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COMPLEX ANALYSIS OF REAL FUNCTIONS I: COMPLEX-ANALYTIC STRUCTURE AND INTEGRABLE REAL FUNCTIONS

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Abstract

A complex-analytic structure within the unit disk of the complex plane is presented. It can be used to represent and analyze a large class of real functions. It is shown that any integrable real function can be obtained by means of the restriction of an analytic function to the unit circle, including functions which are non-differentiable, discontinuous or unbounded. An explicit construction of the analytic functions from the corresponding real functions is given. The complex-analytic structure can be understood as an universal regulator for analytic operations on real functions.

1. Introduction

In this paper, we will exhibit a mathematical structure, based on certain analytic functions within the unit circle of the complex plane, that can be used to represent and analyze a very wide class of real functions. These include analytic and non-analytic integrable real functions, as well

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as unbounded integrable real functions. All these objects will be interpreted as parts of a larger complex-analytic structure, within which they can be treated and manipulated in a robust and unified way.

In order to assemble the mathematical structure a set of mathematical objects must be introduced, and their properties established. This will be done in Section 2, in which all the eight necessary definitions will be given, and all the corresponding properties will be stated and proved. The objects to be defined are elements within complex analysis [1], and include a general scheme for the classification of all possible singularities of analytic functions, as well as the concept of infinite integral-differential chains of functions.

As a first and important application of this complex-analytic structure, in Section 3 we will establish the relation between the complex-analytic structure and integrable real functions. There we will show that every integrable real function defined within a finite interval corresponds to an inner analytic function and can be obtained by means of the restriction of the real part of that analytic function to the unit circle of the complex plane.

This is the first of a series of papers. The discussion of some parts and aspects of this line of work will be postponed to forthcoming papers, in order to keep each paper within a reasonable length. In the second paper of the series we will extend the complex-analytic structure presented in this paper, to include the whole space of singular Schwartz distributions, also known as generalized real functions.

In the third paper of the series we will show that the whole Fourier theory of integrable real functions is contained within that same complex-analytic structure. We will show that this structure induces a very general and powerful summation rule for Fourier series, that can be used to add up Fourier series in a consistent way, even when they are explicitly and strongly divergent. The complex-analytic structure will then allow us to extend the Fourier theory beyond the realm of integrable real functions, with the use of that summation rule.

In the fourth paper of the series we will show that one can include in the same complex-analytic structure a large class of non-integrable real functions, among those that are locally integrable almost everywhere. We will see that the complex-analytic structure allows us to associate to each such function a definite set of Fourier coefficients, despite the fact that the functions are not integrable on the unit circle. There are also other applications of the structure discussed here, for example, in the two-dimensional Dirichlet problem in partial differential equations, a discussion of which will be given in the fifth paper of the series.

The material contained in this paper is a development, reorganization and extension of some of the material found, sometimes still in rather rudimentary form, in the papers [2-6].

2. Definitions and Properties

Here we will introduce the definitions and basic properties of some objects and structures which are not usually discussed in complex analysis [1], and which we will use in the subsequent sections. Consider then the unit circle centered at the origin of the complex plane. Its interior is the open unit disk we will often refer to along the paper. Any reference to the unit disk or to the unit circle should always be understood to refer to those centered at the origin.

Definition 1 (Inner analytic functions).

A complex function $w(z)$ which is analytic in the open unit disk will be named an *inner analytic function*. We will consider the set of all such functions. We will also consider the subset of such functions that have the additional property that $w(0) = 0$, which we will name *proper inner analytic functions*.

Note, in passing, that the set of all inner analytic functions forms a vector space over the field of complex numbers, and so does the subset of all proper inner analytic functions.

The focus of this study will be the set of real objects which are obtained from the real parts of these inner analytic functions when we take the limit from the open unit disk to its boundary, that is, to the unit circle. Specifically, if we describe the complex plane with polar coordinates (ρ, θ), then an arbitrary inner analytic function can be written as

$$w(z) = u(\rho, \theta) + \imath v(\rho, \theta), \tag{1}$$

where $\imath$ is the imaginary unit, and we consider the set of real objects $f(\theta)$ obtained from the set of all inner analytic functions as the limits of their real parts, from the open unit disk to the unit circle,

$$f(\theta) = \lim_{\rho \to 1_{(-)}} u(\rho, \theta), \tag{2}$$

when and where such limits exist, or at least can be defined in a consistent way.

Note that an inner analytic function may have any number of singularities on the unit circle, as well as in the region outside the unit circle. The concept of a singularity is the usual one in complex analysis, namely that a singular point is simply a point where the function fails to be analytic. The singularities on the unit circle will play a particularly important role in the complex-analytic structure to be presented in this paper. If any of these singularities turn out to be branch points, then we assume that the corresponding branch cuts extend outward from the unit circle, either out to infinity or connecting to some other singularity that may exist outside the open unit disk.

Note also that the imaginary parts of the inner analytic functions do not generate an independent set of real objects, since the imaginary part $v(\rho, \theta)$ of the inner analytic function $w(z)$ is also the real part of the inner analytic function $\overline{w}(z)$ given by

$$\overline{w}(z) = -\imath w(z). \tag{3}$$

We thus see, however, that the inner analytic functions do organize the real functions in matched pairs, those originating from the real and imaginary parts of each inner analytic function. The two real functions forming such a pair may be described as mutually *Fourier conjugate* functions. Finally, we will assume that, at all singular points where the functions $w(z)$ can still be defined by continuity, they have been so defined.

In addition to establishing this correspondence between complex functions on the unit disk and real function of the unit circle, we will find it necessary to define analytic operations on the complex functions that correspond to the ordinary operations of differentiation and integration on the real functions. As will be shown in what follows, the next two definitions accomplish this.

Definition 2 (Angular differentiation).

Given an arbitrary inner analytic function $w(z)$, its *angular derivative* is defined by

$$w^{\bullet}(z) = \imath z \frac{dw(z)}{dz}. \tag{4}$$

The angular derivative of $w(z)$ will be denoted by the shifted dot, as shown. The second angular derivative will be denoted by $w^{2\bullet}(z)$, and so on.

Note that this definition has been tailored in order for the following property to hold.

Property 2.1. In terms of the variables (ρ, θ), angular differentiation is equivalent to partial differentiation with respect to θ, taken at constant ρ.

Writing $z = \rho\exp(\imath\theta)$, and considering the partial derivative of z with respect to θ, we have

$$\begin{aligned} w^{\bullet}(z) &= \imath\rho e^{\imath\theta}\,\frac{1}{\imath\rho e^{\imath\theta}}\,\frac{\partial w(z)}{\partial\theta} \\ &= \frac{\partial u(\rho,\,\theta)}{\partial\theta} + \imath\,\frac{\partial v(\rho,\,\theta)}{\partial\theta}, \end{aligned} \tag{5}$$

which establishes this property.

Note that by construction we always have that $w^{\bullet}(0) = 0$, so that we may say that the operation of angular differentiation projects the space of inner analytic functions onto the space of proper inner analytic functions. We may now prove an important property of the angular derivative.

Property 2.2. The angular derivative of an inner analytic function is also an inner analytic function.

Let us recall that the derivative of an analytic function always exists and is also analytic, in the same domain of analyticity of the original function. Since the constant function $w(z) \equiv \imath$ and the identity function $w(z) \equiv z$ are analytic in the whole complex plane, and since the product of analytic functions is also an analytic function, in their common domain of analyticity, it follows at once that the angular derivative of an inner analytic function is an inner analytic function as well, which establishes this property.

In the other words, the operation of angular differentiation stays within the space of inner analytic functions. Note that, since $w^{\bullet}(0) = 0$, angular differentiation always results in *proper* inner analytic functions, and therefore that this operation also stays within the space of proper inner analytic functions.

Definition 3 (Angular integration).

Given an arbitrary inner analytic function $w(z)$, its *angular primitive* is defined by

$$w^{-1\bullet}(z) = -\imath \int_0^z dz' \, \frac{w(z') - w(0)}{z'}, \tag{6}$$

where the integral is taken along any simple curve from 0 to z contained within the open unit disk. Since the integrand is analytic inside the open unit disk, including at the origin, as we will see shortly while proving Property 3.2, due to the Cauchy-Goursat theorem the integral does not depend on the curve along which it is taken. The angular primitive will also be denoted by a shifted dot, this time preceded by a negative integer, as indicated above.

Let us prove that the apparent singularity of the integrand at $z = 0$ is in fact a removable singularity, so that the integrand can be defined at the origin by continuity, thus producing a function which is continuous and well defined there. If we simply take the $z \to 0$ limit of the integrand we get

$$\lim_{z \to 0} \frac{w(z) - w(0)}{z} = \frac{dw}{dz}(0), \tag{7}$$

since this limit is the very definition of the derivative of $w(z)$ at $z = 0$. Since $w(z)$ is an inner analytic function, and is thus analytic in the open unit disk, it is differentiable at the origin, so that this limit exists and is finite. Therefore, the integrand can be defined at the origin to have this particular value, so that it is continuous there. We assume that the integrand is so defined at $z = 0$, as part of the definition of the angular primitive.

Note that this definition has been tailored in order for the following property to hold.

Property 3.1. In terms of the variables (ρ, θ), angular integration can be understood as integration with respect to θ, taken at constant ρ, up to an integration constant.

Given any point z in the open unit disk, and considering that we are free to choose the path of integration from 0 to z, we now choose to go first from the origin along the positive real axis, until we reach the radius ρ, and then to go along an arc of circle of radius ρ, until we reach the angle θ, thus separating the integral in two. In the first integral, the variations of z are given by $dz = d\rho$, and in the second one they are given by $dz = \imath\rho\exp(\imath\theta)d\theta$. Note that as the integrand in Equation (6) we have the *proper* inner analytic function given by

$$
\begin{aligned}
w_p(z) &= w(z) - w(0) \\
&= u_p(\rho, \theta) + \imath v_p(\rho, \theta),
\end{aligned} \tag{8}
$$

where $w_p(0) = 0$. The integral in Equation (6) can therefore be written as

$$
\begin{aligned}
w^{-1\bullet}(z) &= -\imath\int_0^\rho d\rho' \frac{w_p(\rho', 0)}{\rho'} - \imath\int_0^\theta d\theta' \imath\rho e^{\imath\theta'} \frac{w_p(\rho, \theta')}{\rho e^{\imath\theta'}} \\
&= C(\rho) + \int_0^\theta d\theta'\, u_p(\rho, \theta') + \imath\int_0^\theta d\theta'\, v_p(\rho, \theta'),
\end{aligned} \tag{9}
$$

where, in relation to the variable θ, the integral on ρ' becomes the complex function $C(\rho)$, which depends only on ρ and not on θ, while the integral on θ' determines primitives with respect to θ of the real and imaginary parts of $w_p(z)$, which thus establishes this property.

Note that by construction we always have that $w^{-1\bullet}(0) = 0,$ so that we may say that the operation of angular integration projects the space of inner analytic functions onto the space of proper inner analytic functions. We may now establish an important property of the angular primitive.

Property 3.2. The angular primitive of an inner analytic function is also an inner analytic function.

In order to prove that $w^{-1\bullet}(z)$ is an inner analytic function, we use the power-series representation of the inner analytic function $w(z)$. Since this function is analytic within the open unit disk, its Taylor series around $z = 0,$ which is given by

$$w(z) = w(0) + \sum_{k=1}^{\infty} c_k z^k, \tag{10}$$

where $c_k = w^{k'}(0)/k!$ are the Taylor coefficients of $w(z)$ with respect to the origin and where $w^{k'}(z)$ is the k-th derivative of $w(z)$, converges within that disk. We therefore have for the integrand in Equation (6) the power-series representation

$$\begin{aligned} \frac{w(z) - w(0)}{z} &= \sum_{k=1}^{\infty} c_k z^{k-1} \\ &= \sum_{k=0}^{\infty} c_{k+1} z^k. \end{aligned} \tag{11}$$

Since this series has the same set of coefficients as the convergent series of $w(z)$, it is equally convergent, as is implied, for example, by the ratio test. Note that this shows, in particular, that the integrand is analytic at $z = 0.$ Being a convergent power series, this series can be integrated term by term, resulting in an equally convergent power series, so that we have for the angular primitive

$$
\begin{aligned}
w^{-1\bullet}(z) &= -\,\imath\int_0^z dz'\sum_{k=0}^{\infty} c_{k+1}(z')^k \\
&= -\,\imath\sum_{k=0}^{\infty}\frac{c_{k+1}}{k+1}\,z^{k+1} \\
&= -\,\imath\sum_{k=1}^{\infty}\frac{c_k}{k}\,z^k. \qquad (12)
\end{aligned}
$$

Due to the factors of $1/k$, when $k \to \infty$ the coefficients of this series go to zero faster than those of the convergent Taylor series of $w(z)$, and thus it is also convergent, in the same domain of convergence of the Taylor series of $w(z)$. This confirms that this series is convergent within the open unit disk. Being a convergent power series, it converges to an analytic function, thus proving that $w^{-1\bullet}(z)$ is analytic within the open unit disk. We may conclude therefore that the angular primitive of an inner analytic function is an inner analytic function as well, which establishes this property.

In the other words, the operation of angular integration stays within the space of inner analytic functions. Note that, since $w^{-1\bullet}(0) = 0$, angular integration always results in *proper* inner analytic functions, and therefore that this operation also stays within the space of proper inner analytic functions.

Let us now prove that the operations of angular differentiation and of angular integration are inverse operations to one another. Strictly speaking, this is true within the subset of inner analytic functions that have the additional property that $w(0) = 0$, that is, for proper inner analytic functions. Since any inner analytic function can be obtained from a proper inner analytic function by the mere addition of a constant, this is a very weak limitation. Let us consider then the space of proper inner analytic functions.

Property 3.3. The angular primitive of the angular derivative of a proper inner analytic function is that same proper inner analytic function.

We simply compose the two operations in the required order, and calculate in a straightforward manner, merely using the fundamental theorem of the calculus, to get

$$-\imath\int_0^z dz'\,\frac{1}{z'}\left[\imath z'\,\frac{dw}{dz'}(z')-\imath 0\,\frac{dw}{dz'}(0)\right]=\int_0^z dz'\,\frac{dw}{dz'}(z')$$

$$=w(z)-w(0), \tag{13}$$

which is the original inner analytic function $w(z)$ so long as $w(0)=0$, that is, for proper inner analytic functions, thus establishing this property.

Property 3.4. The angular derivative of the angular primitive of a proper inner analytic function is that same proper inner analytic function.

We simply compose the two operations in the required order, and calculate in a straightforward manner, using this time the power-series representation of the inner analytic function $w(z)$. First integrating term by term and then differentiating term by term, both of which are allowed operations for convergent power series, we get

$$\imath z\,\frac{d}{dz}(-\imath)\int_0^z dz'\,\frac{w(z')-w(0)}{z'}=z\,\frac{d}{dz}\int_0^z dz'\sum_{k=1}^{\infty}c_k(z')^{k-1}$$

$$=z\,\frac{d}{dz}\sum_{k=1}^{\infty}\frac{c_k}{k}\,z^k$$

$$=\sum_{k=1}^{\infty}c_k z^k$$

$$=w(z)-w(0), \tag{14}$$

which is the original inner analytic function $w(z)$ so long as $w(0) = 0$, that is, for proper inner analytic functions, thus establishing this property.

With the use of the operations of angular differentiation and angular integration the space of proper inner analytic functions can now be organized as a collection of infinite discrete chains of functions, so that within each chain the functions are related to each other by either angular integrations or angular differentiations. This leads to the definition that follows.

Definition 4 (Integral-differential chains).

Starting from an arbitrary proper inner analytic function $w(z)$, also denoted as $w^{0\bullet}(z)$, one proceeds in the differentiation direction to the functions $w^{1\bullet}(z), w^{2\bullet}(z), w^{3\bullet}(z),$ etc., and in the integration direction to the functions $w^{-1\bullet}(z), w^{-2\bullet}(z), w^{-3\bullet}(z),$ etc. One thus produces an infinite chain of proper inner analytic functions such as

$$\left\{\ldots, w^{-3\bullet}(z), w^{-2\bullet}(z), w^{-1\bullet}(z), w^{0\bullet}(z), w^{1\bullet}(z), w^{2\bullet}(z), w^{3\bullet}(z), \ldots\right\}, \quad (15)$$

in which angular differentiation takes one to the right and angular integration takes one to the left. We name such a structure an *integral-differential chain* of proper inner analytic functions. We may refer to the proper inner analytic functions forming the chain as *links* in that chain.

Note that all the functions in such a chain have exactly the same set of singular points on the unit circle, although the character of these singularities will change from function to function along the chain. Note also that each such integral-differential chain induces, by means of the real parts of their inner analytic functions, a corresponding chain of real objects over the unit circle, when and where the limits from the open unit disk to the unit circle exist, or can be consistently defined. Finally note that, given a singularity at a certain point on the unit circle, the integral-differential chain also induces a corresponding chain of singularities at that point. Let us now prove an important property of these chains, namely that they do not intersect each other.

Property 4.1. Two different integral-differential chains of proper inner analytic functions cannot have a member-function in common.

In order to prove this, we start by proving that, if two proper inner analytic functions have the same angular derivative, then they must be equal. If we have two such proper inner analytic functions $w_1(z)$ and $w_2(z)$, the statement that they have the same angular derivative is expressed as

$$\imath z \frac{d}{dz} w_2(z) - \imath z \frac{d}{dz} w_1(z) = 0 \Rightarrow$$

$$\frac{d}{dz} [w_2(z) - w_1(z)] = 0 \Rightarrow$$

$$w_2(z) - w_1(z) = C, \tag{16}$$

where C is some complex constant, for all z within the open unit disk, including the case $z = 0$, as one can see if one takes the limit $z \to 0$ of the last equation above. However, since at $z = 0$ we have that $w_1(0) = 0$ and $w_2(0) = 0$, it then follows that $C = 0$, so that we may conclude that

$$w_2(z) \equiv w_1(z), \tag{17}$$

thus proving the point. A similar result is valid for two proper inner analytic functions that have the same angular primitive. Since we have already shown that angular integration and angular differentiation are inverse operations to each other, we can prove this by simply noting the trivial fact that the operation of angular differentiation cannot produce two different results for the same function. Therefore, there cannot exist two different proper inner analytic functions whose angular primitives are one and the same function.

We may now conclude that two different integral-differential chains of proper inner analytic functions can never have a member-function in common, because this would mean that two different proper inner analytic functions would have either the same angular derivative or the same angular primitive, neither of which is possible. It follows that each proper inner analytic function appears in one and only one of these integral-differential chains, which establishes this property.

Note, for future use, that there is a *single* integral-differential chain of proper inner analytic functions which is a constant chain, in the sense that all member-functions of the chain are equal, namely, the null chain, in which all members are the null function $w(z) \equiv 0$. It is easy to verify that the differential equation $w^{\bullet}(z) = w(z)$ has no other inner analytic function as a solution. Note also that one may consider all the non-proper inner analytic functions $w(z)$ which are related to a given proper inner analytic function $w_p(z)$ to also belong to the same link of the corresponding integral-differential chain. Since all such functions have the form

$$w(z) = C + w_p(z), \tag{18}$$

where C is a complex constant, this has the effect of associating to each link of the integral-differential chain of $w_p(z)$ a complex plane of constants C in which each point corresponds to a function $w(z)$. In particular, all constant functions are associated to the null function, and therefore to a complex plane of constants at each link of the null chain.

We will now establish a general scheme for the classification of all possible singularities of inner analytic functions. This can be done for analytic functions in general, but we will do it here in a way that is particularly suited for our inner analytic functions.

Definition 5 (Classification of singularities: soft and hard).

Let z_1 be a point on the unit circle. A singularity of an inner analytic function $w(z)$ at z_1 is a *soft singularity* if the limit of $w(z)$ to that point exists and is finite. Otherwise, it is a *hard singularity*.

This is a complete classification of all possible singularities because, given a point of singularity, either the limit of the function to that point from within the open unit disk exists, or it does not. There is no third alternative, and therefore every singularity is either soft or hard. We may now establish the following important property of soft singularities.

Property 5.1. A soft singularity of an inner analytic function $w(z)$ at a point z_1 of the unit circle is necessarily an integrable one.

In order to prove this first note that, since the singularity at z_1 is soft, the function $w(z)$ is defined by continuity there, being therefore continuous at z_1. Consider now a curve contained within the open unit disk, that connects to z_1 along some direction, that has a finite length, and which is an otherwise arbitrary curve. We have at once that $w(z)$ is analytic at all points on this curve except z_1. It follows that $w(z)$ is continuous, and thus that $|w(z)|$ is continuous, everywhere on this curve, *including* at z_1. Hence, the limits of $|w(z)|$ to all points on this curve exist and are finite positive real numbers.

We now note that this set of finite real numbers must be bounded, because otherwise there would be a hard singularity of $w(z)$ somewhere within the open unit disk, where this function is in fact analytic. We conclude therefore that over the curve the function $w(z)$ is a bounded continuous functions on a finite-length domain, which implies that $w(z)$ is integrable in that domain. Therefore, we may state that $w(z)$ is integrable along arbitrary curves reaching the point z_1 from strictly within the open unit disk, which thus establishes this property.

We will now prove a couple of important further properties of the singularity classification, one for soft singularities and one for hard singularities. For this purpose, let z_1 be a point on the unit circle. Let us discuss first a property of soft singularities, which is related to angular integration.

Property 5.2. If an inner analytic function $w(z)$ has a soft singularity at z_1, then the angular primitive of $w(z)$ also has a soft singularity at that point.

In order to prove this we use the fact that a soft singularity is necessarily an integrable one. We already know that if $w(z)$ has a singularity at z_1, then so does its angular primitive $w^{-1\bullet}(z)$. If we now consider the angular primitive of $w(z)$ at $z = z_1$, we have

$$w^{-1\bullet}(z_1) = -\imath\int_0^{z_1} dz\,\frac{w(z) - w(0)}{z}, \tag{19}$$

where the integral can be taken over any simple curve within the open unit disk. We already know that the integrand is regular at the origin. Since the singularity of $w(z)$ at z_1 is soft, that singularity is integrable along any simple curve within the open unit disk that goes from $z = 0$ to $z = z_1$. Therefore, it follows that this integral exists and is finite, and thus that $w^{-1\bullet}(z_1)$ exists and is finite. Since the function $w^{-1\bullet}(z)$ is thus well defined at z_1, as well as analytic around that point, it follows that the singularity of $w^{-1\bullet}(z)$ at z_1 is also soft, which thus establishes this property.

Let us discuss now a property of hard singularities, which is related to angular differentiation.

Property 5.3. If an inner analytic function $w(z)$ has a hard singularity at z_1, then the angular derivative of $w(z)$ also has a hard singularity at that point.

If $w(z)$ has a hard singularity at z_1, then it is not well defined there, implying that it is not continuous there, and therefore that it is also not differentiable there. This clearly implies that the angular derivative $w^{\bullet}(z)$ of $w(z)$, which we already know to also have a singularity at z_1, is not well defined there as well. This in turn implies that the singularity of $w^{\bullet}(z)$ at z_1 must be a hard one.

However, the simplest way to prove this property is to note that it follows from the previous one, that is, from Property 5.2. We can prove it by reductio ad absurdum, using the fact that, as we have already shown, the operations of angular differentiation and angular integration are inverse operations to each other. If we assume that $w(z)$ has a hard singularity at z_1 and that $w^{\bullet}(z)$ has a soft singularity at that point, then we have an inner analytic function, namely, $w^{\bullet}(z)$, that has a soft singularity at z_1, while its angular primitive, namely, $w(z)$ has a hard singularity at that point. However, according to Property 5.2, this is impossible, since angular integration always takes a soft singularity to another soft singularity. This establishes, therefore, that this property holds.

The use of the operations of angular differentiation and of angular integration now leads to a refinement of our general classification of singularities. We will use them to assign to each singularity a *degree of softness* or a *degree of hardness*. Let $w(z)$ be an inner analytic function and z_1 a point on the unit circle, and consider the following two definitions.

Definition 6 (Classification of singularities: gradation of soft singularities).

Let us assume that $w(z)$ has a soft singularity at z_1. If an arbitrarily large number of successive angular differentiations of $w(z)$ always results in a singularity at z_1 which is still soft, then we say that the singularity of $w(z)$ at z_1 is an *infinitely soft* singularity. Otherwise, if n_s is the minimum number of angular differentiations that have to be applied to $w(z)$ in order for the singularity at z_1 to become a hard one, then we define n_s as the *degree of softness* of the original singularity of $w(z)$ at z_1. Therefore, a degree of softness is an integer $n_s \in \{1, 2, 3, \ldots, \infty\}$.

Definition 7 (Classification of singularities: gradation of hard singularities).

Let us assume that $w(z)$ has a hard singularity at z_1. If an arbitrarily large number of successive angular integrations of $w(z)$ always results in a singularity at z_1 which is still hard, then we say that the singularity of $w(z)$ at z_1 is an *infinitely hard* singularity. Otherwise, if $n_h + 1$ is the minimum number of angular integrations that have to be applied to $w(z)$ in order for the singularity at z_1 to become a soft one, then we define n_h as the *degree of hardness* of the original singularity of $w(z)$ at z_1. Therefore, a degree of hardness is an integer $n_h \in \{0, 1, 2, 3, \ldots, \infty\}$.

In order to see that this establishes a complete classification of all possible singularities, let us examine all the possible outcomes when we apply angular differentiations and angular integrations to inner analytic functions. We already saw that, if we apply a angular integration to a soft singularity, then the result is always another soft singularity. Similarly we saw that, if we apply a angular differentiation to a hard singularity, then the result is always another hard singularity. The two remaining alternatives are the application of a angular integration to a hard singularity, and the application of a angular differentiation to a soft singularity. In these two cases, the resulting singularity may be either soft or hard, and the remaining possibilities were dealt with in Definitions 6 and 7. Since this applies to all singularities in all integral-differential chains, it applies to all possible singularities of all inner analytic functions.

In some cases examples of this classification are well known. For instance, a simple example of an infinitely hard singularity is any essential singularity. Examples of infinitely soft singularities are harder to come by, and they are related to integrable real functions which are infinitely differentiable but not analytic. A simple example of a hard singularity with degree of hardness $n \geq 1$ is a pole of order n. Examples of soft singularities are the square root, and products of strictly positive powers with the logarithm.

If a singularity at a given singular point z_1 on the unit circle is either infinitely soft or infinitely hard, then the corresponding integral-differential chain of singularities contains either only soft singularities or only hard singularities. If the singularity is neither infinitely soft nor infinitely hard, then at some point along the corresponding integral-differential chain the character of the singularity changes, and from that point on the soft or hard character remains constant at the new value throughout the rest of the integral-differential chain in that direction. Therefore, in each integral-differential chain that does not consist of either only soft singularities or only hard singularities, there is a single transition between two functions on the chain where the character of the singularity changes.

Let us examine in more detail the important intermediary case in which we assign to the singularity the degree of hardness zero, which we will also describe as that of a *borderline hard* singularity.

Definition 8 (Classification of singularities: borderline hard singularities).

Given an inner analytic function $w(z)$ and a point z_1 on the unit circle where it has a hard singularity, if a *single* angular integration of $w(z)$ results in a function $w^{-1^{\bullet}}(z)$ which has at z_1 a soft singularity, then we say that the original function $w(z)$ has at z_1 a *borderline hard* singularity, that is, a hard singularity with degree of hardness zero.

We establish now the following important property of borderline hard singularities.

Property 8.1. A borderline hard singularity of an inner analytic function $w(z)$ at z_1 must be an integrable one.

This is so because the angular integration of $w(z)$ produces an inner analytic function $w^{-1^{\bullet}}(z)$ which has at z_1 a soft singularity, and therefore is well defined at that point. Since the value of $w^{-1^{\bullet}}(z)$ at z_1 is given by an integral of $w(z)$ along a curve reaching that point, that integral must therefore exist and result in a finite complex number. Therefore, the singularity of $w(z)$ at z_1 must be an integrable one. We may thus conclude that all borderline hard singularities are integrable ones, which establishes this property.

The transition between a borderline hard singularity and a soft singularity is therefore the single point of transition of the soft or hard character of the singularities along the corresponding integral-differential chain. Starting from a borderline hard singularity, n_s angular integrations produce a soft singularity with degree of softness n_s, and n_h angular differentiations produce a hard singularity with degree or hardness n_h. Note that a strictly positive degree of softness given by n can be identified with a negative degree of hardness given by $-n$, and vice-versa. A simple example of a borderline hard singularity is a logarithmic singularity.

Let us end this section with one more important property of hard singularities.

Property 8.2. The borderline hard singularities are the only hard singularities that are integrable.

If a hard singularity of $w(z)$ at z_1 has a degree of hardness of one or larger, then by angular integration it is mapped to another hard singularity, the hard singularity of $w^{-1\bullet}(z)$ at z_1. If the hard singularity of $w(z)$ were integrable, then $w^{-1\bullet}(z)$ would be well defined at z_1, and therefore its singularity would be soft rather than hard. Since we know that the singularity of $w^{-1\bullet}(z)$ is hard, it follows that the singularity of $w(z)$ cannot be integrable. In the other words, all hard singularities with strictly positive degrees of hardness are necessarily non-integrable singularities, which establishes this property.

Note that the structure of the integral-differential chains establishes within the space of proper inner analytic functions what may be described as a structure of discrete fibers, in which the whole space is decomposed as a set of non-intersecting discrete linear structures. The same is then true for the corresponding real objects on the unit circle. One can then reconstruct the space of all inner analytic functions by associating to each link of each integral-differential chain a complex plane of constants to be added to the proper inner analytic function of that link, in order to get all the non-proper inner analytic functions associated to it. In terms of the corresponding real functions, this corresponds to the association to each link of a real line of real constants.

3. Representation of Integrable Real Functions

In this section, we will establish the relation between integrable real functions and inner analytic functions. When we discuss real functions in this paper, some properties will be globally assumed for these functions.

These are rather weak conditions to be imposed on these functions, that will be in force throughout this paper. It is to be understood, without any need for further comment, that these conditions are valid whenever real functions appear in the arguments. These weak conditions certainly hold for any integrable real functions that are obtained as restrictions of corresponding inner analytic functions to the unit circle. The most basic condition is that the real functions must be measurable in the sense of Lebesgue, with the usual Lebesgue measure [7, 8]. In essence, this is basic infrastructure to allow the functions to be integrable.

In order to discuss the other global conditions, we must first discuss the classification of singularities of a real function. The concept of a singularity itself is the same as that for a complex function, namely, a point where the function is not analytic. The concept of a removable singularity is well-known for analytic functions in the complex plane. What we mean by a removable singularity in the case of real functions on the unit circle is a singular point such that both lateral limits of the function to that point exist and result in the same real value, but where the function has been arbitrarily defined to have some other real value. This is therefore a point were the function can be redefined by continuity, resulting in a continuous function at that point. The concepts of soft and hard singularities are carried in a straightforward way from the case of complex functions, discussed in Section 2, to that of real functions. The only difference is that the concept of the limit of the function to the point is now taken to be the real one, along the unit circle.

The second global condition we will impose is that the functions have no removable singularities. Since they can be easily eliminated, these are trivial singularities, which we will simply rule out of our discussions in this paper. Although the presence of even a denumerably infinite set of such trivial singularities does not significantly affect the results to be presented here, their elimination does significantly simplify the arguments to be presented. It is for this reason, that is, for the sake of simplicity, that we rule out such irrelevant singularities. In addition to this we will require, as our third an last global condition, that the number of hard singularities be finite, and hence that they be all isolated from one another. There will be no limitation on the number of soft singularities. In terms of the more immediate characteristics of the real functions, the relevant requirement is that the number of singular points where a given real function diverges to infinity be finite.

In this section, we will prove the following theorem:

Theorem 1. *Every integrable real function defined on a finite interval can be represented by an inner analytic function, and can be recovered almost everywhere by means of the limit to the unit circle of the real part of that inner analytic function.*

Given an arbitrary real function defined within an arbitrary finite closed interval, it can always be mapped to a real function within the periodic interval $[-\pi, \pi]$, by a simple linear change of variables, so it suffices for our purposes here to examine only the set of real functions $f(\theta)$ defined in this standard interval. The interval is then mapped onto the unit circle of the complex z plane. What happens to the values of the function at the two ends of the interval when one does this is irrelevant for our purposes here, but for definiteness we may think that one attributes to the function at the point $z = -1$ the arithmetic average of the values of the function at the two ends of the periodic interval. The further requirements to be imposed on these functions are still quite weak, namely no more than that they be Lebesgue-integrable in the periodic interval, so that one can attribute to them a set of Fourier coefficients [9].

Since for Lebesgue-measurable functions defined within a compact interval plain integrability and absolute integrability are equivalent requirements [7, 8], we may assume that the functions are absolutely integrable, without loss of generality. Note that the functions do not have to be differentiable or even continuous. They may also be unlimited, possibly diverging to infinity at some singular points, so long as they are absolutely integrable. This means, of course, that any hard singularities that they may have at isolated points must be integrable singularities, which we may thus characterize as borderline hard singularities, in a real sense of the term. This means, in turn, that although the functions may diverge to infinity at isolated points, their pairs of asymptotic integrals around these points must still exist and be finite real numbers.

This in turn means that these borderline hard singularities must be surrounded by open intervals where there are no other borderline hard singularities, so that the asymptotic integrals around the singular points can be well defined and finite. It follows that any existing borderline hard singularities must be isolated from any other borderline hard singularities. Note that they do not really have to be isolated singularities in the usual, strict sense of complex analysis, which would require that they be isolated from all other singularities. All that is required is that the borderline hard singularities be isolated from each other. Hence the requirement that the number of hard singularities be finite. Note also that one can have any number of soft singularities, even an infinite number of them. As we pointed out before, in terms of the properties of the real functions $f(\theta)$, the important requirement is that the number of singular points on the unit circle where a given real function diverges to infinity be finite.

Proof 1.1.

With all these preliminaries stated, the first thing that we must do here is, given an arbitrary integrable real function $f(\theta)$ defined within the periodic interval $[-\pi, \pi]$, to build from it an analytic function $w(z)$ within the open unit disk of the complex plane. For this purpose, we will use the Fourier coefficients of the given real function. The Fourier coefficients [9] are defined by

$$\alpha_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta f(\theta),$$

$$\alpha_k = \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta \cos(k\theta) f(\theta),$$

$$\beta_k = \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta \sin(k\theta) f(\theta), \tag{20}$$

where the set of functions $\{1, \cos(k\theta), \sin(k\theta), k \in \{1, 2, 3, \ldots, \infty\}\}$, constitutes the Fourier basis of functions. Since $f(\theta)$ is absolutely integrable, we have that

$$\int_{-\pi}^{\pi} d\theta |f(\theta)| = 2\pi M, \tag{21}$$

where M is a positive and finite real number, namely, the average value, on the periodic interval $[-\pi, \pi]$, of the absolute value of the function. If we use the triangle inequalities, it follows therefore that α_0 exists and that it satisfies

$$\begin{aligned} |\alpha_0| &\leq \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta |f(\theta)| \\ &\leq 2M, \end{aligned} \tag{22}$$

that is, it is limited by $2M$. Since the elements of the Fourier basis are all limited smooth functions, and using again the triangle inequalities, it now follows that all other Fourier coefficients also exist, and are also all limited by $2M$,

$$\begin{aligned} |\alpha_k| &\leq \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta |\cos(k\theta)| \, |f(\theta)| \\ &\leq \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta |f(\theta)| \end{aligned}$$

$$
\begin{aligned}
&\leq 2M, \\
|\beta_k| &\leq \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta |\sin(k\theta)|\,|f(\theta)| \\
&\leq \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta |f(\theta)| \\
&\leq 2M, \qquad (23)
\end{aligned}
$$

for all k, since the absolute values of the sines and cosines are limited by one. Given that we have the coefficients α_k and β_k, the construction of the corresponding inner analytic function is now straightforward. We simply define the set of complex coefficients

$$
\begin{aligned}
c_0 &= \frac{1}{2}\alpha_0, \\
c_k &= \alpha_k - \imath\beta_k, \qquad (24)
\end{aligned}
$$

for $k \in \{1, 2, 3, \ldots, \infty\}$. Note that these coefficients are all limited by $4M$, since, using once more the triangle inequalities, we have

$$
\begin{aligned}
|c_0| &= \frac{1}{2}|\alpha_0| \\
&\leq M, \\
|c_k| &\leq |\alpha_k| + |\beta_k| \\
&\leq 4M. \qquad (25)
\end{aligned}
$$

We now define a complex variable z associated to θ, using an auxiliary positive real variable $\rho \geq 0$,

$$
z = \rho e^{\imath\theta}, \qquad (26)
$$

where (ρ, θ) are polar coordinates in the complex z plane. We then construct the following power series around the origin $z = 0$,

$$S(z) = \sum_{k=0}^{\infty} c_k z^k. \tag{27}$$

According to the theorems of complex analysis [1], where this power series converges in the complex z plane, it converges absolutely and uniformly to an analytic function $w(z)$. It then follows that $S(z)$ is in fact the Taylor series of $w(z)$ around $z = 0$. We must now establish that this series converges within the open unit disk, whatever the values of the Fourier coefficients, given only that they are all limited by $4M$. In order to do this we will first prove that the series $S(z)$ is absolutely convergent, that is, we will establish the convergence of the corresponding series of absolute values

$$\overline{S}(z) = \sum_{k=0}^{\infty} |c_k| \rho^k. \tag{28}$$

Let us now consider the partial sums of this real series, and replace the absolute values of the coefficients by their common upper bound,

$$\begin{aligned} \overline{S}_n(z) &= \sum_{k=0}^{n} |c_k| \rho^k \\ &\le 4M \sum_{k=0}^{n} \rho^k, \end{aligned} \tag{29}$$

where $n \in \{0, 1, 2, 3, \dots, \infty\}$. This is now the sum of a geometric progression, so that we have

$$\overline{S}_n(z) \le 4M \, \frac{1 - \rho^{n+1}}{1 - \rho}. \tag{30}$$

For $\rho < 1$, we may now take the $n \to \infty$ limit of the right-hand side, without violating the inequality, so that we get the sum of a geometric series,

$$\overline{S}_n(z) \leq \frac{4M}{1-\rho}. \tag{31}$$

For $\rho < 1$, the right-hand side is now a positive upper bound for all the partial sums of the series of absolute values. Therefore, since the sequence $\overline{S}_n(z)$ of partial sums is a monotonically increasing sequence of real numbers which is bounded from above, it now follows that this real sequence is necessarily a convergent one.

Therefore, the series of absolute values $\overline{S}(z)$ is convergent on the open unit disk $\rho < 1$, which in turn implies that the original series $S(z)$ is absolutely convergent on that same disk. This then implies that the series $S(z)$ is simply convergent on that same disk. Since $S(z)$ is a convergent power series, it converges to an analytic function on the open unit disk, which we may now name $w(z)$. Since this is an analytic function within the open unit disk, it is an inner analytic function, the one that corresponds to the real function $f(\theta)$ on the unit circle,

$$f(\theta) \to w(z). \tag{32}$$

The coefficients c_k are now recognized as the Taylor coefficients of the inner analytic function $w(z)$ with respect to the origin. We have therefore established that from any integrable real function $f(\theta)$ one can define a unique corresponding inner analytic function $w(z)$. This completes the first part of the proof of Theorem 1.

Proof 1.2.

Next we must establish that $f(\theta)$ can be recovered as the limit $\rho \to 1_{(-)}$, from the open unit disk to the unit circle, of the real part of $w(z)$, so that we can establish the complete correspondence between the integrable real function and the inner analytic function,

$$f(\theta) \leftrightarrow w(z). \tag{33}$$

We start by writing the coefficients c_k in terms of $w(z)$ and discussing their dependence on ρ. Since the complex coefficients c_k are the coefficients of the Taylor series of $w(z)$ around $z = 0$, the Cauchy integral formulas of complex analysis, for the function $w(z)$ and its derivatives, written at $z = 0$ for the k-th derivative of $w(z)$, tell us that we have

$$c_k = \frac{1}{2\pi i} \oint_C dz \, \frac{w(z)}{z^{k+1}}, \tag{34}$$

for all k, where C is any simple closed curve within the open unit disk that contains the origin, which we may now take as a circle centered at $z = 0$ with radius $\rho \in (0, 1)$. We now note that, since $w(z)$ is analytic in the open unit disk, so that the explicit singularity at $z = 0$ is the only singularity of the integrand on that disk, by the Cauchy-Goursat theorem the integral is independent of ρ within the open unit disk, and therefore so are the complex coefficients c_k.

It thus follows that the coefficients c_k are continuous functions of ρ inside the open unit disk, and therefore that their $\rho \to 1_{(-)}$ limits exist and have those same constant values. Since we have the relations in Equation (24), the same is true for the Fourier coefficients α_k and β_k. On the other hand, by construction these are the same coefficients that were obtained from the real function $f(\theta)$ on the unit circle, and we may thus conclude that the coefficients c_k, α_k, and β_k, for all k, are all constant with ρ and therefore continuous functions of ρ in the whole closed unit disk. This means that, at least in the case of the coefficients, the $\rho \to 1_{(-)}$ limit can be taken trivially.

Let us now establish the fact that $f(\theta)$ and the real part $u(1, \theta)$ of $w(z)$ at $\rho = 1$ have exactly the same set of Fourier coefficients. We consider first the case of the coefficient α_0. If we write the Cauchy integral formula in Equation (34) for the case $k = 0$, we get

$$c_0 = \frac{1}{2\pi i} \oint_C dz \, \frac{w(z)}{z}. \tag{35}$$

Recalling that $c_0 = \alpha_0/2$ and writing the integral on the circle of radius ρ using the integration variable θ, we get

$$\frac{\alpha_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta [u(\rho, \theta) + \imath v(\rho, \theta)]. \tag{36}$$

Since α_0 is real, we conclude that the imaginary part in the right-hand side must be zero, and thus obtain

$$\alpha_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} d\theta u(\rho, \theta), \tag{37}$$

thus proving that α_0, which is the $k = 0$ Fourier coefficient of $f(\theta)$, is also the $k = 0$ Fourier coefficient of $u(\rho, \theta)$, for any value of ρ, and thus is, in particular, the $k = 0$ Fourier coefficient of $u(1, \theta)$. This is so because, since the $\rho \to 1_{(-)}$ limit of the coefficient α_0 can be taken, so can the limit of the integral in the right-hand side. Note that this shows, in particular, that $u(1, \theta)$ is an integrable real function. In order to extend the analysis of the coefficients to the case $k > 0$ we must first derive some preliminary relations. Consider therefore the following integral, on the same circuit C we used in Equation (34),

$$\oint_C dz w(z) z^{k-1} = 0, \tag{38}$$

with $k > 0$. The integral is zero by the Cauchy-Goursat theorem, since for $k \geq 1$ the integrand is analytic on the whole open unit disk. As before we write the integral on the circle of radius ρ using the integration variable θ, to get

$$\int_{-\pi}^{\pi} d\theta[u(\rho,\ \theta)\cos(k\theta) - v(\rho,\ \theta)\sin(k\theta)]$$

$$+ \imath\int_{-\pi}^{\pi} d\theta[u(\rho,\ \theta)\sin(k\theta) + v(\rho,\ \theta)\cos(k\theta)] = 0. \tag{39}$$

We are therefore left with the two identities involving $u(\rho, \theta)$ and $v(\rho, \theta)$,

$$\int_{-\pi}^{\pi} d\theta u(\rho,\ \theta)\cos(k\theta) = \int_{-\pi}^{\pi} d\theta v(\rho,\ \theta)\sin(k\theta),$$

$$\int_{-\pi}^{\pi} d\theta u(\rho,\ \theta)\sin(k\theta) = -\int_{-\pi}^{\pi} d\theta v(\rho,\ \theta)\cos(k\theta), \tag{40}$$

which are valid for all $k > 0$ and for all $\rho \in (0, 1)$. If we now write the integrals of the Cauchy integral formulas in Equation (34) explicitly as integrals on θ, we get

$$c_k = \frac{\rho^{-k}}{2\pi}\left\{\int_{-\pi}^{\pi} d\theta[u(\rho,\ \theta)\cos(k\theta) + v(\rho,\ \theta)\sin(k\theta)]\right.$$

$$\left. + \imath\int_{-\pi}^{\pi} d\theta[-u(\rho,\ \theta)\sin(k\theta) + v(\rho,\ \theta)\cos(k\theta)]\right\}. \tag{41}$$

Using the identities in Equation (40) in order to eliminate $v(\rho, \theta)$ in favor of $u(\rho, \theta)$ and recalling that $c_k = \alpha_k - \imath\beta_k$, we get

$$\alpha_k - \imath\beta_k = \frac{\rho^{-k}}{\pi}\int_{-\pi}^{\pi} d\theta[u(\rho,\ \theta)\cos(k\theta) - \imath u(\rho,\ \theta)\sin(k\theta)], \tag{42}$$

so that we have the relations for the Fourier coefficients,

$$\alpha_k = \frac{\rho^{-k}}{\pi} \int_{-\pi}^{\pi} d\theta u(\rho, \theta) \cos(k\theta),$$

$$\beta_k = \frac{\rho^{-k}}{\pi} \int_{-\pi}^{\pi} d\theta u(\rho, \theta) \sin(k\theta). \tag{43}$$

Since the $\rho \to 1_{(-)}$ limits of the coefficients in the left-hand sides can be taken, so can the $\rho \to 1_{(-)}$ limits of the integrals in the right-hand sides. Therefore, taking the limit we have for the Fourier coefficients,

$$\alpha_k = \frac{1}{\pi} \int_{-\pi}^{\pi} d\theta u(1, \theta) \cos(k\theta),$$

$$\beta_k = \frac{1}{\pi} \int_{-\pi}^{\pi} d\theta u(1, \theta) \sin(k\theta), \tag{44}$$

thus completing the proof that the real functions $u(1, \theta)$ and $f(\theta)$ have exactly the same set of Fourier coefficients. Note, in passing, that due to the identities in Equation (40) these same coefficients can also be written in terms of $v(1, \theta)$,

$$\alpha_k = \frac{1}{\pi} \int_{-\pi}^{\pi} d\theta v(1, \theta) \sin(k\theta),$$

$$\beta_k = -\frac{1}{\pi} \int_{-\pi}^{\pi} d\theta v(1, \theta) \cos(k\theta), \tag{45}$$

in which the $\cos(k\theta)$ was exchanged for $\sin(k\theta)$ and the $\sin(k\theta)$ was exchanged for $-\cos(k\theta)$. In fact, this is one way to state that $u(1, \theta)$ and $v(1, \theta)$ are two mutually Fourier-conjugate real functions.

Let us now examine the limit $\rho \to 1_{(-)}$ that allows us to recover from the real part $u(\rho, \theta)$ of the inner analytic function $w(z)$ the original real function $f(\theta)$. We want to establish that we may state that

$$f(\theta) = \lim_{\rho \to 1_{(-)}} u(\rho, \theta) \tag{46}$$

almost everywhere. Let us prove that $u(1, \theta)$ and $f(\theta)$ must coincide almost everywhere. Simply consider the real function $g(\theta)$ given by

$$g(\theta) = u(1, \theta) - f(\theta), \tag{47}$$

where

$$u(1, \theta) = \lim_{\rho \to 1_{(-)}} u(\rho, \theta). \tag{48}$$

Since it is the difference of two integrable real functions, $g(\theta)$ is itself an integrable real function. However, since the expression of the Fourier coefficients is linear on the functions, and since $u(1, \theta)$ and $f(\theta)$ have exactly the same set of Fourier coefficients, it is clear that all the Fourier coefficients of $g(\theta)$ are zero. Therefore, for the integrable real function $g(\theta)$ we have that $c_k = 0$ for all k, and thus the inner analytic function that corresponds to $g(\theta)$ is the identically null complex function $w_\gamma(z) \equiv 0$. This is an inner analytic function which is, in fact, analytic over the whole complex plane, and which, in particular, is zero over the unit circle, so that we have $g(\theta) \equiv 0$. Note, in particular, that the $\rho \to 1_{(-)}$ limits exist at *all* points of the unit circle in the case of the inner analytic function associated to $g(\theta)$. Since our argument is based on the Fourier coefficients α_k and β_k, which in turn are given by integrals involving these functions, we can only conclude that

$$f(\theta) = \lim_{\rho \to 1_{(-)}} u(\rho, \theta), \tag{49}$$

is valid *almost everywhere* over the unit circle. Therefore, we have concluded that the $\rho \to 1_{(-)}$ limit of $w(z)$ exists and that the limit of its real part $u(\rho, \theta)$ results in the values of $f(\theta)$ almost everywhere. This concludes the proof of Theorem 1.

Regarding the fact that the proof above is valid only almost everywhere, it is possible to characterize, up to a certain point, the set of points where the recovery of the real function $f(\theta)$ may fail, using the character of the possible singularities of the corresponding inner analytic function $w(z)$. Wherever $w(z)$ is either analytic or has only soft singularities on the unit circle, the $\rho \to 1_{(-)}$ limit exists, and therefore the values of $f(\theta)$ can be recovered. At points on the unit circle where $f(\theta)$ has hard singularities, $w(z)$ necessarily also has hard singularities, and therefore the limit does not exist and thus the values of $f(\theta)$ cannot be recovered. However, in this case this fact is irrelevant, since $f(\theta)$ is not well defined at these points to begin with. In any case, by hypothesis there can be at most a finite number of such points, which therefore form a zero-measure set.

Therefore, the only points where $f(\theta)$ may exist but not be recoverable from the real part of $w(z)$ are those singular points on the unit circle where $f(\theta)$ has a *soft* singularity, in the real sense of the term, while $w(z)$ has a *hard* singularity, in the complex sense of the term. In principle this is possible because the requirement for a singularity to be soft in the complex case is more restrictive than the corresponding requirement in the real case. For a singularity to be soft in the real case, it suffices that the limits of the function to the point exist and be the same only along two directions, coming from either side along the unit circle, but for the singularity to be soft in the complex case the limits must exist and be the same along *all* directions.

It is indeed possible for a hard complex singularity on the unit circle to be so oriented that the limit exists along the two particular directions along the unit circle, but does not exist along other directions. For example, consider the rather pathological real function

$$f(\theta) = \theta \sin\left(\frac{\pi^2}{\theta}\right), \tag{50}$$

for $-\pi \leq \theta < 0$ and $0 < \theta \leq \pi$. It is well-known that this function has an essential singularity at $\theta = 0$ in the complex θ plane, which is an infinitely hard singularity. However, if defined by continuity at $\theta = 0$ the function is continuous there, and therefore the singularity at $\theta = 0$ is a soft one in the real sense of the term. The function is also continuous at all other points on the unit circle. We now observe that, despite having an infinitely hard complex singularity at $\theta = 0$, this is a limited real function on a finite domain and therefore an integrable real function, which means that we may still construct an inner analytic function that corresponds to it. Presumably, this inner analytic function also has an essential singularity at the point $z = 1$, which corresponds to $\theta = 0$ on the unit circle. This fact would then prevent us from obtaining the value of the function at $\theta = 0$ as the $\rho \to 1_{(-)}$ limit of the real part of that inner analytic function.

The mere fact that one can establish that there is a well defined inner analytic function for such a pathological real function is in itself rather unexpected and surprising. Furthermore, one can easily see that this is not the only example. One can also consider the related example, this time one in which the singularity is *not* soft in the real sense of the term,

$$f(\theta) = \sin\left(\frac{\pi^2}{\theta}\right), \tag{51}$$

which is still a limited real function on a finite domain and therefore an integrable real function, which again means that we may still construct an inner analytic function that corresponds to it. Many other variations of these examples can be constructed without too much difficulty.

Excluding all such exceptional cases, we may consider that the recovery of the real function $f(\theta)$ as the $\rho \to 1_{(-)}$ limit of the real part of the inner analytic function holds everywhere in the domain of definition of $f(\theta)$, that is, wherever it is well defined. In order to exclude all such exceptional cases, all we have to do is to exchange the condition that there be at most a finite number of hard singularities, in the real sense of the term, of the integrable real function $f(\theta)$, for the condition that there be at most a finite number of hard singularities with finite degrees of hardness, in the complex sense of the term, of the corresponding inner analytic function $w(z)$.

Once we have the inner analytic function that corresponds to a given integrable real function, we may consider the integral-differential chain to which it belongs. There are two particular cases that deserve mention here. One is that in which the inner analytic functions in the chain do not have any singularities at all on the unit circle, in which case the corresponding real functions are all analytic functions of θ in the real sense of the term. The other is that in which the inner analytic functions in the chain have only infinitely soft singularities on the unit circle, in which case the corresponding real functions are all infinitely differentiable functions of θ, although they are not analytic. In this case, one can go indefinitely along the chain in either direction without any change in the soft character of the singularities.

If, on the other hand, one does have borderline hard singularities or soft singularities with finite degrees of softness, then at some point along the chain there will be a transition to one or more hard singularities with strictly positive degrees of hardness, which do not necessarily correspond to integrable real functions. It can be shown that most of these singularities are instead associated to either singular distributions or non-integrable real functions. Their discussion will be postponed to the aforementioned forthcoming papers.

4. Behavior Under Analytic Operations

Let us now discuss how the correspondence between inner analytic functions and integrable real functions behaves under the respective operations of differentiation and integration, that take us along the corresponding integral-differential chain. There are two issues here, one being the existence of the $\rho \to 1_{(-)}$ limit at each point on the unit circle, the other being whether or not the correspondence between the real function $f(\theta)$ and the inner analytic function $w(z)$, established by the construction of the inner analytic function from the integrable real function, and by the $\rho \to 1_{(-)}$ limit of the real part of the inner analytic function, survives the operation unscathed.

The existence of the limit $\rho \to 1_{(-)}$ hinges on whether the point at issue is a singular point or not, and then on whether the singularity at the point is either soft or hard. If a point on the unit circle is *not* a singularity of the inner analytic function $w(z)$, then the $\rho \to 1_{(-)}$ limit always exists at that point, no matter how many angular integrations or angular differentiations are performed on the inner analytic function, that is, the limit exists throughout the corresponding integral-differential chain. The same is true if the point is an infinitely soft singularity of $w(z)$. On the other hand, if it is an infinitely hard singularity of $w(z)$, then the limit at that point never exists, in the complex sense, throughout the integral-differential chain. Note, however, that in some cases the limit may still exist, in the real sense, along the unit circle.

If the point on the unit circle is a soft singularity of $w(z)$ with a finite degree of softness n_s, then the $\rho \to 1_{(-)}$ limit exists no matter how many angular integrations are performed, since the operation of angular integration takes soft singularities to other soft singularities. However, since the operation of angular differentiation may take soft singularities to hard singularities, the limit will only exist up to a certain number of angular differentiations, which is given by $n_s - 1$. Again, we note that in some cases the limit may still exist beyond this point, in the real sense, even if it does not exist in the complex sense.

If the point on the unit circle is a hard singularity of $w(z)$ with a finite degree of hardness n_h, including zero, then the $\rho \to 1_{(-)}$ limit does not exist in the complex sense, and will also fail to exist in that sense for any of the angular derivatives of $w(z)$, since the operation of angular differentiation takes hard singularities to other hard singularities. Once more we note that in some cases the limit may still exist in the real sense, even if it does not exist in the complex sense. However, since the operation of angular integration may take hard singularities to soft singularities, the limit will in fact exist after a certain number of angular integrations of $w(z)$, which is given by $n_h + 1$.

Whatever the situation may be, if after a given set of analytic operations is performed there is at most a finite number of hard singularities, then the $\rho \to 1_{(-)}$ limit exists almost everywhere, and therefore the corresponding real function can be recovered at almost all points on the unit circle. Note, by the way, that the same is true if there is a denumerably infinite number of hard singularities, so long as they are *not* densely distributed on the unit circle or any part of it, so that almost all of then can be isolated.

The next question is whether or not the relation between the real function $f(\theta)$ and the inner analytic function $w(z)$ implies the corresponding relation between the corresponding functions after an operation of integration or differentiation is applied. This is always true from a strictly local point of view, since we have shown in Section 2 that the operation of angular differentiation on the open unit disk reduces to the operation of differentiation with respect to θ on the unit circle, and that the operation of angular integration on the open unit disk reduces to the operation of integration with respect to θ on the unit circle, up to an integration constant.

There are, however, some global concerns over the unit circle, since the operations of angular integration and of angular differentiation always result in *proper* inner analytic functions, and there is no corresponding property of the operations of integration and differentiation with respect to θ on the unit circle. Note that the condition $w(0) = 0$, which holds for a proper inner analytic function, is translated, on the unit circle, to the global condition that the corresponding real function $f(\theta)$ have zero average value over that unit circle. This is so because $w(0) = 0$ is equivalent to $c_0 = 0$, and therefore to $\alpha_0 = 0$. However, according to the definition of the Fourier coefficients in Equation (20), the coefficient $\alpha_0/2$ is equal to that average value.

One way to examine this issue is to use the correspondence between the Taylor coefficients c_k of the inner analytic function $w(z)$ and the Fourier coefficients α_k and β_k of the integrable real function $f(\theta)$, which according to our construction of $w(z)$ are related by the relations in Equation (24). Since we have that

$$w(z) = \sum_{k=0}^{\infty} c_k z^k, \tag{52}$$

it follows from the definition of angular differentiation that under that operation the coefficients c_k transform as

$$\begin{aligned} c_0 &\to 0, \\ c_k &\to \imath k c_k, \end{aligned} \tag{53}$$

for $k \in \{1, 2, 3, \ldots, \infty\}$, and it also follows that from the definition of angular integration that under that operation they transform as

$$\begin{aligned} c_0 &\to 0, \\ c_k &\to -\frac{\imath}{k} c_k, \end{aligned} \tag{54}$$

for $k \in \{1, 2, 3, \ldots, \infty\}$. If we now look at the Fourier coefficients, considering their definition in Equation (20), in the case $k = 0$, we have that under differentiation α_0 transforms as

$$\begin{aligned}\alpha_0 &\rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta f'(\theta)\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} df(\theta),\end{aligned} \tag{55}$$

which is zero so long as $f(\theta)$ is a continuous function, since we are integrating on a circle. Note that, if $f(\theta)$ is not continuous, then $f'(\theta)$ is not even a well defined integrable real function, and we therefore cannot even write the integral, with what we know so far. In the case $k > 0$, we have that under differentiation the Fourier coefficients transform as

$$\begin{aligned}\alpha_k &\rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta f'(\theta)\cos(k\theta)\\ &= \frac{k}{\pi}\int_{-\pi}^{\pi} d\theta f(\theta)\sin(k\theta)\\ &= k\beta_k,\\ \beta_k &\rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta f'(\theta)\sin(k\theta)\\ &= -\frac{k}{\pi}\int_{-\pi}^{\pi} d\theta f(\theta)\cos(k\theta)\\ &= -k\alpha_k,\end{aligned} \tag{56}$$

where we have integrated by parts, noting that the integrated terms are zero because we are integrating on a circle. We therefore have, so long as $f(\theta)$ is a continuous function, that

$$\alpha_0 \to 0,$$
$$\alpha_k - \imath\beta_k \to \imath k(\alpha_k - \imath\beta_k), \tag{57}$$

for $k \in \{1, 2, 3, \ldots, \infty\}$, which are, therefore, the same transformations undergone by c_k. In the case of integration operations, the change in α_0 is indeterminate due to the presence of an arbitrary integration constant on θ and, considering once more the definition of the Fourier coefficients in Equation (20), we have that for $k > 0$ the Fourier coefficients transform under integration as

$$\begin{aligned}
\alpha_k &\to \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta f^{-1'}(\theta)\cos(k\theta) \\
&= -\frac{1}{k\pi}\int_{-\pi}^{\pi} d\theta f(\theta)\sin(k\theta) \\
&= -\frac{\beta_k}{k}, \\
\beta_k &\to \frac{1}{\pi}\int_{-\pi}^{\pi} d\theta f^{-1'}(\theta)\sin(k\theta) \\
&= \frac{1}{k\pi}\int_{-\pi}^{\pi} d\theta f(\theta)\cos(k\theta) \\
&= \frac{\alpha_k}{k},
\end{aligned} \tag{58}$$

where we have again integrated by parts, noting once more that the integrated terms are zero because we are integrating on a circle. We therefore have, so long as $f(\theta)$ is an integrable function, and so long as one *chooses* the integration constant of the integration on θ leading to $f^{-1'}(\theta)$ so that α_0 is mapped to zero, that

$$\alpha_0 \rightarrow 0,$$

$$\alpha_k - \imath\beta_k \rightarrow -\frac{\imath}{k}(\alpha_k - \imath\beta_k), \tag{59}$$

for $k \in \{1, 2, 3, \ldots, \infty\}$, which are, once more, the same transformations undergone by c_k. We therefore see that, from the point of view of the respective coefficients, the correspondence between the real function $f(\theta)$ and the inner analytic function $w(z)$ survives the respective analytic operations, so long as the operations produce integrable real functions on the unit circle, and so long as one chooses appropriately the integration constant on θ.

Let us discuss the situation in a little more detail, starting with the operation of integration. As we saw in Property 3.1 of Section 2, angular integration is translated, up to an integration constant, to integration with respect to θ on the unit circle, when we take the $\rho \rightarrow 1_{(-)}$ limit. In addition to this, angular integration never produces new hard singularities out of soft ones, so that the $\rho \rightarrow 1_{(-)}$ limit giving $f^{-1'}(\theta)$ exists at all points where those giving $f(\theta)$ exist. We see therefore that, so long as the integration constant is chosen so as to satisfy the condition that the function $f^{-1'}(\theta)$ have zero average value over the unit circle, it follows that the correspondence between the real function $f(\theta)$ and the inner analytic function $w(z)$ implies the correspondence between the real function $f^{-1'}(\theta)$ and the inner analytic function $w(z)^{-1\bullet}$,

$$f(\theta) \leftrightarrow w(z) \Rightarrow$$

$$f^{-1'}(\theta) \leftrightarrow w^{-1\bullet}(z). \tag{60}$$

This is valid so long as $f(\theta)$ is an integrable real function. Let us now discuss the case of the operation of differentiation. As we saw in Property

2.1 of Section 2, angular differentiation corresponds to differentiation with respect to θ on the unit circle, when we take the $\rho \to 1_{(-)}$ limit. However, angular differentiation can produce new hard singularities out of soft ones, and can also produce non-integrable hard singularities out of borderline hard ones. Therefore, we may conclude only that, if all the singularities of $w(z)$ are soft, which implies that $f(\theta)$ is continuous, then the correspondence between the real function $f(\theta)$ and the inner analytic function $w(z)$ does imply the correspondence between the real function $f'(\theta)$ and the inner analytic function $w^{\bullet}(z)$,

$$f(\theta) \leftrightarrow w(z) \Rightarrow$$

$$f'(\theta) \leftrightarrow w^{\bullet}(z), \tag{61}$$

with the exception of the points where $f'(\theta)$ has hard singularities produced out of soft singularities of $f(\theta)$. Note, however, that this statement is true even if $w^{\bullet}(z)$ has borderline hard singularities and therefore $f'(\theta)$ is not continuous.

On the other hand, if $f(\theta)$ is discontinuous at a finite set of borderline hard singularities of $w(z)$, then $f'(\theta)$ is not even well defined everywhere, by the usual definition of the derivative of a real function. In fact, if $w(z)$ has borderline hard singularities then $w^{\bullet}(z)$ has hard singularities with degrees of hardness equal to one, which are non-integrable singularities, so that $f'(\theta)$ is not necessarily an integrable real function. The same is true if the inner analytic function $w(z)$ has hard singularities with strictly positive degrees of hardness. The discussion of cases such as these will be given in the aforementioned forthcoming papers.

Given any inner analytic function that has at most a finite number of borderline hard singular points and no singularities harder than that, and the corresponding integral-differential chain, the results obtained here allow us to travel freely along the integration side of the chain, without damaging the correspondence between each inner analytic function and the corresponding real function. The part of the chain where this is valid is the part to the integration side starting from the link where all the singularities are either soft or at most borderline hard. What happens when one travels in the other direction along the chain, starting from this link, will be discussed in the aforementioned forthcoming papers.

5. Conclusions and Outlook

We have shown that there is a close and deep relationship between real functions and complex analytic functions in the unit disk centered at the origin of the complex plane. This close relation between real functions and complex analytic functions allows one to use the powerful and extremely well-known machinery of complex analysis to deal with the real functions in a very robust way, even if the real functions are very far from being analytic. For example, the $\rho \to 1_{(-)}$ limit can be used to define the values of the functions or the values of their derivatives, at points where these quantities cannot be defined by purely real means. The concept of inner analytic functions played a central role in the analysis presented. The integral-differential chains of inner analytic functions, as well as the classification of singularities of these functions, which we introduced here, also played a significant role.

One does not usually associate non-differentiable, discontinuous and unbounded real functions with single analytic functions. Therefore, it may come as a bit of a surprise that *all* integrable real functions are given by the real parts of certain inner analytic functions on the open unit disk when one approaches the unit circle. Note, however, that there are many more inner analytic functions within the open unit disk than those that were examined here, generated by integrable real functions. This leads to extensions of the relationship between inner analytic functions and real functions or related objects on the unit circle, which will be tackled in the aforementioned forthcoming papers.

One important limitation in the arguments presented here is that requiring that there be only a finite number of borderline hard singularities. It may be possible, perhaps, to lift this limitation, allowing for a denumerably infinite set of such integrable singularities. It is probably not possible, however, to allow for a densely distributed set of such singularities. Possibly, the limitation that the number of borderline hard singularities be finite may be exchanged for the limitation that the number of *accumulation points* of a denumerably infinite set of singular points with borderline hard singularities be finite.

It is quite apparent that the complex-analytic structure presented here can be used to discuss the Fourier series of real functions, as well as other aspects of the structure of the Fourier theory of real functions. The study of the convergence of Fourier series was, in fact, the way in which this structure was first unveiled. Parts of the arguments that were presented can be seen to connect to the Fourier theory, such as the role played by the Fourier coefficients, and the sufficiency of these Fourier coefficients to represent the functions, which relates to the question of the completeness of the Fourier basis of functions. This is a rather extensive discussion, which will be presented in a forthcoming paper.

It is interesting to note that the structure presented here may go some way towards explaining the rather remarkable fact that physicists usually operate with singular objects and divergent series in what may seem, from a mathematical perspective, a rather careless way, while very rarely getting into serious trouble while doing this. The fact that there is a robust underlying complex-analytic structure, that in fact explains how many such murky operations can in fact be rigorously justified, helps one to understand the unexpected success of this way to operate within the mathematics used in physics applications. In the parlance of physics, one may say that the complex-analytic structure within the unit disk functions as a universal regulator for all real functions, and related singular objects, which are of interest in physics applications.

We believe that the results presented here establish a new perspective for the analysis of real functions. The use of the theory of complex analytic functions makes it a rather deep and powerful point of view. Since complex analysis and analytic functions constitute such a powerful tool, with so many applications in almost all areas of mathematics and physics, it is to be hoped that other applications of the ideas explored here will in due time present themselves.

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