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COMPLEX ANALYSIS OF REAL FUNCTIONS III: EXTENDED FOURIER THEORY

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Abstract

In the context of the complex-analytic structure within the unit disk centered at the origin of the complex plane, that was presented in a previous paper, we show that the complete Fourier theory of integrable real functions is contained within that structure, that is, within the structure of the space of inner analytic functions on the open unit disk. We then extend the Fourier theory beyond the realm of integrable real functions, to include for example singular Schwartz distributions, and possibly other objects.

1. Introduction

In a previous paper [1], we introduced a certain complex-analytic structure within the unit disk of the complex plane, and showed that one can represent essentially all integrable real functions within that structure. The construction leading to this result started with the use of the Fourier coefficients α_k and β_k of the integrable real function $f(\theta)$,

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from which we defined a set of complex Taylor coefficients c_k, thus leading to the corresponding inner analytic function $w(z)$. It is therefore clearly apparent that there is a close relation between that complex-analytic structure and the Fourier theory [2] of integrable real functions.

In this paper we will make that relation explicit by showing, in Sections 2-5, that *all* the elements of the Fourier theory of integrable real functions are contained within the complex-analytic structure. What we mean by these elements is the set of mathematical objects including the Fourier basis of functions, the Fourier series, the scalar product for integrable real functions, the relations of orthogonality and norm of the basis elements, and the completeness of the Fourier basis, including its so-called completeness relation.

The fact that one can recover the real functions from their Fourier coefficients almost everywhere, even when the corresponding Fourier series are divergent, as we showed in [1], leads to a powerful and very general summation rule for *all* Fourier series. Furthermore, we will show in Section 6 that the complex-analytic structure allows us to extend the Fourier theory beyond the realm of integrable real functions, to include the singular Schwartz distributions that we examined in detail in another previous paper [3], as well as at least some non-integrable real functions, and possibly other objects.

For ease of reference, we include here a one-page synopsis of the complex-analytic structure introduced in [1]. It consists of certain elements within complex analysis [4], as well as of their main properties.

Synopsis: The Complex-Analytic Structure

An *inner analytic function* $w(z)$ is simply a complex function which is analytic within the open unit disk. An inner analytic function that has the additional property that $w(0) = 0$ is a *proper inner analytic function*. The *angular derivative* of an inner analytic function is defined by

$$w^{\bullet}(z) = \imath z \frac{dw(z)}{dz}. \tag{1}$$

By construction we have that $w^{\bullet}(0) = 0$, for all $w(z)$. The *angular primitive* of an inner analytic function is defined by

$$w^{-1\bullet}(z) = -\imath \int_0^z dz' \, \frac{w(z') - w(0)}{z'}. \tag{2}$$

By construction we have that $w^{-1\bullet}(0) = 0$, for all $w(z)$. In terms of a system of polar coordinates (ρ, θ) on the complex plane, these two analytic operations are equivalent to differentiation and integration with respect to θ, taken at constant ρ. These two operations stay within the space of inner analytic functions, they also stay within the space of proper inner analytic functions, and they are the inverses of one another. Using these operations, and starting from any proper inner analytic function $w^{0\bullet}(z)$, one constructs an infinite *integral-differential chain* of proper inner analytic functions,

$$\left\{\ldots, w^{-3\bullet}(z), w^{-2\bullet}(z), w^{-1\bullet}(z), w^{0\bullet}(z), w^{1\bullet}(z), w^{2\bullet}(z), w^{3\bullet}(z), \ldots\right\}. \tag{3}$$

Two different such integral-differential chains cannot ever intersect each other. There is a *single* integral-differential chain of proper inner analytic functions which is a constant chain, namely, the null chain, in which all members are the null function $w(z) \equiv 0$.

A general scheme for the classification of all possible singularities of inner analytic functions is established. A singularity of an inner analytic function $w(z)$ at a point z_1 on the unit circle is a *soft singularity* if the limit of $w(z)$ to that point exists and is finite. Otherwise, it is a *hard singularity*. Angular integration takes soft singularities to other soft singularities, and angular differentiation takes hard singularities to other hard singularities.

Gradations of softness and hardness are then established. A hard singularity that becomes a soft one by means of a single angular integration is a *borderline hard* singularity, with degree of hardness zero. The *degree of softness* of a soft singularity is the number of angular differentiations that result in a borderline hard singularity, and the *degree of hardness* of a hard singularity is the number of angular integrations that result in a borderline hard singularity. Singularities which are either soft or borderline hard are integrable ones. Hard singularities which are not borderline hard are non-integrable ones.

Given an integrable real function $f(\theta)$ on the unit circle, one can construct from it a unique corresponding inner analytic function $w(z)$. Real functions are obtained through the $\rho \to 1_{(-)}$ limit of the real and imaginary parts of each such inner analytic function and, in particular, the real function $f(\theta)$ is obtained from the real part of $w(z)$ in this limit. The pair of real functions obtained from the real and imaginary parts of one and the same inner analytic function are said to be mutually Fourier-conjugate real functions.

Singularities of real functions can be classified in a way which is analogous to the corresponding complex classification. Integrable real functions are typically associated with inner analytic functions that have singularities which are either soft or at most borderline hard. This ends our synopsis.

When we discuss real functions in this paper, some properties will be globally assumed for these functions, just as was done in [1] and [3]. These are rather weak conditions to be imposed on these functions, that will be in force throughout this paper. It is to be understood, without any need for further comment, that these conditions are valid whenever real functions appear in the arguments. These weak conditions certainly hold for any integrable real functions that are obtained as restrictions of corresponding inner analytic functions to the unit circle.

The most basic condition is that the real functions must be measurable in the sense of Lebesgue, with the usual Lebesgue measure [5, 6]. The second global condition we will impose is that the functions have no removable singularities. The third and last global condition is that the number of hard singularities on the unit circle be finite, and hence that they be all isolated from one another. There will be no limitation on the number of soft singularities.

The material contained in this paper is a development, reorganization and extension of some of the material found, sometimes still in rather rudimentary form, in the papers [7-11].

2. Fourier Series

In [1] we showed that, given any integrable real function $f(\theta)$, one can construct a corresponding inner analytic function $w(z) = u(\rho, \theta) + \imath v(\rho, \theta)$, from the real part of which $f(\theta)$ can be recovered almost everywhere on the unit circle, through the use of the $\rho \to 1_{(-)}$ limit, where (ρ, θ) are polar coordinates on the complex plane. In that construction we started by calculating the Fourier coefficients [2] of the real function, which is always possible given that the function is integrable, using the usual integrals defining these coefficients,

$$\alpha_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta f(\theta),$$

$$\alpha_k = \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta \cos(k\theta) f(\theta),$$

$$\beta_k = \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta \sin(k\theta) f(\theta), \qquad (4)$$

for $k \in \{1, 2, 3, \ldots, \infty\}$. We then defined a set of complex Taylor coefficients

$$c_0 = \frac{1}{2}\alpha_0,$$

$$c_k = \alpha_k - \imath\beta_k, \tag{5}$$

for $k \in \{1, 2, 3, \ldots, \infty\}$. Next we defined a complex variable z associated to θ, using the positive real variable ρ, by $z = \rho\exp(\imath\theta)$. Using all these elements we then constructed the power series

$$S(z) = \sum_{k=0}^{\infty} c_k z^k, \tag{6}$$

which we showed to be convergent to an inner analytic function $w(z) = S(z)$ within the open unit disk. This power series is therefore the Taylor series of $w(z)$. We also proved that one recovers the real function $f(\theta)$ almost everywhere on the unit circle from the $\rho \to 1_{(-)}$ limit of the real part $u(\rho, \theta)$ of $w(z)$. It is now very easy to show that the Fourier series of an integrable real function $f(\theta)$ is simply given by the real part of this Taylor series, when restricted to the unit circle. Writing the series explicitly in terms of the polar coordinates (ρ, θ) of the complex plane, we get

$$\begin{aligned} w(z) &= \frac{\alpha_0}{2} + \sum_{k=1}^{\infty}(\alpha_k - \imath\beta_k)\rho^k[\cos(k\theta) + \imath\sin(k\theta)] \\ &= \frac{\alpha_0}{2} + \sum_{k=1}^{\infty}\rho^k[\alpha_k\cos(k\theta) + \beta_k\sin(k\theta)] \\ &\quad + \imath\sum_{k=1}^{\infty}\rho^k[\alpha_k\sin(k\theta) - \beta_k\cos(k\theta)], \end{aligned} \tag{7}$$

where $w(z) = u(\rho, \theta) + \imath v(\rho, \theta)$. Taking now the $\rho \to 1_{(-)}$ limit we get

$$u(1, \theta) + \imath v(1, \theta) = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty}[\alpha_k \cos(k\theta) + \beta_k \sin(k\theta)]$$

$$+ \imath \sum_{k=1}^{\infty}[\alpha_k \sin(k\theta) - \beta_k \cos(k\theta)]. \tag{8}$$

It follows, therefore, that the real part of $w(z)$ for $\rho = 1$ is the Fourier series of $f(\theta)$,

$$u(1, \theta) = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty}[\alpha_k \cos(k\theta) + \beta_k \sin(k\theta)], \tag{9}$$

and that the imaginary part of $w(z)$ for $\rho = 1$ is the Fourier series of the real function which is the Fourier conjugate of $f(\theta)$,

$$v(1, \theta) = \sum_{k=1}^{\infty}[\alpha_k \sin(k\theta) - \beta_k \cos(k\theta)]. \tag{10}$$

Here we see that, with respect to the Fourier series of $u(1, \theta)$, the $k = 0$ term is missing, all the other coefficients are the same, while the $\cos(k\theta)$ were exchanged for $\sin(k\theta)$, and the $\sin(k\theta)$ were exchanged for $-\cos(k\theta)$. In [1] we proved that $u(1, \theta)$ is equal to $f(\theta)$ almost everywhere, irrespective of the convergence or lack of convergence of the Fourier series, so that it now becomes clear that, when and where this trigonometric series converges at all, it converges to the original integrable real function,

$$f(\theta) = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty}[\alpha_k \cos(k\theta) + \beta_k \sin(k\theta)]. \tag{11}$$

The convergence of this Fourier series can be characterized in terms of the singularities of the inner analytic function $w(z)$ on the unit disk. If there are no singularities of $w(z)$ on the unit circle, then the maximum convergence disk of its Taylor series is larger than the unit disk, and contains it. Therefore, in this case the Fourier series is always convergent, as well as absolutely and uniformly convergent. On the other hand, if there is at least one singularity of $w(z)$ on the unit circle, then the unit disk is the maximum disk of convergence of the Taylor series, and in this case the Fourier series may or may not be convergent. In this case we see that, given any integrable real function, the issue of the convergence of its Fourier series is thus identified completely with the issue of the convergence of the Taylor series of the corresponding inner analytic function, at the border of its maximum convergence disk.

From the expansion in Equation (7) we see that the recovery of $f(\theta)$ from its Fourier coefficients via the inner analytic function $w(z)$, as we discussed in [1], which works even when the Fourier series diverges almost everywhere, is equivalent to taking the $\rho \to 1_{(-)}$ limit of the following modified or *regulated* Fourier series,

$$f(\theta) = \lim_{\rho \to 1_{(-)}} \left\{ \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} \rho^k [\alpha_k \cos(k\theta) + \beta_k \sin(k\theta)] \right\}$$

$$= \frac{\alpha_0}{2} + \lim_{\rho \to 1_{(-)}} \sum_{k=1}^{\infty} \rho^k [\alpha_k \cos(k\theta) + \beta_k \sin(k\theta)], \tag{12}$$

which of course is always convergent, so long as $\rho < 1$, for all integrable real functions $f(\theta)$, given that it is the real part of the convergent Taylor series of $w(z)$. The limit indicated will exist when and where $f(\theta)$ can be recovered from the real part of the corresponding inner analytic function. This holds for all the points on the unit circle where the inner analytic function $w(z)$ is either analytic or has only soft singularities. This recipe constitutes, therefore, a very general *summation rule* for Fourier series.

3. Orthogonality Relations

The Fourier series of an integrable real function can be understood as the expansion of that real function in the Fourier basis of functions, which consists of the set of functions

$$\left\{1,\ \cos(k\theta),\ \sin(k\theta),\ k \in \{1, 2, 3, \ldots, \infty\}\right\}. \tag{13}$$

Let us now show that this is an orthogonal basis. Of course this can be done using the standard form of the scalar product between two real functions on the unit circle, by simply calculating a set of easy integrals by elementary means. However, what we want to do here is to show that both the form of the scalar product and the relations of orthogonality and norm are contained within the structure of the inner analytic functions, and can be derived from that structure. In fact, we will show that these elements can be obtained from a particular set of functions, the powers z^k, with $k \geq 0$, and their multiplicative inverses z^{-k}. We start by noting that, if C is any circle centered at the origin, including the unit circle, then from the residues theorem we have that

$$\frac{1}{2\pi\imath}\oint_C dz\, z^{p-1} = \delta_{p,0}, \tag{14}$$

where p is an arbitrary integer, and where $\delta_{p,0}$ is the Kronecker delta. This is so because the integral can be calculated by residues, and a function which is a simple power, either positive or negative, is its own Laurent series, which has only one term. Therefore, its residue at $z = 0$ is zero unless $p = 0$, in which case it is equal to one. Using this result for the case $p = k - k'$, where the integers k and k' are in the set $\{0, 1, 2, 3, \ldots, \infty\}$, we have

$$\frac{1}{2\pi\imath}\oint_C dz\, z^{k-k'-1} = \delta_{k,k'}, \tag{15}$$

while using the same expression for $p = k + k'$, with the limitation that $k + k' > 0$, which means that k and k' cannot both be zero, we have

$$\frac{1}{2\pi\imath}\oint_C dz\, z^{k+k'-1} = 0. \tag{16}$$

This is also a consequence of the Cauchy-Goursat theorem, since in this case the integrand is analytic within the unit disk. Note that the power z^k with $k \geq 0$ is itself an inner analytic function. Writing these two relations in terms of the integration variable θ we have

$$\frac{1}{2\pi}\rho^{k-k'}\int_{-\pi}^{\pi} d\theta\, e^{\imath k\theta} e^{-\imath k'\theta} = \delta_{k,k'},$$

$$\frac{1}{2\pi}\rho^{k+k'}\int_{-\pi}^{\pi} d\theta\, e^{\imath k\theta} e^{\imath k'\theta} = 0, \tag{17}$$

since $z = \rho\exp(\imath\theta)$, where in the second equation we must have $k + k' > 0$. So long as $\rho \neq 0$ the powers of ρ can be eliminated from the second equation, and since the right-hand term of the first equation is zero unless $k = k'$, they can also be eliminated from the first equation, so that we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, e^{\imath k\theta} e^{-\imath k'\theta} = \delta_{k,k'},$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, e^{\imath k\theta} e^{\imath k'\theta} = 0, \tag{18}$$

where in the second equation we must have $k + k' > 0$. Note that this is valid not only on the unit circle, but for all values of ρ so long as $\rho \neq 0$. Expanding the complex exponentials, with the use of the Euler formula, and collecting real and imaginary parts, we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} d\theta\left[\cos(k\theta)\cos(k'\theta) + \sin(k\theta)\sin(k'\theta)\right]$$

$$+\, \imath\,\frac{1}{\pi}\int_{-\pi}^{\pi} d\theta\left[\sin(k\theta)\cos(k'\theta) - \cos(k\theta)\sin(k'\theta)\right] = 2\delta_{k,k'},$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} d\theta\left[\cos(k\theta)\cos(k'\theta) - \sin(k\theta)\sin(k'\theta)\right]$$

$$+\, \imath\,\frac{1}{\pi}\int_{-\pi}^{\pi} d\theta\left[\sin(k\theta)\cos(k'\theta) + \cos(k\theta)\sin(k'\theta)\right] = 0, \tag{19}$$

where in the second equation we must have $k + k' > 0$. Since the right-hand sides are real, we have the four real equations

$$\frac{1}{\pi}\int_{-\pi}^{\pi} d\theta\left[\cos(k\theta)\cos(k'\theta) + \sin(k\theta)\sin(k'\theta)\right] = 2\delta_{k,k'},$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} d\theta\left[\sin(k\theta)\cos(k'\theta) - \cos(k\theta)\sin(k'\theta)\right] = 0,$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} d\theta\left[\cos(k\theta)\cos(k'\theta) - \sin(k\theta)\sin(k'\theta)\right] = 0,$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} d\theta\left[\sin(k\theta)\cos(k'\theta) + \cos(k\theta)\sin(k'\theta)\right] = 0, \tag{20}$$

where we must have $k + k' > 0$ in the last two equations. In the case $k + k' = 0$, which implies that $k = 0$ and $k' = 0$, we obtain from the first equation the identity

$$\frac{1}{\pi}\int_{-\pi}^{\pi} d\theta\cos(0)\cos(0) = 2, \tag{21}$$

which is a part of the relations of orthogonality and norm of the Fourier basis, namely, the one giving the squared norm of the constant function which is equal to one for all θ. The second equation is just a trivial identity when we have $k = 0$ and $k' = 0$, which we may therefore ignore.

We may now assume that we have $k + k' > 0$ for all the four equations. Adding and subtracting the first and third equations, we get

$$\frac{1}{\pi}\int_{-\pi}^{\pi} d\theta \cos(k\theta)\cos(k'\theta) = \delta_{k,k'},$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} d\theta \sin(k\theta)\sin(k'\theta) = \delta_{k,k'}, \tag{22}$$

for $k + k' > 0$, while adding and subtracting the other two equations we get

$$\frac{1}{\pi}\int_{-\pi}^{\pi} d\theta \sin(k\theta)\cos(k'\theta) = 0,$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} d\theta \cos(k\theta)\sin(k'\theta) = 0, \tag{23}$$

for $k + k' > 0$, which are just two copies of the same relation. We have therefore the complete set of orthogonality relations, which also includes those relations giving the norms of the basis functions,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} d\theta \cos(k\theta)\cos(k'\theta) = \delta_{k,k'},$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} d\theta \sin(k\theta)\sin(k'\theta) = \delta_{k,k'},$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} d\theta \sin(k\theta)\cos(k'\theta) = 0, \tag{24}$$

where $k + k' > 0$, which includes all the relevant cases, that is, all the relevant pairs of elements of the basis in Equation (13), except for the single case for $k = 0$ and $k' = 0$, which we examined separately before, leading to Equation (21). Note that this derivation included the determination of the form of the scalar product for the basis elements.

Given two integrable real functions $f(\theta)$ and $g(\theta)$, their scalar product is given by

$$(f|g) = \int_{-\pi}^{\pi} d\theta f(\theta) g(\theta), \tag{25}$$

which induces a positive-definite norm in the space of all integrable real functions defined on the periodic interval, which is thus seen to constitute a Hilbert space. We may therefore conclude that the whole structure of orthogonality and norm of the Fourier basis is contained in the structure of the inner analytic function within the unit disk of the complex plane.

Note that, since all possible inner analytic functions are given by convergent power series within the open unit disk, and since these power series can be understood as infinite linear combinations of the particular set of inner analytic functions given by the non-negative powers $\left\{z^k,\ k \in \{0, 1, 2, 3, \ldots, \infty\}\right\}$, we may think that this set of functions forms a *basis* of the space of inner analytic functions, which we may call the *Taylor basis*. Since the orthogonality of the Fourier basis was obtained above from the properties of this set of non-negative powers, it becomes clear that the orthogonality of the Fourier basis is a consequence of similar properties that must hold for the Taylor basis. In fact, it is possible to define a complex scalar product within the space of inner analytic functions, according to which this Taylor basis is orthogonal. Since this constitutes a considerable detour from our main line of reasoning here, it will be presented as an Appendix. As one can see in Appendix A, this complex scalar product induces in the space of inner analytic functions a positive-definite norm. As was observed in [1], this space forms a vector space over the field of complex numbers, and we thus see that it constitutes in fact a complex Hilbert space.

4. Completeness Relation

Let us now prove the completeness of the Fourier basis. In this context the concept of completeness is that of a basis within a vector space. We will first give a simple and direct proof of completeness, which is however subject to a slight limitation regarding the vector space for which the basis is shown to be complete, using the analytic structure within the open unit disk, and later establish the relation of the concept of completeness with the so-called completeness relation. The proof of completeness using the completeness relation is not subject to any such limitation.

In this section we will prove the following completeness theorem.

Theorem 1. *The basis of real functions* $\{1, \cos(k\theta), \sin(k\theta), k \in \{1, 2, 3, \ldots, \infty\}\}$, *is complete to represent the space of all integrable real functions defined on the unit circle.*

The proof consists of establishing that, given an arbitrary integrable real function $\psi(\theta)$ on the unit circle, which is orthogonal to all the elements of the Fourier basis, according to the scalar product defined in Equation (25), it then follows that $\psi(\theta)$ must be zero almost everywhere. Note that the orthogonality to the elements of the basis means that $\psi(\theta)$ is such that all its Fourier coefficients, as defined in Equation (4), are zero.

Proof 1.1.

Let $\psi(\theta)$ be a real function on the unit circle which can be obtained as the $\rho \to 1_{(-)}$ limit of an inner analytic function. We assume that it is orthogonal to all the elements of the basis, so that all its Fourier coefficients are zero, that is, we assume that for this function we have $\alpha_0 = 0, \alpha_k = 0,$ and $\beta_k = 0,$ for all $k \in \{1, 2, 3, \ldots, \infty\}$. Since we thus have all the Fourier coefficients of $\psi(\theta)$, we may use the construction presented in [1] in order to determine the corresponding inner analytic function. However, since all the Fourier coefficients are zero, it follows at once from the step of that construction given in Equation (5) that for $\psi(\theta)$ the complex coefficients c_k are zero for all k. Therefore, the power series $S(z)$ constructed in the next step of the process, given in Equation (6), is identically zero and thus converges trivially to the identically zero complex function $w_\psi(z) \equiv 0$ on the whole complex plane.

The analyticity region of $w_\psi(z)$ includes the unit circle, and therefore the series converges to zero there. Since on the one hand the series converges to zero, and on the other hand we know that for $\rho = 1$ it necessarily converges to the restriction of $w_\psi(z)$ to the unit circle, it follows that the restriction, including both real and imaginary parts, must be zero everywhere on the unit circle. Therefore, it follows that $\psi(\theta)$ and the identically zero real function coincide everywhere on the unit circle, and therefore we conclude that $\psi(\theta) = 0$ everywhere on that circle. This establishes that the Fourier basis is complete for the space of all integrable real functions defined on the periodic interval, which can be obtained as the $\rho \to 1_{(-)}$ limits of inner analytic functions. This completes the first version of the proof of Theorem 1, which is valid for the vector space of real functions just described.

Note that, since all possible inner analytic functions are given by convergent power series within the open unit disk, and since these power series can be understood as expansions of those inner analytic functions in the Taylor basis of functions given by the non-negative powers $\left\{z^k, k \in \{0, 1, 2, 3, \ldots, \infty\}\right\}$, we may say that this Taylor basis is complete for the space of all inner analytic functions. Since the proof of the completeness of the Fourier basis given above was obtained from the complex-analytic structure within the open unit disk, it becomes clear that the completeness of the Fourier basis on the unit circle is a consequence of the completeness of the Taylor basis within the open unit disk. This adds to the relationship between the Fourier basis on the unit circle and the

Taylor basis on the unit disk, which was first established during the discussion involving the orthogonality of the Fourier basis, in Section 3. In addition to all this, within the spaces generated by either basis one may define scalar products that induce positive-definite norms, thus making them both Hilbert spaces, as is discussed in Appendix A.

Let us now turn to the usual completeness relation. Let us first write it down and then exhibit its usefulness. The relation can be understood as the expression, as a Fourier series, of the Dirac delta "function" defined with respect to a point given by the angle θ_1 on the unit circle, which we examined in great detail in [3], and which we denote by $\delta(\theta - \theta_1)$. As we have shown in [3], using the usual rules for the manipulation of the delta "function", one finds that the corresponding Fourier coefficients are given by

$$\begin{aligned}
\alpha_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta \delta(\theta - \theta_1) \\
&= \frac{1}{\pi}, \\
\alpha_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta \cos(k\theta)\delta(\theta - \theta_1) \\
&= \frac{1}{\pi}\cos(k\theta_1), \\
\beta_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta \sin(k\theta)\delta(\theta - \theta_1) \\
&= \frac{1}{\pi}\sin(k\theta_1),
\end{aligned} \tag{26}$$

for $k \in \{1, 2, 3, \ldots, \infty\}$, so that the completeness relation is given by the Fourier expansion, that turns out to be a bi-linear form on the elements of the Fourier basis,

$$\delta(\theta - \theta_1) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}[\cos(k\theta_1)\cos(k\theta) + \sin(k\theta_1)\sin(k\theta)], \tag{27}$$

which is manifestly divergent, but which can be made to converge for all values of θ, so that we may recover the delta "function" almost everywhere, in fact everywhere but at θ_1, through the use of the summation rule given in Equation (12),

$$\delta(\theta - \theta_1) = \frac{1}{2\pi} + \frac{1}{\pi} \lim_{\rho \to 1_{(-)}} \sum_{k=1}^{\infty} \rho^k [\cos(k\theta_1)\cos(k\theta) + \sin(k\theta_1)\sin(k\theta)]. \quad (28)$$

This is equivalent to the definition of the delta "function" as the $\rho \to 1_{(-)}$ limit of the real part of the inner analytic function given by

$$w_\delta(z, z_1) = \frac{1}{2\pi} - \frac{1}{\pi} \frac{z}{z - z_1}, \quad (29)$$

as was discussed in detail in [3]. One can use the expansion in Equation (27), possibly regulated as in Equation (28), to prove the completeness of the basis, while operating strictly in terms of real objects on or near the unit circle. Here is how this can be done.

Proof 1.2.

If we assume that an arbitrary integrable real function $\psi(\theta)$ on the unit circle is given, which is such that its scalar products with all the elements of the basis are zero, then we have the infinite set of equations

$$\int_{-\pi}^{\pi} d\theta \psi(\theta) = 0,$$

$$\int_{-\pi}^{\pi} d\theta \cos(k\theta) \psi(\theta) = 0,$$

$$\int_{-\pi}^{\pi} d\theta \sin(k\theta) \psi(\theta) = 0, \quad (30)$$

for all $k \in \{1, 2, 3, \ldots, \infty\}$. We may therefore construct an infinite linear combination of all these equations, with the coefficients carefully chosen as shown below, involving an arbitrary parameter θ_1 in the interval

$[-\pi, \pi]$ and an auxiliary strictly positive real variable $\rho < 1$, where the right-hand side is still zero,

$$\left[\frac{1}{2\pi}\right]\int_{-\pi}^{\pi} d\theta\psi(\theta) + \sum_{k=1}^{\infty}\left[\rho^k \frac{1}{\pi}\cos(k\theta_1)\right]\int_{-\pi}^{\pi} d\theta \cos(k\theta)\psi(\theta)$$

$$+ \sum_{k=1}^{\infty}\left[\rho^k \frac{1}{\pi}\sin(k\theta_1)\right]\int_{-\pi}^{\pi} d\theta \sin(k\theta)\psi(\theta) = 0 \Rightarrow$$

$$\int_{-\pi}^{\pi} d\theta\left\{\frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\rho^k[\cos(k\theta_1)\cos(k\theta) + \sin(k\theta_1)\sin(k\theta)]\right\}\psi(\theta) = 0. \tag{31}$$

Since the expression within curly brackets in this last integral is now seen to be the regulated expansion of $\delta(\theta - \theta_1)$ in the Fourier basis, shown in Equation (28), we may therefore take the $\rho \to 1_{(-)}$ limit and write that

$$\int_{-\pi}^{\pi} d\theta\delta(\theta - \theta_1)\psi(\theta) = 0. \tag{32}$$

Finally, using the rules of manipulation of the delta "function", when and where $\psi(\theta)$ is continuous, which it therefore must be almost everywhere, we have

$$\psi(\theta_1) = 0. \tag{33}$$

Since θ_1 is an arbitrary value of θ, we conclude that $\psi(\theta)$ is zero everywhere. This completes the second version of the proof of Theorem 1, which is valid for the vector space of all integrable real functions defined on the unit circle, regardless of whether or not they can be obtained from an inner analytic function.

Note that, in a sense, this method of proof of the completeness of the Fourier basis is a little more limited than the direct proof using the analytic structure within the open unit disk, because we must assume during the argument that $\psi(\theta)$ is continuous almost everywhere. However, since this hypothesis does get confirmed a posteriori by the result obtained, this is not a true limitation.

On the other hand, this second proof is *less* limited than the first one because in this case the vector space of functions for which one shows that the basis is complete is the space of integrable real functions without removable singularities defined on the interval $[-\pi, \pi]$, with *no reference* to whether or not these functions can be obtained as the $\rho \to 1_{(-)}$ limits of inner analytic functions.

In fact, by establishing the completeness of the Fourier basis without any recourse to the $\rho \to 1_{(-)}$ limit for the real functions, as a corollary of this second proof we have shown that there is no integrable real function on the unit circle, other that the identically zero real function, which corresponds to the identically zero inner analytic function. As a consequence of this, there is no integrable real function defined on the unit circle that cannot be represented by an unique inner analytic function.

5. Notes on the Convergence Problem

In this paper we have made the deliberate choice of not discussing the question of the convergence of Fourier series in any amount of detail, that is, we have not discussed any of the many existing so-called Fourier theorems. The reason for this is that we believe that this would constitute a rather long and complex discussion, best left for a separate paper. Instead, we have focused our attention on the summation rule given in Equation (12), according to which *all* Fourier series of integrable real functions, without any further restrictions, can be added up in such a way that one is able to recover the functions from their Fourier coefficients, even if the Fourier series themselves diverge. However, we may make a few comments about the issue of convergence, without going too far afield in that subject, in order to exhibit the relation between our complex analytic structure and the convergence problem.

First of all, let us recall that, as was shown in [1], the real function $f(\theta)$ is equal almost everywhere to the real part of the corresponding inner analytic function $w(z)$, taken in the $\rho \to 1_{(-)}$ limit, and also that, as we have shown in Section 2 of this paper, the Fourier series of $f(\theta)$ is given by the real part of the Taylor series $S(z)$ of $w(z)$ in that same limit. Therefore, it is clearly apparent that, as was already noted in Section 2, the problem of the convergence of real Fourier series is completely identified with the problem of the convergence of the corresponding complex power series on the unit circle, including the cases in which it is the rim of their maximum disks of convergence. Whatever is established for one type of series is also valid for the other. As was also noted in Section 2, the convergence properties on the unit circle will depend on the existence and nature of the singularities of $w(z)$ on that circle.

One way to discuss the issue of convergence is to observe that the summation rule given in Equation (12) involves two limits, one being the series summation limit and the other being the $\rho \to 1_{(-)}$ limit from the interior of the unit disk to the unit circle. What has been shown so far in this series of papers is that if one takes the series summation limit first, and only after that the $\rho \to 1_{(-)}$ limit, then it is always possible to recover the real function from its Fourier coefficients. It is therefore immediately apparent that the statement that the Fourier series converges over the unit circle is equivalent to the statement that the order of these two limits can be inverted. In fact, by first taking the $\rho \to 1_{(-)}$ limit one obtains the usual Fourier series over the unit circle, and if one is then able to take the series summation limit, then that series converges to the corresponding real function.

The general problem of deciding under what conditions the order of the two limits can be inverted is not a simple one. However, it is not too difficult to use our analytic structure to write the partial sums of the Fourier series in terms of real integrals which are similar to the Dirichlet integrals usually involved in some of the Fourier theorems. This can then be used as the starting point for further discussions of the convergence problem, including in particular discussions establishing the connection of the analytic structure with specific Fourier theorems. In order to do this, let $f(\theta)$ be an integrable real function on $[-\pi, \pi]$ and let the real numbers α_0, α_k, and β_k, for $k \in \{1, 2, 3, \ldots, \infty\}$, be its Fourier coefficients. We may define the complex coefficients c_0 and c_k shown in Equation (5), and thus construct the corresponding inner analytic function $w(z)$ within the open unit disk, using the power series $S(z)$ given in Equation (6), which, as was shown in [1], always converges for $|z| < 1$. The partial sums of the first N terms of this series are given by

$$S_N(z) = \sum_{k=0}^{N-1} c_k z^k, \tag{34}$$

a complex sequence which, for $|z| < 1$, we already know to converge to $w(z)$ in the $N \to \infty$ limit. Note however that, since $S_N(z)$ is in fact an analytic function over the whole complex plane, this expression itself can be consistently considered for all z, and in particular for z on the unit circle, where $|z| = 1$. Note also that the function $w(z)$ may have singularities on the unit circle, but that these must be integrable ones, at least along that circle. In addition to this, the complex coefficients c_k may be written as integrals involving $w(z)$, with the use of the Cauchy integral formulas,

$$c_k = \frac{1}{2\pi\imath} \oint_C dz \, \frac{w(z)}{z^{k+1}}, \tag{35}$$

where C can be taken as a circle centered at the origin, with radius $\rho \leq 1$. The reason why we may include the case $\rho = 1$ here is that, as was shown in [1], as a function of ρ the expression above for c_k is not only constant within the open unit disk, but also continuous from within at the unit circle. In this way the coefficients c_k may be written back in terms of the inner analytic function $w(z)$. If we substitute this expression for c_k back in the partial sums of the series we get

$$\begin{aligned} S_N(z) &= \sum_{k=0}^{N-1} z^k \frac{1}{2\pi\imath} \oint_C dz_1 \frac{w(z_1)}{z_1^{k+1}} \\ &= \frac{1}{2\pi\imath} \oint_C dz_1 \frac{w(z_1)}{z_1} \sum_{k=0}^{N-1} \left(\frac{z}{z_1}\right)^k, \end{aligned} \tag{36}$$

where we must have $|z_1| \leq 1$. The sum is now a finite geometric progression, so that we have

$$\begin{aligned} S_N(z) &= \frac{1}{2\pi\imath} \oint_C dz_1 \frac{w(z_1)}{z_1} \frac{1-(z/z_1)^N}{1-(z/z_1)} \\ &= \frac{1}{2\pi\imath} \oint_C dz_1 \frac{w(z_1)}{z_1 - z} - \frac{z^N}{2\pi\imath} \oint_C dz_1 \frac{w(z_1)}{z_1^N (z_1 - z)}. \end{aligned} \tag{37}$$

A careful discussion of this formula is now in order. There are two relevant cases to consider. In the first case we see that, if we have that $|z_1| > |z|$, then in the first term above we obtain the expression of the Cauchy integral formula for $w(z)$, which then allows us to write an explicit expression for the remainder of the complex power series after one adds up its first N terms,

$$\begin{aligned} R_N(z) &= w(z) - S_N(z) \\ &= \frac{z^N}{2\pi\imath} \oint_C dz_1 \frac{w(z_1)}{z_1^N (z_1 - z)}, \end{aligned} \tag{38}$$

where $|z| < |z_1| \leq 1$. This expression of the remainder in closed form, an expression which, as one can easily show, goes to zero in the $N \to \infty$ limit, is what makes it easy to discuss the convergence of complex power series. However, this expression does *not* give us an equivalent expression for the remainder of the Fourier series, because this would require us to make $|z| = |z_1| = 1,$ which is not allowed by the strict inequality $|z| < |z_1|,$ a restriction which is due to the use of the Cauchy integral formulas. In the second case we observe that, if we have that $|z_1| < |z|,$ then the first term in Equation (37) is simply zero, and therefore we get a modified expression for the partial sums of the series,

$$S_N(z) = -\frac{z^N}{2\pi\imath}\oint_C dz_1\,\frac{w(z_1)}{z_1^N(z_1 - z)}, \tag{39}$$

where $|z| > |z_1|.$ Note that in this case we are unable to write an explicit expression in closed form for the remainder of the series, a fact which seems to be related to the remarkable difficulty in finding a necessary and sufficient condition for the convergence of Fourier series. Since z may have any complex value in this expression, we may now make $z = \rho\exp(\imath\theta)$ with $\rho = 1,$ as well as $z_1 = \rho_1\exp(\imath\theta_1),$ and thus write the integral explicitly in terms of the variable θ_1 on the circle of radius $\rho_1,$

$$\begin{aligned} S_N(1,\,\theta) &= -\frac{e^{\imath N\theta}}{2\pi\imath}\int_{-\pi}^{\pi} d\theta_1\imath\rho_1 e^{\imath\theta_1}\,\frac{w(\rho_1,\,\theta_1)}{\rho_1^N e^{\imath N\theta_1}(\rho_1 e^{\imath\theta_1} - e^{\imath\theta})} \\ &= -\frac{1}{2\pi\rho_1^{N-1}}\int_{-\pi}^{\pi} d\theta_1 e^{\imath N(\theta-\theta_1)}\,\frac{w(\rho_1,\,\theta_1)}{\rho_1 - e^{\imath(\theta-\theta_1)}}. \end{aligned} \tag{40}$$

Making $\Delta\theta = \theta_1 - \theta$ we have

$$S_N(1,\,\theta) = -\frac{1}{2\pi\rho_1^{N-1}}\int_{-\pi}^{\pi} d\theta_1 e^{-\imath N\Delta\theta}\,\frac{w(\rho_1,\,\theta_1)}{\rho_1 - e^{-\imath\Delta\theta}}. \tag{41}$$

In order to be able to write explicitly the real and imaginary parts of the partial sums, we must now rationalize this expression,

$$S_N(1,\,\theta) = -\frac{1}{2\pi\rho_1^{N-1}}\int_{-\pi}^{\pi} d\theta_1 e^{-\imath N\Delta\theta}\,\frac{w(\rho_1,\,\theta_1)(\rho_1 - e^{\imath\Delta\theta})}{(\rho_1 - e^{-\imath\Delta\theta})(\rho_1 - e^{\imath\Delta\theta})}$$

$$= \frac{1}{2\pi\rho_1^{N-1}}\int_{-\pi}^{\pi} d\theta_1 w(\rho_1,\,\theta_1) e^{-\imath(N-1/2)\Delta\theta}$$

$$\times\,\frac{e^{\imath\Delta\theta/2} - \rho_1 e^{-\imath\Delta\theta/2}}{1 + \rho_1^2 - 2\rho_1\cos(\Delta\theta)}. \tag{42}$$

The expression can be somewhat simplified if we write most things in terms of $\Delta\theta/2$, as well as in terms of $N_1 = N - 1/2$,

$$S_N(1,\,\theta) = \frac{1}{2\pi\rho_1^{N-1}}\int_{-\pi}^{\pi} d\theta_1[u(\rho_1,\,\theta_1) + \imath v(\rho_1,\,\theta_1)]$$

$$\times\,[\cos(N_1\Delta\theta) - \imath\sin(N_1\Delta\theta)]$$

$$\times\,\frac{(1-\rho_1)\cos(\Delta\theta/2) + \imath(1+\rho_1)\sin(\Delta\theta/2)}{(1-\rho_1)^2 + 4\rho_1\sin^2(\Delta\theta/2)}. \tag{43}$$

In this context, a Fourier theorem is one which states sufficient conditions on $f(\theta)$ under which it follows that the real part of the corresponding sequence of partial sums $S_N(1,\,\theta)$ converges in the $N \to \infty$ limit, after one takes the $\rho_1 \to 1$ limit, so that the integral is written over the unit circle. In any circumstances in which one managed to calculate these integrals explicitly in terms of ρ_1, for $\rho_1 < 1$, one would then be able to consider taking the $\rho_1 \to 1$ limit of the resulting expression. However, despite the facts that $f(\theta) = u(1,\,\theta_1)$ and that $g(\theta) = v(1,\,\theta_1)$, almost everywhere over the unit circle, as well as the fact that these are integrable real functions, we cannot simply take the $\rho_1 \to 1$ limit of this expression as it stands, because it was derived under the hypothesis that $|z| > |z_1|$, and therefore that $\rho > \rho_1$, which at this point implies the strict inequality $1 > \rho_1$. We may however put $\rho_1 = 1$ in the integrand simply in order to simplify the integrals, so as to exhibit their structure more clearly. If one does that one obtains

$$\int_{-\pi}^{\pi} d\theta_1 [f(\theta_1) + \imath g(\theta_1)] \frac{\sin[(N - 1/2)\Delta\theta] + \imath \cos[(N - 1/2)\Delta\theta]}{\sin(\Delta\theta/2)}, \tag{44}$$

which clearly reduces to Dirichlet integrals and other similar integrals. A more complete discussion of the issue of convergence would require considerable development of the ideas and structures involved in these arguments. It is currently not entirely clear how useful the analytic structure within the open unit disk can be in regards to proving known Fourier theorems or discovering new ones.

6. Extension of the Theory

Up to this point we have been examining only the Fourier theory of integrable real functions. In addition to this, a small extension of the theory has already been considered when we wrote the Fourier expansion of the Dirac delta "function" in Equations (27) and (28) of Section 4, with the help of the summation rule given in Equation (12). This "function" has in common with the integrable real functions the fact that its Fourier coefficients α_k and β_k are limited when we take the limit $k \to \infty$. The same is true for the corresponding complex Taylor coefficients c_k in either case. However, the correspondence between real Fourier coefficients and complex Taylor coefficients given by the relations in Equation (5) can be generalized, independently of any concerns about the behaviour of these coefficients when $k \to \infty$, and independently of any concerns about the convergence of the corresponding series.

We will now discuss the extension of the Fourier theory beyond the realm of integrable real functions. One way to look at this, which is probably the most general possible way, is to simply consider the set of *all* inner analytic functions. Given *any* inner analytic function $w(z)$ and its complex Taylor series around the origin, which is therefore convergent within the open unit disk, and irrespective of whether or not $w(z)$ corresponds to an integrable real function, one can define a corresponding real Fourier series on the unit circle. In all such cases, the issues of convergence of the resulting Fourier series are then completely identified with the corresponding issues for the Taylor series restricted to the unit circle, which is often the border of its maximum disk of convergence. Important examples which are not related to integrable real functions are the cases of the Dirac delta "function" and of its derivatives of all orders, which were discussed in detail in [3].

Another way to look at this issue is through the properties of the sets of complex coefficients c_k of the Taylor series. Given any set of complex coefficients c_k, regardless of whether or not they follow from a known inner analytic function, one can construct both a complex power series $S(z)$ and the corresponding real coefficients α_k and β_k, using the relations in Equations (5) and (6). In many cases the Fourier series generated by these real coefficients will not converge, even if the complex power series converges to an inner analytic function within the open unit disk. However, if the complex power series is indeed convergent on that disk, then one can discuss whether or not a real object can be defined on the unit circle, through the $\rho \to 1_{(-)}$ limit from the open unit disk, for example using the summation rule for Fourier series given in Equation (12).

If we examine that summation rule, it is apparent that it will work for much more than just integrable real functions, which always have bounded Fourier coefficients. For example, one may have unbounded Fourier coefficients α_k and β_k, such as those of the n-th derivative of the delta "function", which diverge to infinity as the power k^n when $k \to \infty$, and still have a well-defined inner analytic function, as was shown in detail in [3]. In fact, one can show that the summation rule can be used for all sets of Fourier coefficients that do *not* diverge exponentially fast with

k. In order to develop this idea, let us first define a very general condition on the sequences of complex coefficients that guarantees that the corresponding power series are convergent within the open unit disk, and thus converge to inner analytic functions.

Definition 1 (Exponentially bounded coefficients).

Given an arbitrary ordered set of complex coefficients a_k, for $k \in \{0, 1, 2, 3, \dots, \infty\}$, if they satisfy the condition that

$$\lim_{k \to \infty} |a_k| e^{-Ck} = 0, \tag{45}$$

for all real $C > 0$, then we say that the sequence of coefficients a_k is *exponentially bounded*.

What this means is that a_k may or may not go to zero as $k \to \infty$, may approach a non-zero complex number, and may even diverge to infinity as $k \to \infty$, so long as it does not do so exponentially fast. This includes therefore not only the sequences of complex Taylor coefficients corresponding to all possible convergent Fourier series, but many sequences that correspond to Fourier series that diverge almost everywhere. Also, it not only includes the sequences of complex Taylor coefficients corresponding to all possible integrable real functions, but many sequences of coefficients that cannot be obtained at all from a real function, such as those associated to the Dirac delta "function" and its derivatives of arbitrarily high orders, as was shown in [3]. We see therefore that this is a very weak condition on the complex sequence of coefficients a_k.

Before we proceed to the extension of the Fourier theory, let us establish a preliminary result, which can be understood as a property of the sequences of complex coefficients a_k which satisfy the condition stated in Definition 1. We will show that the condition expressed in Equation (45) implies an infinite collection of other similar conditions involving the $k \to \infty$ limit, that express modified bounds on these sequences of coefficients.

Property 1.1. If the sequence of complex coefficients a_k is exponentially bounded, then we also have that

$$\lim_{k\to\infty}|a_k|k^p e^{-Ck} = 0, \tag{46}$$

for all real $C > 0$ and for all real powers $p > 0$.

This is just a formalization of the well-known fact that the negative-exponent real exponential function of k goes to zero faster than any positive real power of k goes to infinity, as $k \to \infty$. In order to prove this, we observe that for $k > 0$ we may write the function of k on the left-hand side of Equation (46) as

$$|a_k|k^p e^{-Ck} = |a_k|e^{p\ln(k)}e^{-Ck}. \tag{47}$$

Note that this is a positive real quantity. Recalling the properties of the real logarithm function, we now observe that, given an arbitrary real number $A > 0$, there is always a sufficiently large finite value k_m of k above which $\ln(k) < Ak$. Due to this we may write, for all $k > k_m$,

$$|a_k|k^p e^{-Ck} < |a_k|e^{pAk}e^{-Ck}, \tag{48}$$

since the exponential with a strictly positive real exponent is a monotonically increasing function. If we now choose $A = C/(2p)$, which we may do because this value is positive and not zero, we get that, for all $k > k_m$,

$$\begin{aligned} |a_k|k^p e^{-Ck} &< |a_k|e^{Ck/2}e^{-Ck} \\ &= |a_k|e^{-Ck/2}. \end{aligned} \tag{49}$$

According to our hypothesis about the coefficients a_k, the $k \to \infty$ limit of the expression in the right-hand side is zero for any strictly positive value of $C' = C/2$, so that taking the $k \to \infty$ limit we establish our preliminary result,

$$\lim_{k\to\infty} |a_k| k^p e^{-Ck} = 0, \tag{50}$$

for all real $C > 0$ and all real $p > 0$. Therefore, we have established this property.

Let us now show that the condition that the sequence of complex coefficients c_k in Equation (5) is exponentially bounded is equivalent to the condition that the sequences of real coefficients α_k and β_k are both exponentially bounded. First, if we assume that the sequences α_k and β_k are both exponentially bounded, and since from Equation (5) we have that

$$|c_k| = \sqrt{|\alpha_k|^2 + |\beta_k|^2}, \tag{51}$$

it follows at once that

$$\begin{aligned}
\lim_{k\to\infty} |c_k| e^{-Ck} &= \lim_{k\to\infty} \sqrt{(|\alpha_k| e^{-Ck})^2 + (|\beta_k| e^{-Ck})^2} \\
&= \sqrt{\left(\lim_{k\to\infty} |\alpha_k| e^{-Ck}\right)^2 + \left(\lim_{k\to\infty} |\beta_k| e^{-Ck}\right)^2} \\
&= 0,
\end{aligned} \tag{52}$$

since both limits in the right-hand side are zero, thus establishing that the sequence c_k is exponentially bounded. Second, if we assume that the sequence c_k is exponentially bounded, and since from Equation (5) we have that

$$\begin{aligned}
|c_k| &= \sqrt{|\alpha_k|^2 + |\beta_k|^2} \\
&\geq |\alpha_k| \Rightarrow \\
|c_k| e^{-Ck} &\geq |\alpha_k| e^{-Ck},
\end{aligned} \tag{53}$$

taking the $k \to \infty$ limit and using the assumed property of the sequence of coefficients c_k it follows that

$$\lim_{k\to\infty} |c_k| e^{-Ck} \geq \lim_{k\to\infty} |\alpha_k| e^{-Ck} \Rightarrow$$

$$0 \geq \lim_{k\to\infty} |\alpha_k| e^{-Ck} \Rightarrow$$

$$\lim_{k\to\infty} |\alpha_k| e^{-Ck} = 0, \tag{54}$$

thus establishing that the sequence α_k is exponentially bounded. Clearly, an identical argument can be made for the sequence β_k. This establishes that the statement that the sequence of complex coefficients c_k is exponentially bounded is equivalent to the statement that the sequences of real coefficients α_k and β_k are both exponentially bounded.

Let us now prove the following theorem about the convergence of the power series constructed out of a given arbitrary sequence of complex coefficients c_k.

Theorem 2. *If the sequence of complex coefficients* c_k, *for* $k \in \{0, 1, 2, 3, \ldots, \infty\}$, *is exponentially bounded, then the power series constructed from this sequence of coefficients converges within the open unit disk.*

Given the arbitrary sequence of complex coefficients c_k, we may construct the complex power series in the complex z plane, just as we did in [1],

$$S(z) = \sum_{k=0}^{\infty} c_k z^k. \tag{55}$$

We will first show that, if the sequence of coefficients c_k is exponentially bounded, then this series is absolutely convergent inside the open unit disk, which then implies that it is simply convergent there.

Proof 2.1.

In order to prove that $S(z)$ is absolutely convergent, we consider the real power series $\overline{S}(z)$ of the absolute values of the terms of that series, which we write as

$$\begin{aligned}\overline{S}(z) &= \sum_{k=0}^{\infty}|c_k|\rho^k\\ &= \sum_{k=0}^{\infty}|c_k|e^{k\ln(\rho)}.\end{aligned} \tag{56}$$

Since $\rho < 1$ inside the open unit disk, the logarithm shown is strictly negative, and we may put $\ln(\rho) = -C$ with real $C > 0$. We can now see that, according to our hypothesis about the coefficients c_k, the terms of this series go to zero as $k \to \infty$,

$$\overline{S}(z) = \sum_{k=0}^{\infty}|c_k|e^{-Ck}, \tag{57}$$

since C is real and strictly positive. In order to establish the convergence of this real series, we write

$$\overline{S}(z) = |c_0| + \sum_{k=1}^{\infty}\frac{k^2|c_k|e^{-Ck}}{k^2}. \tag{58}$$

According to the property expressed in Equation (46), with $p = 2$, the numerator shown above goes to zero as $k \to \infty$, and therefore above a sufficiently large value k_m of k it is less that one, so that we may write that

$$\begin{aligned}\overline{S}(z) &= \sum_{k=0}^{k_m}|c_k|e^{-Ck} + \sum_{k=k_m+1}^{\infty}\frac{k^2|c_k|e^{-Ck}}{k^2}\\ &< \sum_{k=0}^{k_m}|c_k|e^{-Ck} + \sum_{k=k_m+1}^{\infty}\frac{1}{k^2}.\end{aligned} \tag{59}$$

The first term on the right-hand side is a finite sum and therefore is finite, and the second term can be bounded from above by a convergent asymptotic integral on k, so that we have

$$\begin{aligned}\overline{S}(z) &< \sum_{k=0}^{k_m} |c_k| e^{-Ck} + \int_{k_m}^{\infty} dk \, \frac{1}{k^2} \\ &= \sum_{k=0}^{k_m} |c_k| e^{-Ck} + \frac{-1}{k} \Big[_{k_m}^{\infty} \\ &= \sum_{k=0}^{k_m} |c_k| e^{-Ck} + \frac{1}{k_m}.\end{aligned} \tag{60}$$

This last expression is therefore a finite upper bound for all the partial sums of the series $\overline{S}(z)$. It follows that $\overline{S}(z)$, which is a real sum of positive terms, so that its partial sums form a monotonically increasing real sequence which is now found to be bounded from above, is therefore convergent. It then follows that $S(z)$ is absolutely convergent and therefore convergent. Since this is valid for all $\rho < 1$, we may conclude that $S(z)$ converges on the open unit disk. This completes the proof of Theorem 2.

Since the series $S(z)$ considered above is a convergent power series within the open unit disk, it converges to an analytic function $w(z)$ in that domain, which is therefore an inner analytic function. We therefore conclude that, if the sequence of complex coefficients c_k in Equation (5) is exponentially bounded, then it is the set of Taylor coefficients of an inner analytic function. It now follows that, if the corresponding Fourier coefficients α_k and β_k are both exponentially bounded, then the corresponding complex coefficients c_k are also exponentially bounded, and therefore the corresponding Fourier series can be regulated by the use of the summation rule in Equation (12). Unless the Fourier coefficients go to zero as $k \to \infty$, the Fourier series on the unit circle is sure to diverge almost everywhere. One can then consider defining the corresponding real object on the unit circle using the $\rho \to 1_{(-)}$ limit from the open unit disk, for example through the use of the summation rule for the Fourier series, given in Equation (12).

In this way the Fourier theory of integrable real functions on the unit circle can be extended to a much larger set of real objects, including for example all the singular distributions discussed in [3], as well as the examples of non-integrable real functions mentioned in that paper. In fact, this extension of the Fourier theory includes a large class of non-integrable real functions, as will be shown in the fourth paper of this series. In this extended Fourier theory the real objects can be considered as representable directly by their sequences of Fourier coefficients, even when the corresponding Fourier series diverge. All operations involving these divergent Fourier series can be mapped to absolutely and uniformly convergent series and analytic operations within the open unit disk, whose results are then taken to the unit circle through the use of the $\rho \to 1_{(-)}$ limit. In many simple cases the mere values of the real objects on the unit circle will be recovered in this way, and in other more abstract cases global properties of the real objects may be obtained in this way, such as in the case of the Dirac delta "function" and its derivatives of all orders, as was discussed in detail in [3].

7. Conclusions and Outlook

We have shown that the complex-analytic structure within the unit disk of the complex plane established in a previous paper [1], which leads to a close and deep relationship between integrable real functions on the unit circle and inner analytic functions within the unit disk centered at the origin of the complex plane, includes the whole structure of the Fourier theory of integrable real functions. This fact leads to the definition of a very general and powerful summation rule for Fourier series, which allows one to still use and manipulate in a consistent way divergent

Fourier series, even when they are explicitly and strongly divergent. The connection of the complex-analytic structure with the usual Fourier theorems was exhibited.

The Fourier theory was then extended to include all the inner analytic functions associated to singular Schwartz distributions, which were discussed in detail in another previous paper [3], in which the discussion of the complex-analytic structure was generalized to include those singular distributions. In fact, the Fourier theory can be extended to essentially the whole space of inner analytic functions. This includes at least some non-integrable real functions, as was pointed out in [3]. The generalization to a much wider class of non-integrable real functions will be tackled in a future paper.

As part of this process of extension, we introduced the concept of an exponentially bounded sequence of complex coefficients c_k, and proved that any such sequence is the set of Taylor coefficients of some inner analytic function. As interesting open question is whether or not the reverse of this statement is true, that is, whether or not the criterion that the sequences of complex coefficients of the power series be exponentially bounded includes all possible inner analytic functions. At this time this seems rather unlikely, and in that case the problem poses itself of what more general condition on the coefficients could cover the whole space of inner analytic functions.

We believe that the results presented here establish a new perspective for the study of the Fourier theory of real functions and related objects. It provides a simple and complete account of all the mathematical structures involved, as well as of all the main results of that theory, including in particular a simple and solid proof of the completeness of the basis. Due to this, it might also constitute a simpler and more efficient way to teach the subject.

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Appendix A: Scalar Product for Inner Analytic Functions

Given two inner analytic functions $w_1(z)$ and $w_2(z)$, we consider the complex contour integral over the circle C_0 of radius ρ_0, with $0 < \rho_0 < 1$, given by

$$(w_1|w_2) = \frac{1}{2\pi\imath}\oint_{C_0} dz\,\frac{1}{z}\,w_1^*(z)w_2(z). \tag{61}$$

Since the integrand in this expression is *not* analytic, the integral depends on the circuit, and therefore on ρ_0. Therefore, what we have here is in fact a one-parameter family of integrals. We will show that for each value of ρ_0 this integral defines a scalar product within the space of inner analytic functions, which induces in that space a positive-definite norm. If we write the integral in terms of the integration variable θ, with constant ρ_0, we get for this scalar product

$$(w_1|w_2) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta w_1^*(\rho_0, \theta)w_2(\rho_0, \theta). \tag{62}$$

If we now make both $w_1(z)$ and $w_2(z)$ equal to $w(z) = u(\rho, \theta) + \imath v(\rho, \theta)$, we get

$$\begin{aligned}(w|w) &= \|w\|^2 \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta|w(\rho_0, \theta)|^2 \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta[u^2(\rho_0, \theta) + v^2(\rho_0, \theta)] \\ &\geq 0,\end{aligned} \tag{63}$$

which is a manifestly real and positive quantity, that is zero if and only if $w(\rho_0, \theta) = 0$ for all θ, which in turn is equivalent to $w(\rho, \theta) = 0$ for all θ and all ρ within the open unit disk, because all zeros of an analytic function must be isolated, unless it is the identically zero function. Therefore, for each value of the parameter ρ_0 the real quantity $\|w\|$ is a positive-definite norm on the space of all inner analytic functions which, as was observed in [1], forms a vector space over the field of complex numbers. That vector space is thus seen to constitute a complex Hilbert space, with this scalar product and the associated positive-definite norm.

We can also see from the equation above that the scalar product and the norm reduce naturally to the corresponding definitions for the real functions $u(1, \theta)$ and $v(1, \theta)$ on the unit circle, when we take the $\rho_0 \to 1_{(-)}$ limit, thus establishing a close correspondence between these two identical real Hilbert spaces on the unit circle and the complex Hilbert space on the unit disk. In addition to this, for any value of ρ_0 within the open interval $(0, 1)$ we also have a pair of identical real Hilbert spaces with the real functions $u(\rho_0, \theta)$ and $v(\rho_0, \theta)$ on the circle of radius ρ_0.

We may now show that the Taylor basis of functions around the origin, which is complete to generate the whole space of inner analytic functions, and which consists of the set of non-negative powers

$$\left\{z^k,\ k \in \{0, 1, 2, 3, \ldots, \infty\}\right\}, \tag{64}$$

is in fact an orthogonal basis according to this definition of the scalar product. If we make $w_1(z) = w_{k_1}(z) = z^{k_1}$ and $w_2(z) = w_{k_2}(z) = z^{k_2}$, we get

$$\begin{aligned}(w_{k_1}|w_{k_2}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \left(z^{k_1}\right)^* z^{k_2}\\ &= \rho_0^{k_1+k_2}\,\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta e^{-\imath k_1\theta} e^{\imath k_2\theta}.\end{aligned} \tag{65}$$

Using now the first result shown in Equation (18) we obtain the orthogonality relation for the Taylor basis,

$$(w_{k_1}|w_{k_2}) = \rho_0^{k_1+k_2}\delta_{k_1,k_2}. \tag{66}$$

Since the integer powers are analytic on the whole complex plane, there is no obstruction to taking the $\rho_0 \to 1_{(-)}$ limit, and thus we see that in this case the Taylor basis is not only orthogonal, but also normalized,

$$(w_{k_1}|w_{k_2}) = \delta_{k_1,k_2}, \tag{67}$$

with $\|w_k\| = 1$ for all k, where the scalar product is now defined on the unit circle. If we write the inner analytic functions in terms of their Taylor series around the origin,

$$w_1(z) = \sum_{k=0}^{\infty} c_{1,k} z^k,$$

$$w_2(z) = \sum_{k=0}^{\infty} c_{2,k} z^k, \tag{68}$$

we obtain for the scalar product, since we may always integrate convergent power series term-by-term,

$$\begin{aligned}
(w_1|w_2) &= \frac{1}{2\pi\imath}\oint_{C_0} dz\,\frac{1}{z}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} c_{1,k_1}^* c_{2,k_2}\left(z^{k_1}\right)^* z^{k_2} \\
&= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} c_{1,k_1}^* c_{2,k_2}\,\frac{1}{2\pi\imath}\oint_{C_0} dz\,\frac{1}{z}\left(z^{k_1}\right)^* z^{k_2} \\
&= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} c_{1,k_1}^* c_{2,k_2}(w_{k_1}|w_{k_2}) \\
&= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} c_{1,k_1}^* c_{2,k_2}\rho_0^{k_1+k_2}\delta_{k_1,k_2} \\
&= \sum_{k=0}^{\infty} \rho_0^{2k} c_{1,k}^* c_{2,k},
\end{aligned} \tag{69}$$

where we identified the scalar product $(w_{k_1}|w_{k_2})$ and then used the orthogonality relations of the Taylor basis. So long as $\rho_0 < 1$, and so long as $c_{1,k}$ and $c_{2,k}$ are exponentially bounded, this series converges exponentially fast. We may also write the corresponding expression for the norm, if we make $c_{1,k} = c_{2,k} = c_k$ and $w_1(z) = w_2(z) = w(z)$,

$$\begin{aligned}\|w\|^2 &= (w|w)\\ &= \sum_{k=0}^{\infty}\rho_0^{2k}|c_k|^2, \end{aligned}\tag{70}$$

with the same conditions for the convergence of the series. In all this structure, if we take the $\rho_0 \to 1_{(-)}$ limit, the scalar product and the norm may in general diverge, unlike what happens in the case of the elements of the Taylor basis. However, so long as $\rho_0 < 1$ all the inner analytic functions have finite norms and finite scalar products with one another. In some cases, it may be possible to determine the values of these quantities on the unit circle using the $\rho_0 \to 1_{(-)}$ limit, even if the corresponding series expressions written directly on the unit circle diverge.

Perhaps the best way to characterize this structure is as a one-parameter family of pairs of identical real Hilbert spaces, one associated to the real parts and another associated to the imaginary parts of the inner analytic functions, where the parameter is the radius ρ_0 of each circle within the unit disk, which are connected to each other by a process of analytic continuation. For each value of ρ_0 within the open interval $(0, 1)$, there is a one-to-one mapping between the inner analytic functions on the open unit disk and the real functions obtained as the real parts of these inner analytic function restricted to the circle of radius ρ_0. This one-to-one mapping preserves the scalar product and the norm, as they are defined within each space. This fact is still true even in the $\rho_0 \to 1_{(-)}$

limit, although in that case not every real object at the unit circle, resulting from the limit, is a normal real function, and although in many cases the norms and scalar products may diverge in the limit.

Note that the integral defining the scalar product of the inner analytic functions is a one-dimensional integral over the circle of radius ρ_0, despite the fact that each complex inner analytic function consist of a pair of real functions of two variables. However, this is a natural characteristic of the scalar product in this context, since it is a well-known fact that an analytic function is completely determined on a two-dimensional region of the complex plane by its values only at a one-dimensional boundary of that region. In this way, although only a one-dimensional restriction of the inner analytic function is explicitly taken into account in the integral over the circle of radius ρ_0 that defines the scalar product, that restriction still includes implicitly the whole structure of the inner analytic function within the corresponding disk of radius ρ_0. Therefore, it is perhaps arguable that the most natural definition of the scalar product is that associated to the choice $\rho_0 = 1$, despite the convergence issues that this choice may involve.