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SOME IMPROVEMENTS OF HERMITE-HADAMARD INTEGRAL INEQUALITY

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Abstract

We presented here an improvement of Hermite-Hadamard inequality as a linear combination of its end-points. Improvements of the second order with applications in Theory of Means are also given.

1. Introduction

A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on an non-empty interval I if the inequality

$$f(px + qy) \leq pf(x) + qf(y), \tag{1.1}$$

holds for all $x, y \in I$ and all non-negative $p, q;\ p + q = 1$.

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If the inequality (1.1) reverses, then f is said to be concave on I [1].

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on an interval I and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2}. \tag{1.2}$$

This double inequality is well known in the literature as Hermite-Hadamard (HH) integral inequality for convex functions. See, for example, [2] and references therein.

If f is concave, both inequalities in (1.2) hold in the reversed direction.

Our task in this paper is to improve the inequality (1.2) in a simple manner, i.e., to find some positive constants $\alpha, \beta, \gamma, \delta$ such that the relations

$$\gamma(f(a)+f(b)) + \delta f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)dt \le \alpha(f(a)+f(b)) + \beta f\left(\frac{a+b}{2}\right), \tag{1.3}$$

hold for any convex f.

Taking $f(t) = Ct$, $C \in \mathbb{R} / \{0\}$, it can be easily seen that both conditions

$$2\alpha + \beta = 1;\ 2\gamma + \delta = 1, \tag{1.4}$$

are necessary for (1.3) to hold.

Denote

$$M(\gamma, \delta) = M_f(a, b; \gamma, \delta) =: \gamma(f(a)+f(b)) + \delta f\left(\frac{a+b}{2}\right),$$

and

$$N(\alpha, \beta) = N_f(a, b; \alpha, \beta) =: \alpha(f(a)+f(b)) + \beta f\left(\frac{a+b}{2}\right).$$

Since

$$N(\alpha, \beta) = (2\alpha)\left(\frac{f(a)+f(b)}{2}\right) + \beta f\left(\frac{a+b}{2}\right)$$

$$\leq \max\left\{\frac{f(a)+f(b)}{2}, f\left(\frac{a+b}{2}\right)\right\} = \frac{f(a)+f(b)}{2},$$

and, consequently,

$$M(\gamma, \delta) = (2\gamma)\left(\frac{f(a)+f(b)}{2}\right) + \delta f\left(\frac{a+b}{2}\right)$$

$$\geq \min\left\{\frac{f(a)+f(b)}{2}, f\left(\frac{a+b}{2}\right)\right\} = f\left(\frac{a+b}{2}\right),$$

it follows that the inequality (1.3) represents a refinement of Hermite-Hadamard inequality (1.2).

Now, it can be seen that the bound $M(0, 1)$ is best possible in general case. Indeed, let $\gamma \in (0, 1/2]$ be fixed and the relation

$$M_f(0, 1; \gamma, \delta) \leq \int_0^1 f(t)dt,$$

holds for arbitrary convex *f*.

Then the convex function $f(t) = t^{1/\gamma}$ gives a counter-example.

This means that the left-hand side of Hermite-Hadamard inequality cannot be improved, in general, by the form of (1.3).

Nevertheless, such improvement is possible for some special classes of convex functions (see Corollary 2.9 below).

In case of the bound $N(\alpha, \beta)$, we found the value $N(1/4, 1/2)$ for which the right-hand side of (1.3) holds for any integrable convex function. Since $N(\alpha, \beta)$ is monotone increasing in α, because

$$\frac{d}{d\alpha} N_f(a, b; \alpha, \beta) = f(a) + f(b) - 2f\left(\frac{a+b}{2}\right) \geq 0,$$

it follows that the right-hand side of (1.3) also holds for all $\alpha \in [1/4, 1/2]$.

In general, the bound $N(1/4,\ 1/2)$ is best possible, as the example $f(t) = |t|,\ t \in [-a,\ a]$ shows.

2. Results and Proofs

We shall begin with the known estimation, cf. ([3], Corollary 3.2).

Theorem 2.1. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a convex function on an interval* I *and* $a,\ b \in I$. *Then*

$$\frac{1}{b-a}\int_a^b f(t)dt \le \frac{1}{4}(f(a)+f(b)) + \frac{1}{2}f(\frac{a+b}{2}) =: N(1/4,\ 1/2). \tag{2.2}$$

If f is a concave function on I, then the inequality is reversed.

Proof. We shall derive the proof by Hermite-Hadamard inequality itself. Indeed, applying twice the right part of this inequality, we get

$$\frac{2}{b-a}\int_a^{\frac{a+b}{2}} f(t)dt \le \frac{1}{2}(f(a)+f(\frac{a+b}{2})),$$

and

$$\frac{2}{b-a}\int_{\frac{a+b}{2}}^b f(t)dt \le \frac{1}{2}(f(\frac{a+b}{2})+f(b)).$$

Summing, the result appears. Therefore, HH inequality has the self-improving property.

For the second part, note that concavity of f implies convexity of $-f$ on I. Hence, applying (2.2) we get the result. □

For the sake of further refinements, we shall consider in the sequel functions from the class $C^{(m)}(I),\ m \in \mathbb{N}$, i.e., functions which are continuously differentiable up to m-th order on an interval $I \subset \mathbb{R}$.

We give firstly a sharp improvement of Theorem 2.1.

Theorem 2.3. *Let* $f \in C^{(2)}(I)$ *be convex on I together with its second derivative. Then for each* $a, b \in I,\ a < b,$

$$\frac{(b-a)^2}{48} f''\left(\frac{a+b}{2}\right) \le N(1/4, 1/2) - \frac{1}{b-a}\int_a^b f(t)dt \le \frac{(b-a)^2}{96}[f''(a) + f''(b)].$$

If f is convex and f'' *concave on I, then*

$$\frac{(b-a)^2}{96}[f''(a) + f''(b)] \le N(1/4, 1/2) - \frac{1}{b-a}\int_a^b f(t)dt \le \frac{(b-a)^2}{48} f''\left(\frac{a+b}{2}\right).$$

Proof. We need the following two assertions.

Lemma 2.4 ([4]). *If* h *is convex on* $I = [a, b]$ *and, for* $x, y \in I$, $x + y = a + b,$ *then*

$$2h\left(\frac{a+b}{2}\right) \le h(x) + h(y) \le h(a) + h(b).$$

Remark 2.5. Note that this result is a pre-HH inequality, i.e., HH inequality is its direct consequence. Indeed, let $x = pa + qb,\ y = qa + pb$ for $p, q \ge 0,\ p + q = 1.$ Then $x, y \in I$ and $x + y = a + b.$ Hence,

$$2h\left(\frac{a+b}{2}\right) \le h(pa + qb) + h(qa + pb) \le h(a) + h(b).$$

Integrating this expression over $p \in [0, 1]$ we obtain the HH inequality.

Lemma 2.6. *Let* $f \in C^{(2)}(I)$ *and* $a, b \in I,\ a < b.$ *Then the following identity holds*:

$$N(1/4, 1/2) - \frac{1}{b-a}\int_a^b f(t)dt = \frac{(b-a)^2}{16}\int_0^1 t(1-t)[f''(x) + f''(y)]dt,$$

with $x := a\frac{t}{2} + b\left(1 - \frac{t}{2}\right),\ y := b\frac{t}{2} + a\left(1 - \frac{t}{2}\right).$

It is not difficult to prove this identity by double partial integration of its right-hand side.

Since $x + y = a + b$ and f'' is convex/concave, applying Lemma 2.4 the proof readily follows. □

Another improvement of HH inequality is given in the next

Theorem 2.7. *Let* $f \in C^{(4)}(I)$ *and* $a, b \in I$, $a < b$. *If* f'' *is convex on* I, *then*

$$\frac{1}{b-a}\int_a^b f(t)dt \le N(1/6,\ 2/3),$$

and the coefficients $1/6,\ 2/3$ *are best possible for this class of functions. If* f'' *is concave on* I, *then the reversed inequality takes place.*

Proof. Note that the coefficients $1/6$ and $2/3$ are involved in well-known Simpson's rule which is of importance in numerical integration. It says that

Lemma 2.8 ([5])**.** *For an integrable* f, *we have*

$$\int_{x_1}^{x_3} f(t)dt = \frac{1}{3}h(f_1 + 4f_2 + f_3) - \frac{1}{90}h^5 f^{(4)}(\xi),\ (x_1 < \xi < x_3),$$

where $f_i = f(x_i)$ *and* $h := x_2 - x_1 = x_3 - x_2$.

Now, taking $x_1 = a$, $x_2 = (a+b)/2$, $x_3 = b$, we get $h = (b-a)/2$. Also, convexity/concavity of f'' on I implies that $f^{(4)}(\xi) \gtrless 0$ and the proof follows. □

Combining this theorem with the results of Theorem 2.1, we get

Corollary 2.9. *Let* $f \in C^{(4)}(I)$. *If* f *is convex and* f'' *concave functions on* I, *then*

$$N(1/6,\ 2/3) \le \frac{1}{b-a}\int_a^b f(t)dt \le N(1/4,\ 1/2).$$

Analogously, let f be concave and f'' *a convex function on I, then*

$$N(1/4,\ 1/2) \le \frac{1}{b-a}\int_a^b f(t)dt \le N(1/6,\ 2/3).$$

Those formulae gives a proper answer, regarding this class of functions, to the problem posed in Introduction.

Further refinement of the assertion from Theorem 2.7 is possible.

Theorem 2.10. *For* $f \in C^{(4)}(I)$, *let* f'' *be convex on I. Then*

$$0 \le \frac{1}{6}[f(a) + f(b)] + \frac{2}{3} f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt$$

$$\le \frac{(b-a)^2}{324}\left[f''(a) + f''(b) - 2f''\left(\frac{a+b}{2}\right)\right].$$

If f'' *concave on I, then*

$$0 \le \frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{6}\left[f(a) + f(b) + 4f\left(\frac{a+b}{2}\right)\right]$$

$$\le \frac{(b-a)^2}{324}\left[2f''\left(\frac{a+b}{2}\right) - (f''(a) + f''(b))\right].$$

The above theorem sharply refines Simpson's rule for this class of functions.

Proof. The left part is proved in Theorem 2.7. For the right part we shall use an integral identity.

Lemma 2.11.

$$N(1/6,\ 2/3) - \frac{1}{b-a}\int_a^b f(t)dt = \frac{(b-a)^2}{48}\int_0^1 t(2-3t)[f''(x) + f''(y)]dt,$$

where x and y are the same as in Lemma 2.6.

Writing,

$$\int_0^1 t(2-3t)[\cdot]dt = \int_0^{2/3} t(2-3t)[\cdot]dt - \int_{2/3}^1 t(3t-2)[\cdot]dt,$$

and applying Lemma 2.4 to each integral separately, the result appears since

$$\int_0^{2/3} t(2-3t)dt = \int_{2/3}^1 t(3t-2)dt = \frac{4}{27}.$$

□

3. Applications in Means Theory

A mean is a map $M : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, with a property

$$\min\{a, b\} \le M(a, b) \le \max\{a, b\},$$

for each $a, b \in \mathbb{R}_+$.

Hence M is necessary reflexive, $M(a, a) = a$.

Most known ordered family of means is the following family Δ of elementary means:

$$\Delta : H \le G \le L \le I \le A \le S,$$

where

$$H = H(a, b) =: 2(1/a + 1/b)^{-1};\ G = G(a, b) =: \sqrt{ab};\ L = L(a, b) =: \frac{b-a}{\log b - \log a};$$

$$I = I(a, b) =: \frac{1}{e}(b^b/a^a)^{1/(b-a)};\ A = A(a, b) =: \frac{a+b}{2};\ S = S(a, b) =: a^{\frac{a}{a+b}} b^{\frac{b}{a+b}},$$

are the harmonic, geometric, logarithmic, identric, arithmetic, and Gini mean, respectively.

Generalized arithmetic mean A_α is defined as

$$A_\alpha = A_\alpha(a, b) =: \begin{cases} \left(\dfrac{a^\alpha + b^\alpha}{2}\right)^{1/\alpha}, & \alpha \ne 0. \\ A_0 = G = \sqrt{ab} \end{cases}$$

Power-difference mean K_α is defined as

$$K_\alpha = K_\alpha(a, b) =: \begin{cases} \dfrac{\alpha}{\alpha+1} \dfrac{a^{\alpha+1} - b^{\alpha+1}}{a^\alpha - b^\alpha}, & \alpha \neq 0, -1; \\ K_0(a, b) = L(a, b); & \\ K_{-1}(a, b) = \dfrac{ab}{L(a, b)}. & \end{cases}$$

It is well known that both means are monotone increasing with α and, evidently,

$$A_{-1} = H,\ A_1 = A,\ K_{-2} = H,\ K_{-1/2} = G,\ K_1 = A.$$

As an illustration of our results, we shall give firstly some sharp approximations of logarithmic and identric means.

Theorem 3.1. *The inequality* $G \leq L \leq A$ *can be improved to*

$$\frac{1}{3}(A + 2G) - \frac{2}{81}\left(\frac{A - G}{L}\right)^2 (A + G) \leq L \leq \frac{1}{3}(A + 2G).$$

Similarly, an approximation of $1/L$ *in terms of the arithmetic and harmonic means is given by*

$$\frac{A - H}{6A^2} \leq \frac{1}{2}\left(\frac{1}{A} + \frac{1}{H}\right) - \frac{1}{L} \leq \frac{A(A - H)}{6H^2}\left(\frac{4}{H} - \frac{3}{A}\right).$$

Proof. Applying Theorem 2.10 with $f = e^t$, we obtain

$$0 \leq \frac{1}{6}(e^x + e^y) + \frac{2}{3} e^{\frac{x+y}{2}} - \frac{e^x - e^y}{x - y}$$

$$\leq \frac{(x - y)^2}{324}\left(e^x + e^y - 2e^{\frac{x+y}{2}}\right).$$

Since x and y are arbitrary real numbers, putting $x = \log b$, $y = \log a$, we get

$$0 \le \frac{1}{3}(A+2G) - L \le \frac{(\log b - \log a)^2}{162}(A-G)$$

$$= \frac{4}{162}\left(\frac{\log b - \log a}{b-a}\right)^2 (A^2 - G^2)(A+G) = \frac{2}{81}\left(\frac{A-G}{L}\right)^2 (A+G),$$

and the proof is done.

For the second part, applying Theorem 2.3 with $f = 1/t$, $f'' = 2/t^3$, we get

$$\frac{(b-a)^2}{24}\frac{1}{A^3} \le \frac{1}{4}\left(\frac{1}{a}+\frac{1}{b}\right) + \frac{1}{2A} - \frac{1}{L} \le \frac{(b-a)^2}{48}\left(\frac{1}{a^3}+\frac{1}{b^3}\right).$$

Now, the identities $1/a + 1/b = 2/H$, $(b-a)^2 = 4A(A-H)$, $AH = G^2$ yields the proof. □

Some interesting inequalities for the identric mean follows.

Theorem 3.2. *For arbitrary positive* a, b, *we have*

$$A^{2/3}G^{1/3} \le I \le A^{2/3}G^{1/3}\exp\left(\frac{(A-H)^2}{162H}\left(\frac{1}{A}+\frac{2}{H}\right)\right);$$

$$A^{4/3}S^{-1/3}\exp\left(-\frac{4}{81}\frac{(A-H)^2}{AH}\right) \le I \le A^{4/3}S^{-1/3}.$$

Proof. Applying Theorem 2.10 with $f = -\log t$, we obtain the proof.

For the second part we need the next,

Lemma 3.3. *For* $a, b \in \mathbb{R}^+$, *we have*

$$A^{4/3}(a,b)S^{2/3}(a,b)\exp\left(-\frac{4}{81}\frac{(A(a,b)-H(a,b))^2}{A(a,b)H(a,b)}\right)$$

$$\le I(a^2, b^2) \le A^{4/3}(a,b)S^{2/3}(a,b).$$

Indeed, for $f = t\log t$, we get

$$\frac{1}{b-a}\int_a^b f(t)dt = \frac{1}{4}\left(\frac{b^2\log b^2 - a^2\log a^2}{b-a} - (a+b)\right) = \frac{a+b}{4}\log I(a^2, b^2).$$

Since $f'' = 1/t,$ Theorem 2.10 yields

$$\frac{1}{6}(a\log a + b\log b) + \frac{2}{3}A\log A - \frac{(b-a)^2}{324}\left(\frac{1}{a} + \frac{1}{b} - \frac{2}{A}\right)$$

$$\leq \frac{a+b}{4}\log I(a^2, b^2) \leq \frac{1}{6}(a\log a + b\log b) + \frac{2}{3}A\log A,$$

and the proof follows by dividing the last expression with $a + b = 2A.$

Now, combining this assertion with the identity $I(a^2, b^2) = I(a, b)S(a, b),$ we obtain the desired inequality. □

Finally, we give bounds of power-difference means in terms of the generalized arithmetic mean.

Theorem 3.4. *For* $a, b \in \mathbb{R}^+$ *and* $\alpha \geq 1,$ *we have*

$$\frac{1}{2}(A(a, b) + A_\alpha(a, b)) \leq K_\alpha(a, b) \leq A_\alpha(a, b). \tag{3.5}$$

For $\alpha < 1,$ *the inequality* (3.5) *is reversed.*

Proof. Let $g_\alpha(t) = t^{1/\alpha}, \alpha \neq 0.$ Since g_α is concave for $\alpha \geq 1,$ Theorem 2.1 combined with HH inequality gives

$$\frac{1}{2}\left(\frac{x+y}{2}\right)^{1/\alpha} + \frac{1}{4}(x^{1/\alpha} + y^{1/\alpha})$$

$$\leq \frac{\alpha}{\alpha+1}\,\frac{x^{1+1/\alpha} - y^{1+1/\alpha}}{x-y} \leq \left(\frac{x+y}{2}\right)^{1/\alpha}.$$

Now, simple change of variables $x = a^\alpha,\ y = b^\alpha$ yields the result.

For the second part, note that g_α is convex for $\alpha < 1$ and repeat the procedure.

□

The above inequality is refined by the following:

Theorem 3.6. *We have*,

$$A_\alpha \le K_\alpha \le \frac{1}{3}(A + 2A_\alpha),\ \alpha \in (-\infty, 1/3) \cup (1/2, 1);$$

$$\frac{1}{3}(A + 2A_\alpha) \le K_\alpha \le A_\alpha,\ \alpha \in [1, \infty);$$

$$\frac{1}{3}(A + A_\alpha) \le K_\alpha \le \frac{1}{2}(A + A_\alpha),\ \alpha \in [1/3, 1/2].$$

Proof. Observe that g''_α is convex for $\alpha \in (-\infty, 1/3) \cup (1/2, 1)$ and concave for $\alpha \in (1/3, 1/2) \cup (1, \infty)$. Hence, applying Theorem 2.7 and Corollary 2.9 together with HH inequality, we obtain the result.

□

An inequality for the reciprocals follows.

Theorem 3.7. *For* $\beta \ge -2$, *we have*

$$\frac{1}{A_{\beta+1}} \le \frac{1}{K_\beta} \le \frac{1}{2}\left(\frac{1}{H} + \frac{1}{A_{\beta+1}}\right).$$

For $\beta < -2$, *the inequality is reversed.*

Proof. This is a consequence of Theorem 3.4. Indeed, putting there $\alpha = -\beta - 1$ and using identities

$$K_\alpha = \frac{ab}{K_\beta},\ A_\alpha = \frac{ab}{A_{\beta+1}},\ A = \frac{ab}{H},$$

the proof appears.

□

Further improvements of this type by Theorem 3.6 are possible but it is left to the readers.

4. Addendum

Theorems proved above are the source of a plenty of interesting inequalities from Classical Analysis. As an illustration we shall give here a couple of Cusa-type inequalities.

Theorem 4.1. *The inequality*

$$\frac{1}{2}\cos x + \frac{1}{2} \le \frac{\sin x}{x} \le \frac{1}{3}\cos x + \frac{2}{3},$$

holds for $|x| \le \pi/2$.

Also,

$$\frac{1}{4}\cosh x + \frac{3}{4} \le \frac{\sinh x}{x} \le \frac{1}{3}\cosh x + \frac{2}{3},$$

holds for $|x| \le (3/2)^{3/2}$.

Proof. For the first part one should apply Corollary 2.9 to the function $f(t) = -\cos t$ on a symmetric interval $t \in [-x, x] \subset [-\pi/2, \pi/2]$.

Similarly for the second part, one should apply Theorem 2.10 with $f(t) = e^t$.

$\square$

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