8) that

$$\varphi''(t) < \frac{\left(2t - pt^{p-1}\right)\alpha(u)e^{1-\alpha(u)}}{3.1} \text{ for } t \in \left(0, \frac{1}{3.1}\right). \tag{4.17}$$

Now we distinguish two cases. If $1.5 \leq p < 2$, then from (4.17) and Lemma 4.1 (a), we obtain

$$\varphi''(t) < -\frac{p(p-1)(2-p)\alpha(u)e^{1-\alpha(u)}}{3.1 \times 2^{4-p}} \cdot t^{p-1} < -\frac{\frac{3}{4}(2-p)\alpha(u)e^{1-\alpha(u)}}{3.1 \times 2^{2.5}} \cdot t$$

$$< -\frac{(2-p)\alpha(u)e^{1-\alpha(u)}}{24} \cdot t, \text{ for } t \in \left(0, \frac{1}{4}\right). \tag{4.18}$$

If $1 \le p \le 1.5$, then when estimating the right-hand side of (4.17) for a given $t \in (0, \frac{1}{4})$, Let $p = 1.5$. Using Lemma 4.1 (a), inequality (4.18) to obtain that if $1 \le p \le 1.5$ and $t \in (0, \frac{1}{4})$, then

$$\varphi''(t) < \frac{(2t - pt^{p-1})\alpha(u)e^{1-\alpha(u)}}{3.1} \le \frac{(2t - 1.5t^{0.5})\alpha(u)e^{1-\alpha(u)}}{3.1}$$

$$\le -\frac{(2-1.5)\alpha(u)e^{1-\alpha(u)}}{24} \cdot t \le -\frac{(2-p)\alpha(u)e^{1-\alpha(u)}}{48} \cdot t. \tag{4.19}$$

Second, if $2 < p \le 2.3$ and $t \in (0, \frac{1}{4})$, then we prove that $\varphi''(t) > \frac{(p-2)\alpha(u)^{\frac{1}{p}-1}}{2} \cdot t^{1.3}$.

In this case, $\varphi'(0) > 1$ by (4.15), (i), (iii) and (iv). Since φ' is continuous, there exists a largest $s_p \in (0, \infty]$ such that $\varphi'(t) > 1$ if $0 < t < s_p$. Thus if $t \in (0, s_p)$, then $\varphi(t) > t$, and in turn (4.16) yields that

$$\varphi''(t) = \alpha(u)\left(2\varphi(t)\varphi'(t) - pt^{p-1}\right)\varphi'(t) > \alpha(u)(2t - pt^{p-1})\varphi'(t).$$

For $p > 2$ and $t \in [0, t_p]$, we have $2t - pt^{p-1} \ge 0$. In particular, $\varphi'(t)$ is monotone increasing on $(0, \min\{s_p, t_p\})$, which, in turn, implies that $s_p \ge t_p$. We have that

$$\varphi''(t) > \alpha(u)(2t - pt^{p-1}) \quad \text{if } t \in (0, \frac{1}{2}). \tag{4.20}$$

From (4.20) and Lemma 4.1 (b), we have

$$\varphi''(t) > \frac{\alpha(u)p(p-1)(p-2)}{2^{4-p}} \cdot t^{p-1} > \frac{2\alpha(u)(p-2)}{2^2} \cdot t^{1.3} = \frac{\alpha(u)(p-2)}{2} \cdot t^{1.3}$$

$$\text{if } t \in \left(0, \frac{1}{4}\right).$$

If $p \geq 2.3$ and $t \in \left(0, \frac{1}{8}\right)$, then $\varphi''(t) > 0.2\alpha(u)t^{1.3}$, the proof of (4.12) is completed.

In this case, $\varphi'(0) > \frac{\sqrt{\pi}}{2}$ by (4.15), (i)-(iv). Since φ' is continuous, there exists largest $s_p \in (0, \frac{1}{4}]$ such that $\varphi'(t) > \frac{\sqrt{\pi}}{2}$ if $t \in (0, s_p)$. Thus if $t \in (0, s_p]$, then $\varphi(t) > \frac{\sqrt{\pi}}{2}t$. Since $p \mapsto pt^{p-1}$ is a decreasing function of $p \geq 1$, we get

$$\alpha(u)\left(2\varphi(t)\varphi'(t) - pt^{p-1}\right) \geq \alpha(u)\left(\frac{\pi}{2}t - pt^{p-1}\right) \geq \alpha(u)\left(\frac{\pi}{2}t - 2.3t^{1.3}\right) \geq 0,$$

for $0 < s_p \leq 1/4$. Thus (4.16) implies that

$$\varphi''(t) = \alpha(u)\left(2\varphi(t)\varphi'(t) - pt^{p-1}\right)\varphi'(t) > \alpha(u)\left(\frac{\pi}{2}t - 2.3t^{1.3}\right)\frac{\sqrt{\pi}}{2},$$

for $t \in (0, s_p]$. Particularly, we deduce that $s_p = \frac{1}{4}$. Thus Lemma 4.1(b) gives that

$$\varphi''(t) > \frac{(\sqrt{\pi}/2) \cdot 2.3 \cdot 1.3 \cdot 0.3\alpha(u)}{2^{1.7}} \cdot t^{1.3} > 0.2\alpha(u) \cdot t^{1.3} \text{ for } t \in \left(0, \frac{1}{8}\right).$$

If $p = \infty$ and $t > 0$, then $\varphi''(t) > \alpha(u)t$, which obtains the proof of (4.14). Differentiating (4.1) we prove that for $t \in (-1, 1)$,

$$\varphi'(t) = \Gamma\left(1 + \frac{1}{2}\right)e^{\alpha(u)\varphi(t)^2} = \frac{\sqrt{\pi}}{2}e^{\alpha(u)\varphi(t)^2}, \tag{4.21}$$

$$\varphi''(t) = 2\alpha(u)\varphi(t)\varphi'(t)^2. \tag{4.22}$$

Since $\varphi(t) > 0$ for $t > 0$, we get $\varphi''(t) \geq 0$ by (4.22), this implies that $\varphi'(t)$ is monotone increasing for $t \geq 0$. Therefore $\varphi'(t) \geq \varphi'(0) = \frac{\sqrt{\pi}}{2}$ by (4.21), in turn, which from (4.22) gives that

$$\varphi''(t) \geq 2\alpha(u)\left(\frac{\sqrt{\pi}}{2}\right)^3 t > \alpha(u)t \quad \text{ for } t \in (0, 1).$$

Thus all estimates of Lemma 4.2 have been proved for φ''. □

Lemma 4.3. *Let* $p \in [1, \infty] / \{2\}$ *and* $t \in (0, \frac{1}{10})$, *and let* $\alpha : S^{n-1} \to (0, \infty)$ *be a continuous function. Then*

$$\psi_p''(t) > \frac{\alpha(u)(2-p)}{16} \cdot t \quad \textit{if } p \in [1, 2), \tag{4.23}$$

$$\psi_p''(t) < -\frac{\alpha(u)e^{1-\alpha(u)}(p-2)}{11} \cdot t^{1.3} \quad \textit{if } p \in (2, 3], \tag{4.24}$$

$$\psi_p''(t) < -\frac{\alpha(u)e^{1-\alpha(u)}}{11} \cdot t^{1.3} \quad \textit{if } p \in (3, \infty]. \tag{4.25}$$

Proof. To simplify notation, let $\psi = \psi_p$. Since ψ is odd, we have $\psi(0) = 0$. This gives $\psi(t) > 0$ if $t > 0$. Turning to ψ'', we just sketch the main steps. In this case, differentiating (4.2) obtains the formulas

$$\psi'(t) = \frac{\Gamma\left(1 + \frac{1}{p}\right)}{\Gamma\left(1 + \frac{1}{2}\right)} e^{\alpha(u)(\psi(t)^p - t^2)}, \tag{4.26}$$

$$\psi''(t) = \alpha(u)\left(p\psi(t)^{p-1}\psi'(t) - 2t\right)\psi'(t). \tag{4.27}$$

First, for $1 \leq p < 2$ and $t \in \left(0, \frac{1}{8}\right)$, we obtain that $\psi''(t) > \frac{\alpha(u)(2-p)}{16} \cdot t$, which gives (4.23).

If $p \in [1, 2)$, then $\psi'(0) > 1$ by (i), (ii) and (iv). Similar to the proof of Lemma 4.2, we get

$$\psi''(t) = \alpha(u)\left(p\psi(t)^{p-1}\psi'(t) - 2t\right)\psi'(t) > \alpha(u)\left(pt^{p-1} - 2t\right) \quad \text{for} \in t\left(0, \frac{1}{2}\right). \tag{4.28}$$

If $1.5 \leq p < 2$, then it follows from (4.28) and Lemma 4.1 (a) that

$$\psi''(t) > \frac{\alpha(u)p(p-1)(2-p)}{2^{4-p}} \cdot t^{p-1} > \frac{\frac{3}{4}\alpha(u)(2-p)}{2^{2.5}} \cdot t > \frac{\alpha(u)(2-p)}{8} \cdot t$$

$$\text{for } t \in \left(0, \frac{1}{8}\right)$$

If $1 \leq p \leq 1.5$, then when estimating the right-hand side of (4.28) for a given $t \in (0, \frac{1}{e})$, $p \mapsto pt^{p-1}$ is a decreasing function of $p \geq 1$. We may let $p = 1.5$. In fact, (4.28) obtains that if $1 \leq p \leq 1.5$ and $t \in (0, \frac{1}{e})$, then

$$\psi''(t) > \alpha(u)(pt^{p-1} - 2t) \geq \alpha(u)(1.5t^{0.5} - 2t) \geq \frac{\alpha(u)(2-1.5)}{8} \cdot t \geq \frac{\alpha(u)(2-p)}{16} \cdot t. \tag{4.29}$$

Next, for $2 < p \leq 2.3$ and $t \in (0, \frac{1}{4})$, we will show that $\psi''(t) < -\frac{\alpha(u)(p-2)}{7} \cdot t^{1.3}$.

If $p \in (2, 2.3]$, then $\psi'(0) < 1$ by (i)-(iv). From the arguments similar to the ones used in the proof of Lemma 4.2, we give

$$\psi''(t) = \alpha(u)\left(p\psi(t)^{p-1}\psi'(t) - 2t\right)\psi'(t) < -\alpha(u)(2t - pt^{p-1})\psi'(t)$$

$$< -\frac{\alpha(u)e^{1-\alpha(u)}(2t - pt^{p-1})}{3.1} < 0 \text{ for } t \in \left(0, \frac{1}{3.1}\right).$$

It follows from Lemma 4.1 (b) that

$$\psi''(t) < -\frac{\alpha(u)e^{1-\alpha(u)}p(p-1)(p-2)}{3.1\cdot 2^{p-1}}\cdot t^{p-1} < -\frac{2\alpha(u)e^{1-\alpha(u)}(p-2)}{3.1\cdot 2^{2}}\cdot t^{1.3}$$

$$< -\frac{\alpha(u)e^{1-\alpha(u)}(p-2)}{7}\cdot t^{1.3} \text{ for } t\in\left(0, \frac{1}{8}\right).$$

Set $p \geq 2.3$ and $t \in (0, \frac{1}{10})$. We now prove that $\psi''(t) < -\frac{\alpha(u)e^{1-\alpha(u)}}{11}\cdot t^{1.3}$, which gives the proof of (4.24).

In this case, $\psi'(0) < 2/\sqrt{\pi}$ by (i)-(iv). There exists a maximal $s_p \in (0, \frac{1}{5}]$ such that if $t \in (0, s_p)$, then $\psi'(t) < 2/\sqrt{\pi}$. Thus if $t \in (0, s_p)$, then $\psi(t) < (2/\sqrt{\pi})\cdot t$, and, in turn, (4.27) yields that

$$\psi''(t) = \alpha(u)\left(p\psi(t)^{p-1}\psi'(t) - 2t\right)\psi'(t) < \alpha(u)\left(\left(\frac{2}{\sqrt{\pi}}\right)^{p} pt^{p-1} - 2t\right)\psi'(t). \quad (4.30)$$

Take $t \in (0, \frac{1}{2}]$,

$$\frac{d}{dp}\log\left[\left(\frac{2}{\sqrt{\pi}}\right)^{p} pt^{p-1}\right] = \frac{1}{p} + \log\frac{2t}{\sqrt{\pi}} < 0 \text{ for } p\in(2, \infty),$$

(4.30) gets that if $t \in (0, s_p]$, then

$$\psi''(t) = \alpha(u)\left(p\psi(t)^{p-1}\psi'(t) - 2t\right)\psi'(t)$$

$$< \alpha(u)\left(\left(\frac{2}{\sqrt{\pi}}\right)^{2.3}\cdot 2.3t^{1.3} - 2t\right)\psi'(t) = \alpha(u)\cdot f(t)\left(\frac{2}{\sqrt{\pi}}\right)^{2.3}\psi'(t), \quad (4.31)$$

where

$$f(t) = 2.3t^{1.3} - 2\left(\frac{\sqrt{\pi}}{2}\right)^{-2.3} t.$$

Here $f(\frac{1}{5}) < 0,$ thus with $\tau = \frac{1}{5},$ Lemma 4.1 (b) has

$$f(t) < -\frac{2.3 \cdot 1.3 \cdot 0.3}{2^{1.7}} \cdot t^{1.3} < -0.27 \cdot t^{1.3} \quad \text{for } t \in \left(0, \frac{1}{10}\right).$$

Together (4.10) with (4.31), we obtain

$$\psi''(t) < -\frac{\alpha(u)e^{1-\alpha(u)} \cdot \left(\frac{2}{\sqrt{\pi}}\right)^{2.3} \cdot 0.27 \cdot t^{1.3}}{3.1} < -\frac{\alpha(u)e^{1-\alpha(u)} \cdot t^{1.3}}{11} \text{ for } t \in \left(0, \frac{1}{10}\right).$$

Finally, for $p = \infty$ and $t \in (0, \frac{1}{3.1}),$ we pove $\psi''(t) < -\frac{2\alpha(u)}{3.1} \cdot t,$ this gives the proof of (4.25).

Differentiating (4.2) it follows that if $t > 0,$ then

$$\psi'(t) = \frac{1}{\Gamma\left(1 + \frac{1}{2}\right)} e^{-\alpha(u)t^2} = \frac{2}{\sqrt{\pi}} e^{-\alpha(u)t^2},$$

$$\psi''(t) = -2\alpha(u)\psi'(t)t.$$

From (4.10), we obtain that $\psi''(t) < -\frac{2\alpha(u)e^{1-\alpha(u)}t}{3.1}$ for $t \in \left(0, \frac{1}{3.1}\right).$

In summary, we have established all estimates of Lemma 4.3 for $\psi''.$

□

5. The Volume of $Z_{p,\alpha}^*$

In this section, we establish a stability result for the volume of $Z_{p,\alpha}^*$ stated in Theorem 1.5. The other part of this theorem is given in Section 6. In order to prove the stability theorem, we need some lemmas from literature [17] which are basic estimates on isotropic measures.

For $\alpha \in (0, \frac{\pi}{2}]$ and $v \in S^{n-1}$, we consider the following closed and open spherical caps:

$$\Omega(v, \alpha) = \{u \in S^{n-1} : \langle u, v\rangle \geq \cos\alpha\},$$

$$\widetilde{\Omega}(v, \alpha) = \{u \in S^{n-1} : \langle u, v\rangle > \cos\alpha\}.$$

Lemma 5.1. *If* μ *is an isotropic measure on* S^{n-1}, $v \in S^{n-1}$, *and* $\alpha \in \left(0, \frac{\pi}{2}\right)$, *then*

$$\mu\left(\widetilde{\Omega}(v, \alpha)\right) + \mu\left(\widetilde{\Omega}(-v, \alpha)\right) \geq 1 - n\cos^2\alpha.$$

Lemma 5.2. *Let* $\beta = 2^{-(n+1)}n^{-(n+1)/2}$. *If* μ *is an isotropic measure on* S^{n-1}, *then there exist* $v_1, \cdots, v_n \in S^{n-1}$ *such that* $\mu(\Omega(v_i, \beta)) \geq \beta^n$, *for* $i = 1, \cdots, n$, *and such that if* $w_i \in \Omega(v_i, \beta)$, *for* $i \in \{1, \cdots, n\}$, *then* $|\det[w_1, \cdots, w_n]| \geq 2n\beta$.

Lemma 5.2 states that for any isotropic measure μ on S^{n-1}, there exist spherical caps $X_1, \cdots, X_n \subseteq S^{n-1}$ whose μ-measure is bounded from below and which have the additional property that for any vectors $w_i \in X_i$, $i \in \{1, \cdots, n\}$, also the determinant $|\det[w_1, \cdots, w_n]|$ is bounded from below.

Lemma 5.3. *For an isotropic measure* μ *on* S^{n-1}, *let* $v_1, \cdots, v_n \in S^{n-1}$ *and* β *be as in Lemma* 5.2. *For every* $i \in \{1, \cdots, n\}$ *and* $\eta \in (0, \beta)$,

(i) *there exists* $q_i \in \Omega(v_i, \beta)$ *such that*

$$\mu(\Omega(v_i, \beta) \cap \Omega(q_i, \beta)) \geq \frac{\beta^n}{4n},$$

(ii) *or there exist* $\Psi_1, \Psi_2 \subseteq \Omega(v_i, \beta)$ *such that*

$$\mu(\Psi_j) \geq \frac{\beta^n}{4n} \text{ for } j = 1, 2,$$

$$\|a_1 - a_2\| \geq \frac{\eta}{\sqrt{n}} \text{ for } a_1 \in \Psi_1 \text{ and } a_2 \in \Psi_2.$$

The points q_1, q_2 *and the sets* Ψ_1, Ψ_2 *can be chosen independently of* $\eta \in (0, \beta)$.

Lemma 5.4. *For* $u, u_0 \in S^{n-1}$ *with* $\langle u, u_0 \rangle \geq 0$, *we have* $V(\Xi_{u, u_0}) \geq \kappa_n / 240^n$, *where* κ_n *is the n-dimensional measure of the unit ball in* $\mathbb{R}^n$, *and*

$$\Xi_{u, u_0} = \left\{ y \in 0.1B^n : \langle y, u \rangle \geq \frac{1}{30}, \langle y, u_0 \rangle \geq \frac{1}{30}, \langle y, u - u_0 \rangle \geq \frac{\|u - u_0\|}{120} \right\}. \tag{5.1}$$

Lemma 5.5. *If* $b_1, \cdots, b_n \in S^{n-1}$, *and* $s_1, \cdots, s_n \in \mathbb{R}^n$ *satisfy* $\|s_i\| \leq |\det[b_1, \cdots, b_n]| / 4n$, *then*

$$|\det[b_1 + s_1, \cdots, b_n + s_n]| \geq |\det[b_1, \cdots, b_n]| / 2.$$

Lemma 5.6. *Let* $n \geq 2$, *let* $t \in \left(0, \frac{1}{4^n n!}\right)$, *and let* $u_1, \cdots, u_n \in S^{n-1}$. *If*

$$\Omega\left(u, \operatorname{arecos}\left(\frac{1}{\sqrt{n}} - t\right)\right) \cap \{\pm u_1, \cdots, \pm u_n\} \neq \emptyset,$$

for any $u \in S^{n-1}$, *then there exists a cross measure* ν *such that*

$$\delta_H(\operatorname{supp}\nu, \{\pm u_1, \cdots, \pm u_n\}) \leq 4^n n! \cdot t.$$

Lemma 5.7. *Let* μ *be an even isotropic measure, and let* ν *be a cross measure on* S^{n-1} *with* $\operatorname{supp}\nu = \{\pm w_1, \cdots, \pm w_n\}$. *If* $\delta \in \left[0, \frac{\pi}{4}\right)$ *and* $w \in [0, 1)$ *are such that*

$$\mu\left(S^{n-1} \setminus \bigcup_{i=1}^{n} (\Omega(w_i, \delta) \cup \Omega(-w_i, \delta))\right) \le \omega,$$

then

$$\delta_W(u, \nu) \le 2n\delta + 2\pi n^2 \omega.$$

Suppose that $\alpha : S^{n-1} \to (0, +\infty)$ is an even positive bounded continuous function and

$$\alpha_M = \max_{u \in S^{n-1}} \alpha(u), \ \alpha_m = \min_{u \in S^{n-1}} \alpha(u).$$

Meanwhile, let

$$\delta_\alpha = \min_{1 \le i < j \le k} \{|\alpha(u_i) - \alpha(u_j)| : u_i, u_j \in S^{n-1}\},$$

$$\gamma_1 = \min_{u \in S^{n-1}} \{1, e^{1-\alpha(u)}\} = \begin{cases} 1, & \text{if } 0 < \alpha(u) \le 1, \\ e^{1-\alpha(u)}, & \text{if } \quad \alpha(u) > 1, \end{cases}$$

and $\gamma_2 = \min\{1, \delta_\alpha\}$ as well as $\gamma_3 = \min\{1, \alpha_m\}$. Clearly, $\gamma_1, \gamma_2, \gamma_3 \le 1$.

Proposition 5.8. *If* $p \in [1, \infty) \setminus \{2\}$, μ *is an even discrete isotropic measure on* S^{n-1}, *and*

$$V(Z_{p,\alpha}^*(\mu)) \ge (1 - \varepsilon) V(Z_{p,\alpha}^*(\nu_n)),$$

for some $\varepsilon \in (0, 1)$, *then there exists a cross measure* ν *on* S^{n-1} *such that*

$$\delta_W(\mu, \nu) \le n^{cn^3} \max\left\{|p - 2|^{-\frac{2}{3}}, 1\right\} \cdot \varepsilon^{\frac{1}{3}},$$

for some absolute constant $c > 0$.

Proof. Next, we will prove that for any $0 < \eta < \beta^n/(2n)$,

$$V(Z^*_{p,\alpha}(\mu)) < \left(1 - n^{cn^3} \min\{(p-2)^2, 1\} \cdot \eta^3 \cdot \mathcal{C}_\alpha\right) V(Z^*_{p,\alpha}(\nu_n)), \tag{5.2}$$

or there exists a cross measure ν satisfying

$$\delta_W(\mu, \nu) \le n^{cn}\eta, \tag{5.3}$$

for some absolute constant $c > 0$. Here $\mathcal{C}_\alpha = \mathcal{C}_{\alpha,1}\mathcal{C}_{\alpha,2} \le 1$ and

$$\mathcal{C}_{\alpha,1} = \left(\frac{\alpha_m}{\alpha_M}\right)^2 \cdot \left(\frac{\gamma_1\gamma_2}{\exp\left(\frac{\alpha_M}{e^{2(1-\alpha_m^{-1})}} - 1\right)}\right)^2 \le 1,$$

$$\mathcal{C}_{\alpha,2} = \left(\frac{\gamma_3}{\exp\left(\frac{\pi\alpha_M^2}{e^{2(1-\alpha_m^{-1})}} + \alpha_M - \pi - 1\right)}\right)^n \le 1.$$

Without loss of generality, suppose that μ is discrete, $\operatorname{supp}\mu = \{\overline{u}_1, \cdots, \overline{u}_{\overline{k}}\}$ and $\overline{c}_i = \mu(\{\overline{u}_i\})$. For $c_0 = \min\{\overline{c}_i : i = 1, \cdots, \overline{k}\}$ and $i = 1, \cdots, \overline{k}$, we define $\overline{m}_i = \min\{m \in \mathbb{Z} : m \ge 1 \text{ and } \overline{c}_i / m \le c_0\}$, and let $k = \sum_{i=1}^{\overline{k}} \overline{m}_i$. We consider $\xi : \{1, \cdots, k\} \to \{1, \cdots, \overline{k}\}$ such that $\#\xi^{-1}(\{i\}) = \overline{m}_i$ for $i = 1, \cdots, \overline{k}$, and define

$$u_i = \overline{u}_{\xi(i)} \text{ and } c_i = \overline{c}_{\xi(i)} / \overline{m}_{\xi(i)},$$

for $i = 1, \cdots, k$. The system $(u_1, \cdots, u_k, c_1, \cdots, c_k)$ is even (i.e., origin symmetric) in the following sense: Any $u \in S^{n-1}$ occurs as u_i exactly as many times as $-u$, and if $u_i = -u_j$, then $c_i = c_j$.

In particular, $\sum_{i=1}^{k} c_i u_i \otimes u_i = \mathrm{Id}_n$ and $\sum_{i=1}^{k} c_i = n$, and for any Borel $X \subseteq S^{n-1}$, we have

$$\mu(X) = \sum_{u_i \in X} c_i.$$

The reason for the renormalization is that

$$\frac{1}{2} c_0 < c_i \le c_0 \quad \text{ for } i = 1, \cdots, k. \tag{5.4}$$

Moreover, let $\varphi_u = \varphi_{p,u}$ denoted in (4.1), $g(t) = e^{-\pi t^2}$, f_i be defined as in (3.7), for $i = 1, \cdots, k$. Then from (3.5), it follows that

$$\frac{\alpha(u_i)^{\frac{1}{p}}}{2\Gamma\left(1 + \frac{1}{p}\right)} \int_{-\infty}^{t} e^{-\alpha(u_i)|t|^p} ds = \int_{-\infty}^{\alpha(u_i)\varphi_u(t)} e^{-\pi t^2} ds,$$

namely,

$$\frac{\alpha(u_i)^{\frac{1}{p}}}{2\Gamma\left(1 + \frac{1}{p}\right)} e^{-\alpha(u_i)|t|^p} = e^{-\pi(\alpha(u_i)\varphi_u(t))^2} \varphi'(t)\varphi(u_i). \tag{5.5}$$

Taking the log of both sides of (5.5), we have

$$-\log\Gamma\left(1 + \frac{1}{p}\right) - \log 2 + \frac{1}{p}\log\alpha(u_i) - \alpha(u_i)|t|^p = -\pi(\alpha(u)\varphi_u(t))^2 + \log(\varphi_u'(t)\alpha(u)). \tag{5.6}$$

We define $T : \mathbb{R}^n \to \mathbb{R}^n$ by

$$T(y) = \sum_{i=1}^{k} c_i \varphi_u(\langle y, u_i \rangle)\alpha(u_i)u_i, \tag{5.7}$$

for each $y \in \mathbb{R}^n$. The differential of T is given by

$$dT(y) = \sum_{i=1}^{k} c_i u_i \otimes u_i \varphi_u'(\langle y, u_i \rangle)\alpha(u_i). \tag{5.8}$$

Since $\varphi_u' > 0$ and $\alpha > 0$, the matrix $dT(y)$ is positive definite for each $x \in \mathbb{R}^n$. Therefore, the transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is injective.

By (3.3), (1.10), (3.8) and (5.6), we have

$$V(Z_{p,\alpha}^*(\mu))$$

$$= \frac{1}{\Gamma\left(1+\frac{n}{p}\right)} \int_{\mathbb{R}^n} e^{-\|x\|^p_{Z_{p,\alpha}^*}} dx = \frac{1}{\Gamma\left(1+\frac{n}{p}\right)} \int_{\mathbb{R}^n} e^{-\sum_{i=1}^{k} c_i \alpha(u_i)|\langle x, u_i \rangle|^p} dx$$

$$= \frac{2^n \Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma\left(1+\frac{n}{p}\right)} \left(\prod_{i=1}^{k} \alpha(u_i)^{c_i}\right)^{-\frac{1}{p}}$$

$$\times \int_{\mathbb{R}^n} \exp\left\{-\pi \sum_{i=1}^{k} c_i (\alpha(u_i)\varphi_u(\langle x, u_i \rangle))^2 + \sum_{i=1}^{k} \log(\varphi_u'(\langle x, u_i \rangle)\alpha(u_i))^{c_i}\right\} dx$$

$$= \frac{2^n \Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma\left(1+\frac{n}{p}\right)} \left(\prod_{i=1}^{k} \alpha(u_i)^{c_i}\right)^{-\frac{1}{p}}$$

$$\times \int_{\mathbb{R}^n} \exp\left(-\pi \sum_{i=1}^{k} c_i (\alpha(u_i)\varphi_u(\langle x, u_i \rangle))^2\right) \left(\prod_{i=1}^{k} (\varphi_u'(\langle x, u_i \rangle)\alpha(u_i))^{c_i}\right) dx. \tag{5.9}$$

For each fixed $x \in \mathbb{R}^n$, we estimate the product of the two terms in (5.9) in the integral sign. To estimate the first term in (5.9), we use (2.11) with $\theta_i = \varphi_u(\langle y, u_i \rangle)\alpha(u_i)$, $i = 1, \cdots, k$. Hence the definition of T gives

$$\exp\left(-\pi\sum_{i=1}^{k} c_i(\alpha(u_i)\varphi_u(\langle y, u_i\rangle))^2\right) \le e^{-\pi\|T(y)\|^2}. \tag{5.10}$$

To estimate the second term, we use Lemma 2.2 with $v_i = \sqrt{c_i} \cdot u_i$ and $t_i = \varphi_u'(\langle y, u_i\rangle)\alpha(u_i)$, at each $y \in \mathbb{R}^n$, and write $\theta^*(y)$ and $t_0(y)$ to denote the corresponding $\theta^* \ge 1$ and $t_0 > 0$. In particular, if $\{i_1, \cdots, i_n\} \subseteq \{1, \cdots, k\}$ and $y \in \mathbb{R}^n$, then we assume

$$\mathbf{N}(i_1, \cdots, i_n;\ y)$$

$$= c_{i1} \cdots c_{in} \det[u_{i1}, \cdots, u_{in}]^2 \left(\frac{\sqrt{\varphi'(\langle y, u_{i_1}\rangle)\cdots\varphi'(\langle y, u_{i_n}\rangle)\alpha(u_{i_1})\cdots\alpha(u_{i_n})}}{t_0(y)} - 1\right)^2. \tag{5.11}$$

Therefore, for

$$\theta^*(y) = 1 + \frac{1}{2}\sum_{1\le i_1<\cdots<i_n\le k} \mathbf{N}(i_1, \cdots, i_n;\ y). \tag{4.12}$$

Lemma 2.2 gives

$$\prod_{i=1}^{k}(\varphi_u'(\langle y, u_i\rangle)\alpha(u_i))^{c_i} \le \theta^*(y)^{-1}\det(dT(y)). \tag{5.13}$$

From (5.9), (5.10) and (5.13), we obtain

$$V(Z_{p,\alpha}^*(\mu)) \le \frac{2^n\Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma\left(1+\frac{n}{p}\right)}\left(\prod_{i=1}^{k}\alpha(u_i)^{c_i}\right)^{-\frac{1}{p}}\int_{\mathbb{R}^n}\theta^*(y)^{-1}e^{-\pi\|T(y)\|^2}\det(dT(y))dy. \tag{5.14}$$

To give a lower bound for $\theta^*(y)$, using (4.8) and (4.7) to obtain

$$\frac{1}{3.1e^{\alpha(u)-1}} < \varphi_u'(t) < 3.1e^{\alpha(u)e^{2(\alpha(u)^{-1}-1)}-1}, \tag{5.15}$$

and

$$\varphi_u(t) < \frac{1}{e^{1-\alpha(u)^{-1}}}, \tag{5.16}$$

for $p \in [1, \infty]$ and $t \in [0, \frac{1}{3.1}]$.

We consider the vectors $v_1, \cdots, v_n \in S^{n-1}$ provided by Lemma 5.2 such that

$$\mu(\Omega(v_i, \beta)) > \beta^n \text{ for } i = 1, \cdots, n;$$

$$|\det[w_1, \cdots, w_n]| \geq 2n\beta \text{ for } w_i \in \Omega(v_i, \beta) \text{ and } i \in \{1, \cdots, n\}, \tag{5.17}$$

$$\beta = 2^{-(n+1)} n^{-(n+1)/2}.$$

The remaining discussion is split into three cases, where the first two correspond to the two cases in Lemma 5.3.

Case 1. There exist $l \in \{1, \cdots, n\}$ and $\Psi_1, \Psi_2 \subseteq \Omega(v_l, \beta)$ such that

$$\mu(\Psi_j) \geq \frac{\beta^n}{4n} \text{ for } j = 1, 2, \text{ and}$$

$$\|a_1 - a_2\| \geq \frac{\eta}{\sqrt{n}} \text{ for } a_1 \in \Psi_1 \text{ and } a_2 \in \Psi_2.$$

In this case, we prove

$$V(Z_{p,\alpha}^*(\mu)) \leq \frac{2^n \Gamma\left(1 + \frac{1}{p}\right)^n}{\Gamma\left(1 + \frac{n}{p}\right)} \left(\prod_{i=1}^{k} \alpha(u_i)^{c_i}\right)^{-\frac{1}{p}} \left(1 - n^{-cn^3} \min\{(p-2)^2, 1\} \cdot \eta^2 \cdot \mathcal{C}_\alpha\right), \tag{5.18}$$

for some absolute constant $c > 0$.

We may let that $l = n$ and

$$\prod_j = \{i \in \{1, \cdots, k\} : u_i \in \Psi_j\} \neq \emptyset,$$

for $j = 1, 2$. Possibly after interchanging the roles of Ψ_1 and Ψ_2 we may assume that $\# \prod_1 \leq \# \prod_2$. Let

$$\tau : \prod_1 \to \prod_2,$$

be an injective map. Given $u_{i_j} \in \Omega(v_j, \beta)$ for $j = 1, \cdots, n-1$ and $u_{i_n} \in \Psi_1$, we have $u_{\tau(i_n)} \in \Psi_2$, together (5.4) with (5.17) yields

$$\left.\begin{array}{r} c_{i_1} \cdots c_{i_{n-1}} \cdot c_{i_n} \det[u_{i_1}, \cdots, u_{i_n}]^2 \\ c_{i_1} \cdots c_{i_{n-1}} \cdot c_{\tau(i_n)} \det[u_{i_1}, \cdots, u_{i_{n-1}}, u_{\tau(i_n)}]^2 \end{array}\right\} \geq 4n^2\beta^2 c_{i_1} \cdots c_{i_{n-1}} \cdot (c_{i_n} / 2). \tag{5.19}$$

Since $\beta < \pi / 4$, we have $\langle u_{i_n}, u_{\tau(i_n)} \rangle > 0$ if $u_{i_n} \in \Psi_1$. Lemma 5.4 shows that $V(\Xi_{u, u_0}) \geq \kappa_n / 240^n$ for $u, u_0 \in S^{n-1}$ with $\langle u, u_0 \rangle \geq 0$, where Ξ_{u, u_0} is defined in (5.1). In particular, if $y \in \Xi_{u_{i_n}, u_{\tau(i_n)}}$, then

$$\langle y, u_{in} \rangle, \langle y, u_{\tau(i_n)} \rangle < \frac{1}{8}, \text{ and}$$

$$\langle y, u_{i_n} \rangle - \langle y, u_{\tau(i_n)} \rangle = \langle y, u_{i_n} - u_{\tau(i_n)} \rangle \geq \frac{\eta}{120\sqrt{n}}.$$

In order to simplicity, we still set $\varphi := \varphi_p = \varphi_{p, u}$. Notice that φ'' is continuous, and Lemma 4.2 obtains that if $t \in [\frac{1}{30}, 0.1]$ and $u \in S^{n-1}$, then

$$|\varphi''(t)| \geq \begin{cases} \dfrac{|p-2|}{48}\left(\dfrac{1}{30}\right)^{1.3} \cdot \alpha(u)\gamma_1 > \dfrac{|p-2|}{2^{12}} \cdot \alpha(u)\gamma_1, & \text{if } p \in [1, 3] \setminus \{2\}, \\ 0.2\left(\dfrac{1}{30}\right)^{1.3} \cdot \alpha(u) > 2^{-9} \cdot \alpha(u), & \text{if } p > 3. \end{cases} \tag{5.20}$$

Thus, using the mean-value theorem for derivatives to obtain

$$|\varphi'(\langle y, u_{i_n}\rangle) - \varphi'(\langle y, u_{\tau(i_n)}\rangle)|$$

$$\geq \begin{cases} \dfrac{|p-2|}{2^{12}120\sqrt{n}}\eta \cdot \alpha(u)\gamma_1 > \dfrac{|p-2|}{2^{19}\sqrt{n}}\eta \cdot \alpha_m\gamma_1, & \text{if } p \in [1, 3] \setminus \{2\}, \\ \dfrac{1}{2^9 120\sqrt{n}}\eta \cdot \alpha(u) > \dfrac{1}{2^{19}\sqrt{n}}\eta \cdot \alpha_m, & \text{if } p > 3. \end{cases}$$

The following algebraic inequality ([43], p. 162) will be useful. If $x, y \geq 0$ and $p \geq 1$, then

$$|x - y|^p \leq |x^p - y^p|. \tag{5.21}$$

From Lemma 2.3 and $0 < \varphi'(t) < 3.1e^{\alpha(u)e^{2(\alpha(u)^{-1}-1)}-1}$ for $p \in [1, \infty) \setminus \{2\}$ and $t \in (0, 0.1]$ (cf. (5.15) and (5.16)), Aczel inequality (2.12) and an algebraic inequality (5.21), it follows that

$$\left(\frac{\sqrt{\varphi'(\langle y, u_{i_1}\rangle)\cdots\varphi'(\langle y, u_{i_{n-1}}\rangle)\cdot\varphi'(\langle y, u_{i_n}\rangle)\alpha(u_{i_1})\cdots\alpha(u_{i_{n-1}})\alpha(u_{i_n})}}{t_0(y)} - 1\right)^2$$

$$+\left(\frac{\sqrt{\varphi'(\langle y, u_{i_1}\rangle)\cdots\varphi'(\langle y, u_{i_{n-1}}\rangle)\cdot\varphi'(\langle y, u_{\tau(i_n)}\rangle)\alpha(u_{i_1})\cdots\alpha(u_{i_{n-1}})\alpha(u_{\tau(i_n)})}}{t_0(y)} - 1\right)^2$$

$$\geq \frac{\left(\varphi'(\langle y, u_{i_n}\rangle)\alpha(u_{i_n}) - \varphi'(\langle y, u_{\tau(i_n)}\rangle)\alpha(u_{\tau(i_n)})\right)^2}{2\left(\varphi'(\langle y, u_{i_n}\rangle)\alpha(u_{i_n}) + \varphi'(\langle y, u_{\tau(i_n)}\rangle)\alpha(u_{\tau(i_n)})\right)^2}$$

$$\geq \frac{\left|\varphi'(\langle y, u_{i_n}\rangle)^2 - \varphi'(\langle y, u_{\tau(i_n)}\rangle)^2\right|\,|\alpha(u_{i_n})^2 - \alpha(u_{\tau(u_{i_n})})^2|}{2\alpha_M^2\left(\varphi'(\langle y, u_{i_n}\rangle) + \varphi'(\langle y, u_{\tau(i_n)}\rangle)\right)^2}$$

$$\geq \frac{\left(\varphi'(\langle y, u_{i_n}\rangle) - \varphi'(\langle y, u_{\tau(i_n)}\rangle)\right)^2 |\alpha(u_{i_n}) - \alpha(u_{\tau(u_{i_n})})|^2}{2\alpha_M^2\left(\varphi'(\langle y, u_{i_n}\rangle) + \varphi'(\langle y, u_{\tau(i_n)}\rangle)\right)^2}$$

$$\geq \frac{\min\{1, (p-2)^2\}}{2^{45} n} \cdot \eta^2 \cdot \left(\frac{\alpha_m}{\alpha_M}\right)^2 \cdot \left(\frac{\gamma_1\gamma_2}{\exp\left(\frac{\alpha_M}{e^{2(1-\alpha_m^{-1})}} - 1\right)}\right)^2.$$

Notice that

$$\mathcal{C}_{\alpha,1} = \left(\frac{\alpha_m}{\alpha_M}\right)^2 \cdot \left(\frac{\gamma_1\gamma_2}{\exp\left(\frac{\alpha_M}{e^{2(1-\alpha_m^{-1})}} - 1\right)}\right)^2 \leq 1.$$

Together this estimate with (5.11) and from (5.19), we obtain that if $p \in [1, \infty)\setminus\{2\}$ and $u_{i_j} \in \Omega(v_j, \beta)$ for $j = 1, \cdots, n-1$, $u_{i_n} \in \Psi_1$, and $y \in \Xi_{u_{i_n}, u_{\tau(i_n)}}$, then

$$N(i_1, \cdots, i_{n-1}, i_n; y) + N(i_1, \cdots, i_{n-1}, \tau(i_n); y)$$

$$\geq 2n^2\beta^2 c_{i_1} \cdots c_{i_{n-1}} \cdot c_{in} \frac{\min\{1, (p-2)^2\}}{2^{45} n} \cdot \eta^2 \cdot \mathcal{C}_{\alpha,1}.$$

If $u_{i_n} \in \Psi_1$ and $y \in \mathbb{R}^n$, then we define

$$\varrho(i_n, \alpha; y) = \begin{cases} 0, & \text{if } y \notin \Xi_{u_{i_n}, u_{\tau(i_n)}}; \\ \dfrac{\beta^2 n(p-2)^2}{2^{44}} \eta^2 \cdot \mathcal{C}_{\alpha,1}, & \text{if } y \in \Xi_{u_{i_n}, u_{\tau(i_n)}} \text{ and } p \in [1, 3]\setminus\{2\}; \\ \dfrac{\beta^2 n}{2^{44}} \eta^2 \cdot \mathcal{C}_{\alpha,1}, & \text{if } y \in \Xi_{u_{i_n}, u_{\tau(i_n)}} \text{ and } p > 3. \end{cases}$$

In particular, if $u_{i_j} \in \Omega(v_j, \beta)$ for $j = 1, \cdots, n-1$, $u_{i_n} \in \Psi_1$ and $y \in \mathbb{R}^n$, then

$$N(i_1, \cdots, i_{n-1}, i_n; y) + N(i_1, \cdots, i_{n-1}, \tau(i_n); y) \geq c_{i_1} \cdots c_{i_n} \varrho(i_n, \alpha; y). \tag{5.22}$$

Substituting (5.22) into (5.12), and then by (5.17), we know that if $y \in \mathbb{R}^n$, then

$$\begin{aligned} \theta^*(y) &\geq 1 + \frac{1}{2} \sum_{u_{i_j} \in \Omega(v_j, \beta), j=1,\cdots,n-1} c_{i_1} \cdots c_{i_{n-1}} \cdot c_{i_n} \varrho(i_n, \alpha; y) \\ &= 1 + \frac{1}{2} \left(\prod_{j=1}^{n-1} \mu(\Omega(v_j, \beta)) \right) \sum_{u_{i_n} \in \Psi_1} c_{i_n} \varrho(i_n, \alpha; y) \\ &\geq 1 + \frac{\beta^{n(n-1)}}{2} \sum_{u_{i_n} \in \Psi_1} c_{i_n} \varrho(i_n, \alpha; y). \end{aligned}$$

Here

$$\frac{\beta^{n(n-1)}}{2} \sum_{u_{i_n} \in \Psi_1} c_{i_n} \varrho(i_n, \alpha; y) \leq \frac{\beta^{n(n-1)}}{2} \mu(\Psi_1) \frac{\beta^2 n}{2^{44}} \cdot \eta^2 \cdot \mathcal{C}_{\alpha,1} < 1.$$

Thus if $x \in \mathbb{R}^n$, then

$$\theta^*(y)^{-1} \leq 1 - \frac{\beta^{n(n-1)}}{2} \sum_{u_{i_n} \in \Psi_1} c_{i_n} \varrho(i_n, \alpha; y). \tag{5.23}$$

From (5.14) and (5.23), we will obtain

$$V(Z^*_{p,\alpha}(\mu))$$

$$\leq \frac{2^n \Gamma\left(1 + \frac{1}{p}\right)^n}{\Gamma\left(1 + \frac{n}{p}\right)} \left(\prod_{i=1}^{k} \alpha(u_i)^{c_i} \right)^{-\frac{1}{p}} \int_{\mathbb{R}^n} \theta^*(y)^{-1} e^{-\pi \|T(y)\|^2} \det(dT(y)) dy$$

$$\leq V(Z^*_{p,\alpha}(\nu_n))\left(\prod_{i=1}^{k}\alpha(u_i)^{c_i}\right)^{-\frac{1}{p}}$$

$$\times\int_{\mathbb{R}^n}\left(1-\frac{\beta^{n(n-1)}}{4}\sum_{u_{i_n}\in\Psi_1}c_{i_n}\varrho(i_n,\ \alpha;\ y)\right)e^{-\pi\|T(y)\|^2}\ \det(dT(y))dy$$

$$= V(Z^*_p(\nu_n))\left(\prod_{i=1}^{k}\alpha(u_i)^{c_i}\right)^{-\frac{1}{p}}\int_{\mathbb{R}^n}e^{-\pi\|T(y)\|^2}\ \det(dT(y))dx$$

$$- V(Z^*_p(\nu_n))\left(\prod_{i=1}^{k}\alpha(u_i)^{c_i}\right)^{-\frac{1}{p}}\frac{\beta^{n(n-1)}}{4}\sum_{u_{i_n}\in\Psi_1}c_{i_n}\int_{\mathbb{R}^n}\varrho(i_n,\ \alpha;\ y)\,e^{-\pi\|T(y)\|^2}$$

$$\times\ \det(dT(y))dy.$$

Thus

$$\int_{\mathbb{R}^n}e^{-\pi\|T(y)\|^2}\ \det(dT(y))dy\leq\int_{\mathbb{R}^n}e^{-\pi\|z\|^2}dz = 1. \tag{5.24}$$

If $y\in\Xi_{i_n,\tau(i_n)}$, (in), then (5.10), (5.12), (5.13), (5.15) and (5.16) obtain that

$$e^{-\pi\|T(y)\|^2}\ \geq\ \exp\left(-\pi\sum_{i=1}^{k}c_i(\alpha(u_i)\varphi_u(\langle y,\ u_i\rangle))^2\right)$$

$$>\ \exp\left(-\frac{\pi\alpha_M^2}{e^2(1-\alpha_m^{-1})}\sum_{i=1}^{k}c_i\right)$$

$$= e^{-\frac{\pi n\alpha_M^2}{e^2(1-\alpha_m^{-1})}}, \tag{5.25}$$

$$\det(dT(y)) \geq \prod_{i=1}^{k}(\varphi_u'(\langle y,\, u_i\rangle)\alpha(u_i))^{c_i}$$

$$> \prod_{i=1}^{k}\left(3.1^{-1}e^{1-\alpha(u)}\right)^{c_i}\prod_{i=1}^{k}\alpha(u_i)^{c_i}$$

$$= 3.1^{-n}e^{n(1-\alpha(u))}\prod_{i=1}^{k}\alpha(u_i)^{c_i}$$

$$> 3.1^{-n}e^{n(1-\alpha_M)}\alpha_m^n$$

$$\geq 3.1^{-n}e^{n(1-\alpha_M)}\gamma_3^n. \tag{5.26}$$

Therefore

$$V(Z_{p,\alpha}^*(\mu)) \leq \frac{2^n\Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma\left(1+\frac{n}{p}\right)}\left(\prod_{i=1}^{k}\alpha(u_i)^{c_i}\right)^{-\frac{1}{p}}$$

$$\times\left(1-\sum_{u_{i_n}\in\Psi_1}c_{i_n}\frac{\beta^{n(n-1)}}{4}\cdot\frac{V(\Xi_{i_n,\tau(i_n)})}{(3.1e^{\pi})^n}\cdot\frac{\beta^2 n\min\{(p-2)^2,\,1\}}{2^{44}}\cdot\eta^2\cdot\mathcal{C}_\alpha\right),$$

where $\mathcal{C}_\alpha = \mathcal{C}_{\alpha,1}\mathcal{C}_{\alpha,2} \leq 1$. Since $V(\Xi_{i_n,\tau(i_n)}) \geq \kappa_n / 240^n$, if $u_{i_n} \in \Psi_1$, according to Lemma 5.4, and

$$\sum_{u_{i_n}\in\Psi_1}c_{i_n} = \mu(\Psi_1) > \frac{\beta^n}{4n},$$

we give (5.18).

Case 2. There exists $q_i \in \Omega(v_i,\, \beta)$, for $i = 1, \cdots, n,$ such that

$$\mu(\Omega(q_i,\, \eta)) \geq \frac{\beta^n}{4n} \text{ for } i = 1, \cdots, n, \tag{5.27}$$

and

$$\mu\left(\bigcup_{i=1}^{n} (\Omega(q_i, 2\eta) \cup \Omega(-q_i, 2\eta))\right) \le n - \eta. \tag{5.28}$$

In this case, we show

$$V(Z_{p,\alpha}^{*}(\mu)) < \frac{2^n \Gamma\left(1 + \frac{1}{p}\right)^n}{\Gamma\left(1 + \frac{n}{p}\right)} \left(1 - n^{-cn^3} \min\{(p-2)^2, 1\} \cdot \eta^2 \cdot \mathcal{C}_\alpha\right), \tag{5.29}$$

for some absolute constant $c > 0$. The argument is very similar to the one in Case 1.

Let

$$\widetilde{\Psi} = S^{n-1} \setminus \left(\bigcup_{i=1}^{n} (\Omega(q_i, 2\eta) \cup \Omega(-q_i, 2\eta))\right).$$

It follows from (5.17) that any $x \in \mathbb{R}^n$ can be written in the form

$$x = \sum_{i=1}^{n} \lambda_i(x) q_i.$$

Since $\mu(\widetilde{\Psi}) \le \eta$ by (5.28), the triangle inequality ensures that there exists some $i \in \{1, \cdots, n\}$ satisfying $|\lambda_i(x)| \ge 1/n$. Thus we may reindex $q_1, \cdots, q_n$ in such a way that

$$\mu(\Psi) \ge \frac{\eta}{n} \text{ for } \Psi = \{x \in \widetilde{\Psi} : |\lambda_n(x)| \ge 1/n\}. \tag{5.30}$$

From (5.17), we obtain that if $x \in \Psi$, then

$$|\det[q_1, \cdots, q_{n-1}, x]| \ge |\det[q_1, \cdots, q_{n-1}, q_n]|/n \ge 2\beta.$$

In the following, for $u_{i_j} \in \Omega(q_j, \eta)$ for $j = 1, \cdots, n-1$, from Lemma 5.5 with $b_l = q_l$, $s_l = u_{i_l} - q_l$ for $l = 1, \cdots n-1$, $b_n = x \in \Psi$, and $s_n = 0$, where

$$|s_i| \leq \eta \leq \frac{\beta}{2n} = \frac{2\beta}{4n} \leq \frac{1}{4n} |\det[q_1, \cdots, q_{n-1}, x]|, \; i = 1, \cdots, n,$$

we get

$$|\det[u_{i_1}, \cdots, u_{i_{n-1}}, x]| \geq \frac{1}{2} |\det[q_1, \cdots, q_{n-1}, x]| \geq \beta. \tag{5.31}$$

We observe that $\Psi = -\Psi$. Hence, for

$$\prod_2 = \{i \in \{1, \cdots, k\} : u_i \in \Psi\},$$

there exists $\prod' \subseteq \prod_2$ with $\# \prod' = \frac{1}{2} \# \prod_2$, and a bijection $\tilde{\tau} : \prod' \to \prod_2 \setminus \prod'$ such that if $i \in \prod'$, then $u_{\tilde{\tau}(i)} = -u_i$.

Since $\eta < \beta^n$, (5.27) gets

$$\sum_{u_i \in \Omega(q_n, \eta)} c_i = \mu(\Omega(q_n, \eta)) \geq \frac{\beta^n}{4n} \geq \frac{\eta}{8n}.$$

This implies that we can find a minimal (with respect to inclusion) set $\prod_1 \subseteq \{1, \cdots, k\}$ such that $u_i \in \Omega(q_n, \eta)$ for $i \in \prod_1$ and

$$\sum_{i \in \prod_1} c_i \geq \frac{\eta}{8n}. \tag{5.32}$$

By minimality and (5.4), we have

$$\frac{c_0}{2} (\# \prod_1 - 1) \leq \frac{\eta}{8n}.$$

Further, by (5.30) and again by (5.4), it follows that

$$c_0 \# \prod_2 \geq \sum_{j \in \prod_2} c_j \geq \frac{\eta}{n}.$$

Thus

$$\frac{c_0}{8}\#\prod_2 \ \geq \frac{c_0}{2}(\#\prod_1 - 1),$$

which yields $\#\prod_2 \geq 4(\#\prod_1 - 1)$ if $\#\prod_1 \geq 2$. In any case, we deduce that $\#\prod_2 \geq 2\#\prod_1$.

We prove that there exists an injective map $\tau : \prod_1 \to \prod_2$ such that if $i \in \prod_1$, then

$$\langle u_i, u_{\tau(i_i)}\rangle \geq 0. \tag{5.33}$$

In addition, if $i \in \prod_1$, then $u_i \in \Omega(q_n, \eta)$ and $u_{\tau(i)} \notin \Omega(q_n, 2\eta)$, thus

$$\|u_i - u_{\tau(i)}\| \geq \frac{\eta}{2}.$$

Given $u_{i_j} \in \Omega(q_j, \eta)$ for $j = 1, \cdots, n-1$ and $i_n \in \prod_1$, we have $\tau(i_n) \in \prod_2$. By (5.4), (5.17) and (5.31), we obtain

$$\left.\begin{array}{r} c_{i_1}\cdots c_{i_{n-1}}\cdot c_{i_n}\det[u_{i_1}, \cdots, u_{i_n}]^2 \\ c_{i_1}\cdots c_{i_{n-1}}\cdot c_{\tau(i_n)}\det[u_{i_1}, \cdots, u_{i_{n-1}}, u_{\tau(i_n)}]^2 \end{array}\right\} \geq \beta^2 c_{i_1}\cdots c_{i_{n-1}}\cdot(c_{i_n}/2).$$

(5.34)

By (5.33), we have that Lemma 5.4 applies to $\Xi_{u_{i_n}, u_{\tau(i_n)}}$. In particular, we obtain $V(\Xi_{i_n, \tau(i_n)}) \geq \kappa_n / 240^n$, and if $y \in \Xi_{i_n, \tau(i_n)}$, then

$$\langle y, u_{i_n}\rangle, \langle y, u_{\tau(i_n)}\rangle < \frac{1}{8}, \text{ and}$$

$$\langle y, u_{i_n}\rangle - \langle y, u_{\tau(i_n)}\rangle = \langle y, u_{i_n} - u_{\tau(i_n)}\rangle \geq \frac{\eta}{240} > \frac{\eta}{2^8}.$$

From (5.20), we have

$$|\varphi'(\langle y, u_{i_n}\rangle) - \varphi'(\langle y, u_{\tau(i_n)}\rangle)| > \frac{\min\{|p-2|, 1\}}{2^{20}} \cdot \alpha(u)\gamma_1\eta.$$

If $i_n \in \prod_1$, from Lemma 2.3, $0 < \varphi'(t) < 3.1e^{\alpha(u)e^{2(\alpha(u)^{-1}-1)}-1}$ for $p \in [1, \infty)\setminus\{2\}$ and $t \in (0, 0.1]$ (cf. (5.15) and (5.16)), Aczel inequality (2.3) and a simple algebraic inequality (5.21), we obtain

$$\left(\frac{\sqrt{\varphi'(\langle y, u_{i_1}\rangle)\cdots\varphi'(\langle y, u_{i_{n-1}}\rangle)\cdot\varphi'(\langle y, u_{i_n}\rangle)\alpha(u_{i_1})\cdots\alpha(u_{i_{n-1}})\alpha(u_{i_n})}}{t_0(y)} - 1\right)^2$$

$$+\left(\frac{\sqrt{\varphi'(\langle y, u_{i_1}\rangle)\cdots\varphi'(\langle y, u_{i_{n-1}}\rangle)\cdot\varphi'(\langle y, u_{\tau(i_n)}\rangle)\alpha(u_{i_1})\cdots\alpha(u_{i_{n-1}})\alpha(u_{\tau(i_n)})}}{t_0(y)} - 1\right)^2$$

$$\geq \frac{\left(\varphi'(\langle y, u_{i_n}\rangle)\alpha(u_{i_n}) - \varphi'(\langle y, u_{\tau(i_n)}\rangle)\alpha(u_{\tau(i_n)})\right)^2}{2\left(\varphi'(\langle y, u_{i_n}\rangle)\alpha(u_{i_n}) + \varphi'(\langle y, u_{\tau(i_n)}\rangle)\alpha(u_{\tau(i_n)})\right)^2}$$

$$\geq \frac{\left|\varphi'(\langle y, u_{i_n}\rangle)^2 - \varphi'(\langle y, u_{\tau(i_n)}\rangle)^2\right| |\alpha(u_{i_n})^2 - \alpha(u_{\tau(u_{i_n})})^2|}{2\alpha_M^2\left(\varphi'(\langle y, u_{i_n}\rangle) + \varphi'(\langle y, u_{\tau(i_n)}\rangle)\right)^2}$$

$$\geq \frac{\left(\varphi'(\langle y, u_{i_n}\rangle) - \varphi'(\langle y, u_{\tau(i_n)}\rangle)\right)^2 |\alpha(u_{i_n}) - \alpha(u_{\tau(u_{i_n})})|^2}{2\alpha_M^2\left(\varphi'(\langle y, u_{i_n}\rangle) + \varphi'(\langle y, u_{\tau(i_n)}\rangle)\right)^2}$$

$$\geq \frac{\min\{1, (p-2)^2\}}{2^{47}} \cdot \eta^2 \cdot \mathcal{C}_{\alpha,1}. \tag{5.35}$$

Thus from (5.11) and (5.34), we have that if $u_{i_j} \in \Omega(v_j, \beta)$ for $j = 1, \cdots, n-1$, $i_n \in \prod_1$ and $y \in \Xi_{u_{i_n}, u_{\tau(i_n)}}$, then

$$N(i_1, \cdots, i_{n-1}, i_n;\ y) + N(i_1, \cdots, i_{n-1}, \tau(i_n);\ y)$$

$$\geq \frac{\beta^2 c_{i_1}\cdots c_{i_n}}{2} \cdot \frac{\min\{1, (p-2)^2\}}{2^{47}} \cdot \eta^2 \cdot \mathcal{C}_{\alpha,1}.$$

If $i_n \in \prod_1$ and $y \in \mathbb{R}^n$, then define

$$\varrho(i_n, \alpha; y) = \begin{cases} 0, & \text{if } y \notin \Xi_{u_{i_n}, u_{\tau(i_n)}}; \\ \dfrac{\beta^2 \min\{(p-2)^2, 1\}}{2^{48}} \cdot \eta^2 \cdot \mathcal{C}_{\alpha,1}, & \text{if } y \in \Xi_{u_{i_n}, u_{\tau(i_n)}}. \end{cases}$$

In particular, if $u_{i_j} \in \Omega(v_j, \beta)$ for $j = 1, \cdots, n-1$, $i_n \in \prod_1$ and $y \in \mathbb{R}^n$, then

$$N(i_1, \cdots, i_{n-1}, i_n; y) + N(i_1, \cdots, i_{n-1}, \tau(i_n); y) \geq c_{i_1} \cdots c_{i_n} \varrho(i_n, \alpha; y). \tag{5.36}$$

Substituting (5.36) into (5.12) and then using (5.17), we get for $y \in \mathbb{R}^n$ that

$$\begin{aligned} \theta^*(y) &\geq 1 + \frac{1}{2} \sum_{\substack{u_{i_j} \in \Omega(v_j, \beta), j=1, \cdots, n-1 \\ i_n \in \prod_1}} c_{i_1} \cdots c_{i_{n-1}} \cdot c_{i_n} \varrho(i_n, \alpha; y) \\ &= 1 + \frac{1}{2} \left(\prod_{j=1}^{n-1} \mu(\Omega(v_j, \beta)) \right) \sum_{i_n \in \prod_1} c_{i_n} \varrho(i_n, \alpha; y) \\ &\geq 1 + \frac{\beta^{n(n-1)}}{2} \sum_{i_n \in \prod_1} c_{i_n} \varrho(i_n, \alpha; y). \end{aligned}$$

Similarly as before, we have

$$\frac{\beta^{n(n-1)}}{2} \sum_{i_n \in \prod_1} c_{i_n} \varrho(i_n, \alpha; y) \leq \frac{\beta^{n(n-1)}}{2} \mu(\Psi_1) \frac{\beta^2 n}{2^{48}} \eta^2 \cdot \mathcal{C}_{\alpha,1} < 1.$$

Thus if $y \in \mathbb{R}^n$, then

$$\theta^*(y)^{-1} \leq 1 - \frac{\beta^{n(n-1)}}{4} \sum_{i_n \in \prod_1} c_{i_n} \varrho(i_n, \alpha; y). \tag{5.37}$$

Together (5.14) with (5.37), we deduce

$$V(Z_{p,\alpha}^{*}(\mu))$$

$$\leq \frac{2^n \Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma\left(1+\frac{n}{p}\right)} \left(\prod_{i=1}^{k} \alpha(u_i)^{c_i}\right)^{-\frac{1}{p}} \int_{\mathbb{R}^n} e^{-\pi\|T(y)\|^2} \det(dT(y))dx$$

$$- \frac{2^n \Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma\left(1+\frac{n}{p}\right)} \left(\prod_{i=1}^{k} \alpha(u_i)^{c_i}\right)^{-\frac{1}{p}} \frac{\beta^{n(n-1)}}{4} \sum_{i_n \in \prod_1} c_{i_n} \int_{\mathbb{R}^n} \varrho(i_n, \alpha; y) e^{-\pi\|T(x)\|^2}$$

$$\times \det(dT(y))dx.$$

Now we use again (5.24) as well as the estimates (5.25) and (5.26) if $y \in \Xi_{i_n, \tau(i_n)}$. This implies

$$V(Z_{p,\alpha}^{*}(\mu)) \leq \frac{2^n \Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma\left(1+\frac{n}{p}\right)} \left(\prod_{i=1}^{k} \alpha(u_i)^{c_i}\right)^{-\frac{1}{p}}$$

$$\times \left(1 - \sum_{i_n \in \prod_1} c_{i_n} \frac{\beta^{n(n-1)}}{4} \cdot \frac{V(\Xi_{i_n,\tau(i_n)})}{(3.1e^{\pi})^n} \cdot \frac{\beta^2 \min\{(p-2)^2, 1\}}{2^{48}} \cdot \eta^2 \cdot \mathcal{C}_\alpha\right).$$

Since $V(\Xi_{i_n,\tau(i_n)}) \geq \kappa_n / 240^n$ for $i_n \in \prod_1$, from (5.32) we give (5.29).

Case 3. There exists $q_i \in \Omega(v_i, \beta)$ for $i = 1, \cdots, n$, such that

$$\mu\left(\cup_{i=1}^{n} (\Omega(q_i, 2\eta) \cup \Omega(-q_i, 2\eta))\right) > n - \eta.$$

In this case, we show that there exists a cross measure ν such that

$$\delta_W(\nu, \mu) \leq n^{cn}\eta, \tag{5.38}$$

for some absolute constant $c > 0$.

We claim that $\frac{1}{2}\left(1 - n\left(\frac{1}{\sqrt{n}} - t\right)^2\right) > \eta$ for $t = 2\eta$, for $\eta < 1/(2n)$.

Thus Lemma 5.1 obtains that $\Omega\left(u, \arccos(\frac{1}{\sqrt{n}} - 2\eta)\right)$ intersects $\bigcup_{i=1}^{n} \Omega(\pm q_i, 2\eta)$ for any $u \in S^{n-1}$. In turn, we have that

$$\Omega\left(u, \arccos(\frac{1}{\sqrt{n}} - 4\eta)\right) \cap \{\pm q_1, \cdots, \pm q_n\} \neq \emptyset,$$

for any $u \in S^{n-1}$, since $4\eta < 1/(4^n n!)$. Therefore Lemma 5.6 obtains that there exists a cross measure ν such that

$$\delta_H(\mathrm{supp}\nu, \{\pm q_1, \cdots, \pm q_n\}) \leq 4^n n! \cdot 4\eta.$$

In particular, (5.38) is given from Lemma 5.7.

According to Lemma 5.3, Cases 1, 2 and 3 cover all possible even isotropic measure μ. This implies that we have proved (5.2) in Cases 1 and 2, and (5.3) in Case 3. □

Proof of Theorem 1.5 in the case of $Z^*_{p,\alpha}(\mu)$: Let $p \in [1, \infty)\setminus\{2\}$ and μ is a discrete even isotropic measure on S^{n-1}, and let $\delta_{WO}(\mu, \nu_n) \geq \varepsilon > 0$ for some $\varepsilon \in (0, 1)$. Then Proposition 5.8 gives that

$$V(Z^*_{p,\alpha}(\mu)) \leq (1 - \gamma\varepsilon^3)V(Z^*_p(\nu_n))\left(\prod_{i=1}^{k} \alpha(u_i)^{c_i}\right)^{-\frac{1}{p}}, \tag{5.39}$$

where $\gamma = n^{-cn^3} \min\{|p-2|^2, 1\}$ for an absolute constant $c > 0$. Since any even isotropic measure can be weakly approximated by discrete even isotropic measures (see, e.g., Barthe [6]), we give (5.39), and in turn Theorem 1.5 in the case of $Z^*_{p,\alpha}(\mu)$, for any even isotropic measure μ on S^{n-1} and $p \in [1, \infty)\setminus\{1\}$.

Since for any isotropic measure μ, we obtain

$$\lim_{p\to\infty} Z^*_{p,\alpha}(\mu) = Z^*_{\infty,\alpha}(\mu).$$

Since the factor γ in (5.39) is independent of $p \in (2, \infty)$, we have the case $p = \infty$. □

6. The Case of the General L_p Zonoids in Theorem 1.5

Similar to the argument of $V(Z^*_{p,\alpha}(\mu))$, we give the proof of Theorem 1.5. We suppose again that μ is a discrete even isotropic measure for $p \in (1, \infty)\setminus\{2\}$. For $p^* \in (1, \infty)$, define $\frac{1}{p} + \frac{1}{p^*} = 1$. We will show that if $\eta \in (0, 1)$, then

$$V(Z_{p*,\alpha}(\mu)) > \left(1 - n^{-cn^3} \min\{(p-2)^2, 1\} \cdot \eta^3 \cdot \mathcal{C}_\alpha\right) V(Z_{p^*}(\nu_n)) \left(\prod_{i=1}^{k} \alpha(u_i)^{c_i}\right)^{\frac{1}{p^*}}, \tag{6.1}$$

or there exists a cross measure ν satisfying

$$\delta_W(\mu, \nu) \le n^{cn}\eta, \tag{6.2}$$

for some absolute constant $c > 0$. Since if $p \in [\frac{3}{2}, 3]$, then $p^* \in [\frac{3}{2}, 3]$ and $|p-2|/2 \le |p^* - 2| \le 2|p-2|$, (6.1) and (6.2) obtain Theorem 1.5 for $V(Z_{p,\alpha}(\mu))$.

Again, let $\operatorname{supp}\nu = \{\bar{u}_1, \cdots, \bar{u}_{\bar{k}}\}$, and let $\bar{c}_i = \mu(\{\bar{u}_i\})$. For $c_0 = \min\{\bar{c}_i : i = 1, \cdots, \bar{k}\}$ and $i = 1, \cdots, \bar{k}$, we define $\overline{m}_i = \min\{m \in \mathbb{Z} : m \geq 1$ and $\bar{c}_i / m \leq c_0\}$, and let $k = \sum_{i=1}^{\bar{k}} \overline{m}_i$. We consider $\xi : \{1, \cdots, k\} \to \{1, \cdots, \bar{k}\}$ such that $\# \xi^{-1}(\{i\}) = \overline{m}_i$ for $i = 1, \cdots, \bar{k}$, and define

$$u_i = \bar{u}_{\xi(i)} \text{ and } c_i = \bar{c}_{\xi(i)} / \overline{m}_{\xi(i)},$$

for $i = 1, \cdots, k$. The system $(u_1, \cdots, u_k, c_1, \cdots, c_k)$ is even (i.e., origin symmetric) in the following sense: For any $u \in S^{n-1}$ occurs as u_i exactly as many times as $-u$, and if $u_i = -u_j$, then $c_i = c_j$.

In particular, $\sum_{i=1}^{k} c_i u_i \otimes u_i = \mathrm{Id}_n$ and $\sum_{i=1}^{k} c_i = n$, and for any Borel $X \subseteq S^{n-1}$, we have

$$\mu(X) = \sum_{u_i \in X} c_i.$$

The reason for the renormalization is that

$$\frac{1}{2} c_0 < c_i \leq c_0 \quad \text{for } i = 1, \cdots, k.$$

Furthermore, let $\psi_u = \psi_p = \psi_{p,u}$ defined in (4.2), $\widetilde{f}_i$ is defined in (3.14), and

$$g(t) = \exp\left[-\pi\left(\prod_{i=1}^{k} \alpha(u_i)^{c_i}\right)^{\frac{2(1-p)}{p}} t^2\right],$$

is the Gaussian density.

We define the map $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Lambda y = \sum_{i=1}^{k} c_i \psi_u(\langle y, u_i \rangle)\alpha(u_i)u_i, \tag{6.3}$$

for each $y \in \mathbb{R}^n$. The differential of Λ is given by

$$d\Lambda(y) = \sum_{i=1}^{k} c_i u_i \otimes u_i \psi'_u(\langle y, u_i \rangle)\alpha(u_i). \tag{6.4}$$

Since $\psi'_u > 0$ and $\alpha > 0$, the matrix $d\Lambda(y)$ is positive definite for each $y \in \mathbb{R}^n$. Hence, the transformation $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ is injective.

First applying (3.15) and from (6.4), we have

$$V(Z_{p^*, \alpha}(\mu)) \geq V(M_{p, \alpha}(\mu))$$

$$= \frac{2^n \Gamma\left(1 + \frac{1}{p}\right)^n}{\Gamma\left(1 + \frac{n}{p}\right)} \int_{\mathbb{R}^n}^{*} \sup_{x = \sum_{i=1}^{k} c_i \theta_i u_i} \prod_{i=1}^{k} \tilde{f}_i(\theta_i)^{c_i} \, dx$$

$$\geq \frac{2^n \Gamma\left(1 + \frac{1}{p}\right)^n}{\Gamma\left(1 + \frac{n}{p}\right)} \int_{\mathbb{R}^n}^{*} \left(\sup_{\Lambda(y) = \sum_{i=1}^{k} c_i \theta_i u_i} \prod_{i=1}^{k} \tilde{f}_i(\theta_i)^{c_i} \right) \det(d(\Lambda y)) dy$$

$$\geq \frac{2^n \Gamma\left(1 + \frac{1}{p}\right)^n}{\Gamma\left(1 + \frac{n}{p}\right)} \int_{\mathbb{R}^n} \left(\prod_{i=1}^{k} \tilde{f}_i(\alpha(u_i)\psi_u(\langle y, u_i \rangle))^{c_i} \right)$$

$$\times \det\left(\sum_{i=1}^{k} c_i \psi'_u(\langle y, u_i \rangle)\alpha(u_i)u_i \otimes u_i \right) dy.$$

To estimate the second term, we apply Lemma 2.2 with $v_i = \sqrt{c_i} \cdot u_i$ and $t_i = \psi_u'(\langle y, u_i\rangle)\alpha(u_i)$, at each $y \in \mathbb{R}^n$, and write $\theta^*(y)$ and $t_0(y)$ to denote the corresponding $\theta^* \geq 1$ and $t_0 > 0$. In particular, if $\{i_1, \cdots, i_n\} \subseteq \{1, \cdots, k\}$ and $y \in \mathbb{R}^n$, then we now set

$$\mathbf{N}(i_1, \cdots, i_n;\ y)$$

$$= c_{i_1} \cdots c_{i_n} \det[u_{i_1}, \cdots, u_{i_n}]^2 \left(\frac{\sqrt{\varphi'(\langle y, u_{i_1}\rangle) \cdots \varphi'(\langle y, u_{i_n}\rangle)\alpha(u_{i_1}) \cdots \alpha(u_{i_n})}}{t_0(y)} - 1 \right)^2. \tag{6.5}$$

Therefore, from the notation

$$\theta^*(y) = 1 + \frac{1}{2} \sum_{1 \leq i_1 < \cdots < i_n \leq k} \mathbf{N}(i_1, \cdots, i_n;\ y). \tag{6.6}$$

Lemma 2.2 and (3.11) have

$$V(Z_{p^*, \alpha}(\mu))$$

$$\geq \frac{2^n \Gamma\left(1 + \frac{1}{p}\right)^n}{\Gamma\left(1 + \frac{n}{p}\right)} \int_{\mathbb{R}^n} \theta^*(y) \left(\prod_{i=1}^{k} \widetilde{f}_i(\alpha(u_i)\psi_u(\langle y, u_i\rangle))^{c_i} \right)$$

$$\times \left(\prod_{i=1}^{k} (\psi_u'(\langle y, u_i\rangle)\alpha(u_i))^{c_i} \right) dy$$

$$= \frac{2^n \Gamma\left(1 + \frac{1}{p}\right)^n}{\Gamma\left(1 + \frac{n}{p}\right)} \int_{\mathbb{R}^n} \theta^*(y) \left(\prod_{i=1}^{k} (g(\langle y, u_i\rangle))^{c_i} \right) dy$$

$$= \frac{2^n \Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma\left(1+\frac{n}{p}\right)} \int_{\mathbb{R}^n} \theta^*(y) \exp\left\{-\pi\left(\prod_{i=1}^{k} \alpha(u_i)^{c_i}\right)^{\frac{2(1-p)}{p}} \|y\|^2\right\} dy$$

$$= \frac{2^n \Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma\left(1+\frac{n}{p}\right)} \left(\prod_{i=1}^{k} \alpha(u_i)^{c_i}\right)^{\frac{p-1}{p}} \int_{\mathbb{R}^n} \theta^*(z) e^{-\pi\|z\|^2} dz$$

$$= \frac{2^n \Gamma\left(1+\frac{1}{p}\right)^n}{\Gamma\left(1+\frac{n}{p}\right)} \left(\prod_{i=1}^{k} \alpha(u_i)^{c_i}\right)^{\frac{1}{p^*}} \int_{\mathbb{R}^n} \theta^*(z) e^{-\pi\|z\|^2} dz.$$

Therefore, (6.1) and (6.2) can be proved as (5.2) and (5.3) in Proposition 5.8. This finishes the proof. □

7. The L_p Loomis-Whitney and Reverse L_p Loomis-Whitney inequality

The classical Loomis-Whitney inequality [49] states that for a convex body K in $\mathbb{R}^n$,

$$V(K)^{n-1} \leq \prod_{i=1}^{k} \mathrm{vol}_{n-1}(K|e_i^{\perp}), \tag{7.1}$$

with equality if and only if K is a coordinate box (a rectangular parallelepiped whose facets are parallel to the coordinate hyper planes), where $K|e_i^{\perp}$ denotes the orthogonal projection of K onto the 1-codimensional space $e_i^{\perp}$ perpendicular to e_i and $\{e_1, \cdots, e_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$. Note that the Loomis-Whitney inequality is of isoperimetric type. In fact, let $S(K)$ denote the surface area of K. Then $S(K) \geq 2\mathrm{vol}_{n-1}(K|e_i^{\perp})$ for $i = 1, \cdots, n.$ From (7.1), we obtain

$$V(K)^{n-1} \leq 2^{-n} S(K)^n,$$

an isoperimetric inequality without the best constant. The Loomis-Whitney inequality is one of the fundamental inequalities in convex geometry and has been studied intensively; see, e.g., [1, 4, 6, 12, 13, 18, 29, 23, 47, 70].

In particular, the Loomis-Whitney inequality still holds along a sequence of directions satisfying John's condition [39], which is showed Ball [4]. Especially, for a convex body K in $\mathbb{R}^n$, if there are unit vectors $(u_i)_{i=1}^k$ and positive numbers $(c_i)_{i=1}^k$ satisfying John's condition (1.17), then

$$V(K)^{n-1} \leq \prod_{i=1}^{k} \mathrm{vol}_{n-1}(K|u_i^{\perp})^{c_i}. \tag{7.2}$$

Obviously, the inequality (7.2) reduces to (7.1) when $k = n$ and taking $u_i = e_i$ with $c_i = 1$ for all $i = 1, \cdots, n$.

Recently, Li and Huang [46] established an the L_p version of the Loomis-Whitney inequality related to the support function of L_p projection bodies with complete equality conditions. For $p \geq 1$, $K \in \mathcal{K}_o^n$ and μ is an even isotropic measure on S^{n-1}, we have

$$V(K)^{\frac{n-p}{p}} \leq \exp\left\{\int_{S^{n-1}} \log h_{\Pi_p K}(u) d\mu(u)\right\}. \tag{7.3}$$

For $1 < p \neq 2$, equality in (7.3) holds if and only if $\mu = \nu$ is a cross measure on S^{n-1} and K is the generalized l_{p*}^n-ball formed by ν. For $p = 1$, equality in (7.3) holds if and only if $\mu = \nu$ is a cross measure on S^{n-1} and K is a box formed by ν (up to translations).

This section is mainly to extend Ball's Loomis-Whitney inequality (7.2) to the L_p setting belonging to the L_p Brunn-Minkowski theory (called as the L_p Loomis-Whitney inequality). Further, we give the complete equality conditions for the L_p version of the Loomis-Whitney inequality.

The following intertwining properties of $\prod_p$ and $\prod_p^*$ with linear transformations were obtained by Lutwak et al. [59] for $p > 1$ and by Petty [66] for $p = 1$.

Lemma 7.1. *Suppose* $p \geq 1$ *and* $K \in \mathcal{K}_o^n$. *Then for* $A \in GL(n)$,

$$\prod_p AK = |\det A|^{1/p} A^{-t} \prod_p K \quad \textit{and} \quad \prod_p^* AK = |\det A|^{-1/p} A \prod_p^* K. \tag{7.4}$$

In particular,

$$\prod_p (cK) = c^{\frac{n-p}{p}} \prod_p K, \; c > 0. \tag{7.5}$$

Motivated by the ways of Ball [4] and Li [46], we prove the L_p Loomis-Whitney inequality in Theorem 7.2.

Theorem 7.2. *Let* K *be a convex body in* $\mathbb{R}^n$, $n \geq 2$, *and let* $p \in [1, \infty]$ *with* $p \neq 2$. *If* μ *is an even isotropic measure on* S^{n-1}, *and there are unit vectors* $(u_i)_{i=1}^k (k \geq n)$ *as well as positive numbers* $(c_i)_{i=1}^k$ *satisfying John's condition* (1.17), *then*

$$V(K)^{\frac{n-p}{p}} \leq \prod_{i=1}^{k} h_{\prod_p K}^{c_i}(u_i). \tag{7.6}$$

For $1 < p \neq 2$ *and* $k = n$ *equality in* (7.6) *holds if and only if* μ *is a cross measure on* S^{n-1} *and* $\operatorname{supp}\mu = \{\pm u_1, \cdots, \pm u_n\}$ *with* $(u_i)_{i=1}^n$ *is an orthonormal basis of* $\mathbb{R}^n$, *as well as K is the generalized* l_{p*}^n*-ball formed by* ν; *namely, there are positive numbers* $(\alpha_i)_{i=1}^n$ *such that*

$$K = \left\{ x \in \mathbb{R}^n : \left(\sum_{i=1}^n |\langle x, u_i \rangle|^{p^*} \alpha_i \right)^{\frac{1}{p^*}} \leq 1 \right\},$$

where $\operatorname{supp}\mu = \{\pm u_1, \cdots, \pm u_n\}$ *and* $(u_i)_{i=1}^n$ *is an orthonormal basis of* $\mathbb{R}^n$. *For* $p = 1$ *equality in* (7.6) *holds if and only if* μ *is a cross measure on* S^{n-1} *and K is a box formed by* μ (*up to translations*); *namely, there is a vector* $x_0 \in \mathbb{R}^n$ *and positive numbers* $(\alpha_i)_{i=1}^n$ *such that*

$$K = \sum_{i=1}^n \alpha_i [-u_i, u_i] + x_0,$$

where $\operatorname{supp}\mu = \{\pm u_1, \cdots, \pm u_n\}$ *and* $(u_i)_{i=1}^n$ *is an orthonormal basis of* $\mathbb{R}^n$.

Proof. Suppose

$$\alpha(u) = h_{\prod_p K}^{-p}(u), \tag{7.7}$$

for $u \in S^{n-1}$. By (2.1), (2.2), the definitions of $Z_{p,\alpha}(\mu)$ (1.7) and $\prod_p K$ (2.3), Fubini's theorem and (1.3), it follows that

$$V(K)^{n-p} \leq V(Z_{p,\alpha}(\mu))^{-p} V_p(K, Z_{p,\alpha}(\mu))^n$$

$$= V(Z_{p,\alpha}(\mu))^{-p} \left(\frac{1}{n} \int_{S^{n-1}} h_{Z_{p,\alpha}}^p(v) dS_p(K, v) \right)^n$$

$$= V(Z_{p,\alpha}(\mu))^{-p}\left(\frac{1}{n}\int_{S^{n-1}}\int_{S^{n-1}}|\langle u, v\rangle|^p \alpha(u)d\mu(u)dS_p(K, v)\right)^n$$

$$= V(Z_{p,\alpha}(\mu))^{-p}\left(\frac{1}{n}\int_{S^{n-1}}\int_{S^{n-1}}|\langle u, v\rangle|^p \alpha(u)dS_p(K, v)d\mu(u)\right)^n$$

$$= V(Z_{p,\alpha}(\mu))^{-p}\left(V(B_{p*}^n)^{\frac{p}{n}} h^p_{\prod_p K}(u)\alpha(u)d\mu(u)\right)^n$$

$$= \frac{V(B_{p*}^n)^p}{V(Z_{p,\alpha}(\mu))^p}.$$

From (1.19) of Theorem 1.3, (1.4) and (7.7), we get

$$V(K)^{\frac{n-p}{p}} \leq \frac{V(B_{p*}^n)}{V(Z_{p,\alpha}(\mu))} \leq \prod_{i=1}^{k} h^{c_i}_{\prod_p K}(u_i), \tag{7.8}$$

which is the desired inequality.

If $k = n$, from the equality condition of L_p Minkowski inequality (2.1), the equality of inequality (7.8) holds if and only if K and $Z_{p,\alpha}$ are dilates when $p > 1$ (K and $Z_{p,\alpha}$ are dilates when $p = 1$). Theorem 1.3 implies that equality of the second inequality in (7.8) holds if and only if μ is a cross measure on S^{n-1} when $p \neq 2$ and $k = n$, and thus by (1.12), $Z_{p,\alpha}$ is the generalized l_{p*}^n-ball $B^n_{p*,1/\alpha}$ formed by ν. Hence K is a dilation of the generalized l_{p*}^n-ball formed by the cross measure ν, which is still the generalized l_{p*}^n-ball formed by ν when $2 \neq p > 1$ and $k = n$ (K coincides with the box formed by ν up to translations when $p = 1$).

Conversely, when $1 < p \neq 2$ and $k = n$, we will prove that equality in (7.8) holds if K is the generalized l_{p*}^n-ball formed by ν; i.e., there are positive numbers $(\alpha_i)_{i=1}^n$ such that

$$K = \left\{ x \in \mathbb{R}^n : \left(\sum_{i=1}^n |\langle x, u_i \rangle|^{p^*} \alpha_i \right)^{\frac{1}{p^*}} \leq 1 \right\}, \tag{7.9}$$

where $\operatorname{supp}\nu = \{\pm u_1, \cdots, \pm u_n\}$ and $(u_i)_{i=1}^n$ is an orthonormal basis of $\mathbb{R}^n$. From (7.8), it is sufficient to verify that K and $Z_{p,\alpha}$ are dilates. By (1.13), we obtain

$$K = B_{p*,\alpha_i}^n = OA^{-1}B_{p*}^n,$$

where O is an orthogonal matrix such that $Oe_i = u_i$ for $i = 1, \cdots, n$ and $A = \operatorname{diag}\{\alpha_1^{1/p}, \cdots, \alpha_n^{1/p}\}$ is a diagonal matrix. Together (7.7) with (7.4), we get

$$\alpha(u_i) = h_{\Pi_p K}^{-p}(u_i) = h_{\Pi_p(OA^{-1}B_{p*}^n)}(u_i)$$

$$= h^{-p}_{|\det A|^{-1/p}(OA^{-1})^{-t}\Pi_p(B_{p*}^n)}(u_i)$$

$$= |\det A| h^{-p}_{\Pi_p(B_{p*}^n)}(AO^{-1}u_i)$$

$$= |\det A| h^{-p}_{\Pi_p(B_{p*}^n)}(Ae_i)$$

$$= |\det A| h^{-p}_{\Pi_p(B_{p*}^n)}(\alpha_i^{1/p}e_i)$$

$$= |\det A| h^{-p}_{\Pi_p(B_{p*}^n)}(e_i)\alpha_i^{-1},$$

for every $i = 1, \cdots, n$. Notice that $h^{-p}_{\prod_p (B^n_{p*})}(e_i)$ is a constant for all $i = 1, \cdots, n$. Thus, there exists a constant $c > 0$ such that $\alpha(u_i) = c\alpha_i^{-1}$ for every $k = 1, \cdots, n$. Thus from (1.12) and (7.9), we have

$$\begin{aligned} Z_{p,\alpha}(\nu) &= B^n_{p*,1/\alpha} \\ &= \left\{ x \in \mathbb{R}^n : \left(\sum_{i=1}^n |\langle x, u_i \rangle|^{p^*} \alpha_i (u_i)^{-1} \right)^{\frac{1}{p^*}} \leq 1 \right\} \\ &= \left\{ x \in \mathbb{R}^n : \left(\sum_{i=1}^n |\langle x, u_i \rangle|^{p^*} c^{-1} \alpha_i \right)^{\frac{1}{p^*}} \leq 1 \right\} = c^{\frac{1}{p^*}} K. \end{aligned}$$

That is, K and $Z_{p,\alpha}$ are dilates when $1 < p \neq 2$ and $k = n$. When $p = 1$, the proof is the same, according to the observation that $\prod(K + x_0) = \prod K$ for every $x_0 \in \mathbb{R}^n$. □

Notice that when $p = 1$, it follows from (2.4) and the inequality (7.6) that Ball's Loomis-Whitney inequality (7.2). In addition, if let $k = n$ and $u_i = e_i$ with $c_i = 1$ for all $i = 1, \cdots, n$ in (7.6), the inequality (7.6) can be written as

$$V(K)^{\frac{n-p}{p}} \leq \prod_{i=1}^n h_{\prod_p K}(u_i), \tag{7.10}$$

where K is a convex body in $\mathbb{R}^n$, and ν is a cross measure on S^{n-1} with $\operatorname{supp}\nu = \{\pm u_1, \cdots, \pm u_n\}$. Notice that for every $u \in S^{n-1}$ and $K \in \mathcal{K}^n_o$, $\lim_{p\to\infty} h_{\prod_p K}(u) = h_{K^*}(u) = 1/\rho_K(u)$. Then inequality (7.6) reduces to the following the interesting inequality:

$$V(K) \geq \prod_{i=1}^{k} \rho_K(u_i)^{c_i}, \tag{7.11}$$

with equality in (7.11) if and only if $k = n$ and μ is a cross measure on S^{n-1} and $\operatorname{supp}\mu = \{\pm u_1, \cdots, \pm u_n\}$ with $(u_i)_{i=1}^{n}$ is an orthonormal basis of $\mathbb{R}^n$, as well as K is the generalized l_{p*}^n-ball formed by μ.

The following lemmas is to establish a dual version of the Loomis-Whitney inequality for isotropic measures with complete equality conditions.

Lemma 7.3. *Let* $K \in \mathcal{S}_o^p$ *and let* $p \in (0, \infty]$. *If* μ *is an even isotropic measure on* S^{n-1}, *and there are unit vectors* $(u_i)_{i=1}^{k}$ *as well as positive numbers* $(c_i)_{i=1}^{k}$ *satisfying John's condition* (1.17), *then*

$$V(K) \leq (n+p)^{\frac{n}{p}} V(B_p^n) \prod_{i=1}^{k} \|u_i\|_{\Gamma_p^* K}^{c_i}. \tag{7.12}$$

For $p \neq 2$ *and* $k = n$, *there is equality if and only if* μ *is a cross measure on* S^{n-1} *and K is a generalized* l_p^n*-ball formed by* ν.

Proof. Set

$$\alpha(u)^{-1} = \int_{S^{n-1}} |\langle u, v\rangle|^p \rho_K^{n+p}(v) dv, \tag{7.13}$$

for $u \in S^{n-1}$. From (2.5), (2.7), the definition of $Z_{p,\alpha}^*$ (1.10), Fubini's theorem, and (1.3), it follows that

$$V(K)^{n+p} \leq V(Z_{p,\alpha}^*(\mu))^p \widetilde{V}_{-p}(K, Z_{p,\alpha}^*(\mu))^n = V(Z_{p,\alpha}^*(\mu))^p \left(\frac{1}{n}\int_{S^{n-1}} \rho_K^{n+p}(v)\rho_{Z_{p,\alpha}^*}(v)dv\right)^n$$

$$= V(Z_{p,\alpha}^*(\mu))^p \left(\frac{1}{n}\int_{S^{n-1}} \rho_K^{n+p}(v)\|v\|_{Z_{p,\alpha}^*(\mu)}^p dv\right)$$

$$= V(Z^*_{p,\alpha}(\mu))^p \left(\frac{1}{n}\int_{S^{n-1}}\left(\int_{S^{n-1}} |\langle u, v\rangle|^p \alpha(u) d\mu(u)\right)\rho_K^{n+p}(v) dv\right)^n$$

$$= V(Z^*_{p,\alpha}(\mu))^p \left(\frac{1}{n}\int_{S^{n-1}}\left(\int_{S^{n-1}} |\langle u, v\rangle|^p \rho_K^{n+p}(v) d\mu(u)\right)\alpha(u) dv\right)^n$$

$$= V(Z^*_{p,\alpha}(\mu))^p.$$

Together this with (1.19) of Theorem 1.3 and from (2.9), we obtain

$$V(K)^{\frac{n+p}{p}} \le V(Z^*_{p,\alpha}(\mu)) \le V(B_p^n)\left(\prod_{i=1}^{n} \alpha(u_i)^{c_i}\right)^{-\frac{1}{p}}$$

$$= V(B_p^n)(n+p)^{\frac{n}{p}} V(K)^{\frac{n}{p}} \prod_{i=1}^{k} \|u_i\|^{c_i}_{\Gamma_p^* K},$$

namely,

$$V(K) \le (n+p)^{\frac{n}{p}} V(B_p^n) \prod_{i=1}^{k} \|u_i\|^{c_i}_{\Gamma_p^* K}, \tag{7.14}$$

which is the desired inequality.

Now, we give the characterization of equalities in (7.14). According to the dual Minkowski inequality (2.7), the equality of the first inequality (7.14) holds if and only if K and $Z^*_{p,\alpha}$ are dilates. Theorem 1.3 implies that equality of the second inequality in (7.14) holds if and only if μ is a cross measure on S^{n-1} when $p \ne 2$ and $k = n$. Namely, $Z^*_{p,\alpha}$ is a generalized l_p^n-ball $B^n_{p,\alpha}$ formed by ν when $p \ne 2$ and $k = n$. Therefore, we obtain that equality in (7.14) holds if and only if K is a dilation of the generalized l_p^n-ball formed by the cross measure ν, which is still a generalized l_p^n-ball formed by the cross measure μ, when $p \ne 2$ and $k = n$.

Conversely, if $k = n$ we will prove that equality in (7.14) holds if K is a generalized l_p^n-ball formed by the cross measure ν, i.e., there are positive numbers $(\alpha_i)_{i=1}^n$ such that

$$K = \left\{x \in \mathbb{R}^n : \left(\sum_{i=1}^n |\langle x, u_i\rangle|^p \alpha_i\right)^{\frac{1}{p}} \leq 1\right\},$$

where $\operatorname{supp}\nu = \{\pm u_1, \cdots, \pm u_n\}$ and $(u_i)_{i=1}^n$ is an orthonormal basis of $\mathbb{R}^n$. By (7.14), it is sufficient to verify that K and $Z_{p,\alpha}^*$ are dilates when $p \neq 2$ and $k = n$. From (1.13), we have

$$K = OA^{-1}B_p^n,$$

where $O \in O(n)$ such that $Oe_i = u_i$ for $i = 1, \cdots, n$ and $A = \operatorname{diag}\{\alpha_1^{1/p}, \cdots, \alpha_n^{1/p}\}$ is a diagonal matrix. Together (7.13) with (2.9), we have

$$\alpha(u_i)^{-1} = \int_{S^{n-1}} |\langle u_i, v\rangle|^p \rho_K^{n+p}(v)dv = (n + p)\int_K |\langle u_i, y\rangle|^p dy$$

$$= (n + p)\int_{OA^{-1}B_p^n} |\langle u_i, y\rangle|^p dy$$

$$= (n + p)|\det A|^{-1} \int_{B_p^n} |\langle A^{-t}O^t u_i, z\rangle|^p dz$$

$$= (n + p)|\det A|^{-1} \alpha_i^{-1} \int_{B_p^n} |e_i, z\rangle|^p dz,$$

for every $i = 1, \cdots, n$. Note that $\int_{B_p^n} |\langle e_i, z\rangle|^p dz$ is a constant for all $i = 1, \cdots, n$. This implies that there exists a constant $c > 0$ such that $\alpha(u_i) = c\alpha_i$ for every $i = 1, \cdots, n$. Recall that

$$Z_{p,\alpha}^{*} = \left\{ x \in \mathbb{R}^n : \left(\int_{S^{n-1}} |\langle x, u \rangle|^p \alpha(u) d\mu(u) \right)^{\frac{1}{p}} \leq 1 \right\}$$

$$= \left\{ x \in \mathbb{R}^n : \left(\sum_{i=1}^{n} |\langle x, u_i \rangle|^p \alpha(u_i) \right)^{\frac{1}{p}} \leq 1 \right\}$$

$$= \left\{ x \in \mathbb{R}^n : \left(\sum_{i=1}^{n} |\langle x, u_i \rangle|^p c\alpha_i \right)^{\frac{1}{p}} \leq 1 \right\} = c^{-1/p} K;$$

i.e., K and $Z_{p,\alpha}^{*}$ are dilates when $p \neq 2$ and $k = n$. □

Notice that for every $u \in S^{n-1}$ and $K \in \mathcal{K}_o^n$, $\lim_{p \to \infty} \|u\|_{\Gamma_p^* K} = \|u\|_{K^*} = h_K(u)$, $\lim_{p \to \infty} V(B_p^n) = 2^n$, and $\lim_{p \to \infty} (n + p)^{\frac{n}{p}} = 1$, then inequality (7.12) deduces to the following the interesting inequality:

$$V(K) \leq 2^n \prod_{i=1}^{k} h_K(u_i)^{c_i} = 2^n \prod_{i=1}^{k} \mathrm{vol}_{n-1}(K|u_i^{\perp})^{c_i}, \tag{7.15}$$

with equality if and only if $k = n$ and μ is a cross measure on S^{n-1}, and $\mathrm{supp}\mu = \{\pm u_1, \cdots, \pm u_n\}$ with $(u_i)_{i=1}^n$ is an orthonormal basis of $\mathbb{R}^n$, as well as K is a generalized l_p^n-ball formed by μ.

Fortunately, the comparison between inequality (7.2) and inequality (7.15) is a very interesting result.

Fradelizi [26] established the following sharp estimates which is vital to the proof of our theorem. The symmetric case $(p > 0)$ of (7.16) was given by Milman and Pajor [63]. The case $p = 2$ is due to Hensley [37]. We also see [5] for related inequalities.

Lemma 7.4. *Let* $p \geq 1$ *and let K be a convex body in* $\mathbb{R}^n$ *whose centroid is at the origin. For* $u \in S^{n-1}$, *we have*

$$\|u\|_{\Gamma_p^* K} \leq \frac{nV(K)}{2\text{vol}_{n-1}(K \cap u^{\perp})} \binom{n+p}{n}^{-1/p}, \tag{7.16}$$

with equality if and only if K is a double cone in the direction u; *and*

$$\|u\|_{\Gamma_p^* K} \leq \frac{c_{n,p} V(K)}{\max\limits_{x \in \mathbb{R}^n} \text{vol}_{n-1}((K + x) \cap u^{\perp})}, \tag{7.17}$$

with equality if and only if K is a cone in the direction u, where

$$c_{n,p} = \left(\left(\frac{n}{n+1} \right)^{n+p} \int_{-1}^{n} |t|^p \left(1 - \frac{t}{n} \right)^{n-1} dt \right)^{\frac{1}{p}}.$$

Combining (7.12), (7.16), (1.4) and (1.6), we directly have the following dual Loomis-Whitney inequality.

Lemma 7.5. *Let K be a convex body in* $\mathbb{R}^n$, *and let* $p \in [1, \infty]$. *If* μ *is an even isotropic measure on* S^{n-1}, *and there are unit vectors* $(u_i)_{i=1}^k$ *as well as positive numbers* $(c_i)_{i=1}^k$ *satisfying John's condition* (1.17), *then*

$$V(K)^{n-1} \geq \frac{\Gamma\left(1 + \frac{n}{p}\right) \binom{n+p}{n}^{\frac{n}{p}}}{n^n (n+p)^{\frac{n}{p}} \Gamma\left(1 + \frac{1}{p}\right)} \prod_{i=1}^{k} \text{vol}_{n-1}(K \cap u_i^{\perp})^{c_i}. \tag{7.18}$$

Lemma 7.5 gives a whole family of inequalities when p varies. This includes the pase $p = 1$. The equality conditions in the results above lead to a sharp inequality for $p = 1$, and the equality condition is also characterized.

Theorem 7.6. *Let K be a convex body in* $\mathbb{R}^n (n \geq 2)$. *If* μ *is an even isotropic measure on* S^{n-1}, *and there are unit vectors* $(u_i)_{i=1}^k$ *as well as positive numbers* $(c_i)_{i=1}^k$ *satisfying John's condition* (1.17), *then*

$$V(K)^{n-1} \geq \frac{n!}{n^n} \prod_{i=1}^{k} \mathrm{vol}_{n-1}(K \cap u_i^{\perp})^{c_i}. \tag{7.19}$$

If $k = n$, *equality holds if and only if* μ *is a cross measure on* S^{n-1} *and* $\mathrm{supp}\mu = \{\pm u_1, \cdots, \pm u_n\}$, *and* $(u_i)_{i=1}^n$ *is an orthonormal basis of* $\mathbb{R}^n$ *and K is a generalized* l_1^p *-ball formed by* μ.

Proof. We need to examine only the equality conditions of (7.18). Since (7.18) follows from (7.12) and (7.16), the equality condition of (7.12) has that μ is a cross measure on S^{n-1} and K is a generalized l_p^n-ball formed by μ when $p \neq 2$ and $k = n$. This is because the centroid of the generalized l_p^n-ball lies on the origin. The equality condition of (7.16) gets that K is a double cone in the direction of the support of μ. Clearly, only the generalized l_1^p-ball formed by the cross measure μ is satisfied. Hence K is a generalized l_1^p-ball formed by the cross measure μ. The equality condition of (7.19) is proved. □

Similarly, together (7.12) with (7.17) has the following inequality.

Theorem 7.7. *Let K be a convex body in* $\mathbb{R}^n (n \geq 2)$, *and let* $p \in [1, \infty]$. *If* μ *is an even isotropic measure on* S^{n-1}, *and there are unit vectors* $(u_i)_{i=1}^k$ *as well as positive numbers* $(c_i)_{i=1}^k$ *satisfying John's condition* (1.17), *then*

$$V(K)^{n-1} \geq \frac{\Gamma\left(1+\frac{n}{p}\right)}{n^n (n+p)^{\frac{n}{p}} \Gamma\left(1+\frac{1}{p}\right) c_{n,p}^n} \prod_{i=1}^{k} \max_{x \in \mathbb{R}^n} \mathrm{vol}_{n-1}((K+x) \cap u_i^{\perp})^{c_i}. \tag{7.20}$$

Conflict of Interests

The author declare that they have no competing interests.

Authors' Contribution

All authors contributed equally to the paper and read and approved its final version.

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