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SOME UPPER BOUNDS FOR THE INCIDENCE ENERGY OF A CONNECTED GRAPH

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Abstract

Let G be a graph of order n. The incidence energy, denoted $IE(G)$, of G is defined as the sum of the singular values of the incidence matrix of G. It has been showed that $IE(G) = \sum_{i=1}^{n} \sqrt{q_i}$, where q_i, $1 \leq i \leq n$, are the signless Laplacian eigenvalues of G. In this note, we present some upper bounds for the incidence energy of a graph.

1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow that in [1]. Let G be a graph with n vertices and m edges. We use $\delta(G)$ and $\Delta(G)$ to denote the minimum and maximum degrees of the vertices in the graph G, respectively. The distance between two distinct vertices in a connected

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graph G is defined as the number of edges in a shortest path that connects the two vertices in G. The diameter of a connected graph G is defined as the largest distance among the distances between all pairs of distinct vertices in G. The eigenvalues of G are the eigenvalues of the adjacency matrix, denoted $A(G)$, of G. The signless Laplacian matrix, denoted $Q(G)$, of G is defined as $A(G) + D(G)$, where $D(G)$ is a diagonal matrix such that the (i, i)-entries of $D(G)$ are the degrees of vertices in G. The eigenvalues, denoted q_i with $1 \leq i \leq n$, of $Q(G)$ are called the signless Laplacian eigenvalues of G. For a matrix M, we use M^t to denote its transpose of M.

Gutman [5] introduced the concept of energy of a graph. The energy of a graph G is defined as the sum of the absolute values of the eigenvalues of G. Nikiforov [14] extended the concept of energy of a graph to the energy of any matrix M. The energy of a matrix is defined as the sum of the singular values of M, where the singular values of M are the square roots of the eigenvalues of the matrix MM^t. Based on Nikiforov's definition of the energy of a matrix, Jooyandeh et al. [8] introduced the concept of incidence energy of a graph. The incidence energy, denoted $IE(G)$, of a graph G is defined as the energy of the incidence matrix of G. Namely, $IE(G)$ is the sum of the singular values of the incidence matrix of G. Gutman et al. [6] showed that in fact $IE(G) = \sum_{i=1}^{n} \sqrt{q_i}$.

The upper bounds for $IE(G)$ of a graph G have been obtained in recent years. Some of them can be found in [7], [18], [17], [4], [13], and [9]. In this note, we will present additional upper bounds for $IE(G)$ of a graph G. The remainder of this note is organized as follows. In Section 2, we will present our main result and its proofs. Our main result gives a generic upper bound for $IE(G)$ of a connected graph G. In Section 3, we will use our main result and some existing upper bounds of the largest signless Laplacian eigenvalue of a graph to obtain some concrete upper bounds for $IE(G)$ of a graph G.

2. The Main Result and its Proofs

The main result of this note is as follows.

Theorem 1. *Let G be a connected graph with* $n \geq 4$ *vertices and m edges. Then*

$$IE \leq \sqrt{q_1} + \sqrt{\frac{2m(n-1)(n-2)}{n}}$$

with equality if and only if G is a complete graph.

Proof of Theorem 1. Notice that $q_1 \geq \frac{4m}{n}$ with equality if and only if G is a regular graph (see Conjecture 5 on page 17 in [3]). From Cauchy-Schwartz inequality and $\sum_{i=1}^{n} q_i = 2m,$ we have that

$$IE = \sum_{i=1}^{n} \sqrt{q_i} = \sqrt{q_1} + \sqrt{q_2} + \sum_{i=3}^{n} \sqrt{q_i}$$

$$\leq \sqrt{q_1} + \sqrt{q_2} + \sqrt{(n-2)\sum_{i=3}^{n} q_i}$$

$$= \sqrt{q_1} + \sqrt{q_2} + \sqrt{(n-2)\left(\sum_{i=1}^{n} q_i - q_1 - q_2\right)}$$

$$= \sqrt{q_1} + \sqrt{q_2} + \sqrt{(n-2)(2m - q_1 - q_2)}$$

$$\leq \sqrt{q_1} + \sqrt{q_2} + \sqrt{(n-2)(2m - \frac{4m}{n} - q_2)}.$$

Now consider the function

$$f(x) = \sqrt{x} + \sqrt{(n-2)(2m - \frac{4m}{n} - x)}.$$

It can be verified that $f(x)$ attains its maximum when $x = \frac{2m(n-2)}{n(n-1)}$.

Thus

$$\begin{aligned}
&\sqrt{q_2} + \sqrt{(n-2)(2m - \frac{4m}{n} - q_2)} \\
&\quad \le \sqrt{\frac{2m(n-2)}{n(n-1)}} + \sqrt{(n-2)\left(2m - \frac{4m}{n} - \frac{2m(n-2)}{n(n-1)}\right)} \\
&\quad = \sqrt{\frac{2m(n-1)(n-2)}{n}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
IE &\le \sqrt{q_1} + \sqrt{q_2} + \sqrt{(n-2)(2m - \frac{4m}{n} - q_2)} \\
&\le \sqrt{q_1} + \sqrt{\frac{2m(n-1)(n-2)}{n}}.
\end{aligned}$$

If

$$IE = \sqrt{q_1} + \sqrt{\frac{2m(n-1)(n-2)}{n}},$$

then, from the above proofs, we have that G is regular, $q_1 = \frac{4m}{n}$, $q_2 = \frac{2m(n-2)}{n(n-1)}$, and $q_3 = \cdots = q_n$. Thus, from $\sum_{i=1}^{n} q_i = 2m$, we have that

$$q_3 = \cdots = q_n = \frac{2m - q_1 - q_2}{n-2} = \frac{2m(n-2)}{n(n-1)}.$$

Therefore G has two distinct signless Laplacian eigenvalues. Recall that the diameter of a connected graph is less than or equal to the number of the distinct signless Laplacian eigenvalues minus one (see Proposition 2.3 on page 508 in [11]). Hence the diameter of G is one. So G is a complete graph.

If G is a complete graph, then $q_1 = 2(n-1)$, $q_2 = \cdots = q_n = (n-2)$ and therefore

$$IE = \sum_{i=1}^{n} q_i = \sqrt{2(n-1)} + (n-1)\sqrt{n-2} = \sqrt{q_1} + \sqrt{\frac{2m(n-1)(n-2)}{n}}.$$

Therefore the proof of Theorem 1 is completed. □

3. Additional Upper Bounds for IE

Theorem 1 implies that every upper bound for q_1 can yield an upper bound for IE. Recall the following upper bounds for the largest signless Laplacian eigenvalues.

Theorem 2. *Let G be a connected graph with n vertices and m edges. Then*

$$q_1 \leq u_1 := \frac{\delta - 1 + \sqrt{(\delta - 1)^2 + 8(2m + \Delta^2 - (n-1)\delta)}}{2}$$

with equality if and only if G is a regular graph.

Theorem 2 above is Theorem 2.1 on page 910 in [2] (also see Theorem 3.1 on page 805 in [10]).

Theorem 3. *Let G be a connected graph with n vertices and m edges. Then*

$$q_1 \leq u_2 := \frac{\Delta + \delta - 1 + \sqrt{(\Delta + \delta - 1)^2 + 8(2m - (n-1)\delta)}}{2}$$

with equality if and only if G is a regular graph.

Theorem 3 above is Theorem 2.2 on page 910 in [2] (also see the proofs of Theorem 4 on page 137 in [12]).

Theorem 4. *Let G be a connected graph with n vertices and m edges. Then*

$$q_1 \leq u_3 := \frac{2m + \sqrt{m(n^3 - n^2 - 2mn + 4m)}}{n}$$

with equality if and only if G is a complete graph K_n.

Theorem 4 above is Theorem 2.3 on page 910 in [2] (also see [15]).

Theorem 5. *Let G be a connected graph with n vertices and m edges. Then*

$$q_1 \leq u_4 := \frac{\delta - 1}{2} + \sqrt{2(\Delta^2 + \delta) + (2m - n\delta) + \frac{(\delta - 1)^2}{4}}$$

with equality if and only if G is a regular graph.

Theorem 5 above is Lemma 2.3 on page 2860 in [16].

From Theorems 1, 2, 3, 4, and 5, we have the following corollary.

Corollary 1. *Let G be a connected graph of order* $n(n \geq 4)$ *and m edges. Then, for each i with* $1 \leq i \leq 4$,

$$IE \leq \sqrt{u_i} + \sqrt{\frac{2m(n-1)(n-2)}{n}}$$

with equality if and only if G is a complete graph K_n.

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