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SOME CHARACTERIZATION OF LACUNARY UNIFORM STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES

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Abstract

In the [3] is proven that sequence S_{ij} uniformly statistically converges to L if and only if it there is a subset A of the set $\mathbb{N} \times \mathbb{N}$ uniform density zero and subsequence $S(x)$ defined by, $S_{ij}(x) = S_{ij}$ for $(i, j) \in A^c$, converges to L, in the Pringsheim's sense. In this paper, it is proven that analog is valid and for lacunary uniformly statistical convergence. Double sequence S_{ij} lacunary uniformly statistically converges to L if and only if it there is a subset A of the set $\mathbb{N} \times \mathbb{N}$ lacunary uniform density zero and subsequence $S(x)$ defined by, $S_{ij}(x) = S_{ij}$ for $(i, j) \in A^c$, converges to L, in the Pringsheim's sense. The subsequence $S(x)$ lacunary uniformly statistically converges to L if and only if it there is a subset A of the set $\mathbb{N} \times \mathbb{N}$ lacunary uniform density zero and subsequence $S(y)$ defined by, $S_{ij}(y) = S_{ij}(x)$ for $(i, j) \in A^c$, such that $\lim_{i \to \infty} (\lim_{j \to \infty} S_{ij}(y)) = L$.

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1. Introduction

The concept of the statistical convergence of a sequences of real numbers was introduced by Fast [9]. Furthermore, Gökhan et al. [12] introduced the notion of pointwise and uniform statistical convergent of double sequences of real-valued function. Dündar and Atay [4-8] investigated the relation between I-convergence of double sequences. Fridy and Orhan [11] have studied lacunary statistical convergence of single sequences. Petterson and Savaş in [13] defined the lacunary statistical analogue for double sequences. Now, we recall that the definitions of concepts of ideal convergence and basic concepts [1, 2, 10]. The sequence S_{ij} of real numbers converges to L in the Pringsheim's sense, if for $\forall \varepsilon > 0, \exists K > 0$ such that

$$|S_{ij} - L| \leq \varepsilon, \ \forall i, \ j \geq K.$$

We write $\lim_{i, j \to \infty} S_{ij} = L.$

Let $K \subset \mathbb{N} \times \mathbb{N}$. Let K_{nm} be the number of $(i, j) \in K$ such that $i \leq n, \ j \leq m.$ If

$$d_2(K) = \lim_{n, m \to \infty} \frac{K_{nm}}{nm}$$

in the Pringsheim's sense then, we say that K has double natural density. Let is sequence S_{ij} of real numbers and $\varepsilon > 0.$ Let

$$A(\varepsilon) = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |S_{ij} - L| \geq \varepsilon\}.$$

The sequence $S = S_{ij}$ statistically converges to $L \in \mathbb{R}$ if $d_2(A(\varepsilon)) = 0$ for $\forall \varepsilon > 0.$

We write $st - \lim S_{ij} = L.$ Let is set $X \neq \emptyset.$ A class I of subsets of X is said to be an ideal in X provided the following statements hold:

(i) $\emptyset \in I$;

(ii) $A, B \in I \Rightarrow A \cup B \in I$;

(iii) $A \in I, B \subset A \Rightarrow B \in I.$

The ideal is called nontrivial if $I \neq \{\emptyset\}$ and $X \in I^c$. A nontrivial ideal I is called admissible if it contains all the singleton sets. A nontrivial ideal I on $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to I for each $i \in \mathbb{N}$.

A nonempty family F of subsets of a set X is called a *filter* if

(i) $\emptyset \in F^c$;

(ii) $A, B \in F \Rightarrow A \bigcap B \in F$;

(iii) $A \in F,\ A \subset B \Rightarrow B \in F$.

In this paper, the focus is put on ideal $I_u \subset 2^{\mathbb{N} \times \mathbb{N}}$ defined by: subset A belongs to the I_u if

$$\lim_{p, q \to \infty} \frac{1}{pq} \left|\{i < p,\ j < q : (n + i,\ m + j) \in A\}\right| = 0$$

uniformly on $n, m \in \mathbb{N}$ in the Pringsheim's sense. That is subset A of the set $\mathbb{N} \times \mathbb{N}$ is uniformly density zero.

The sequence $S = S_{ij}$ uniformly statistically converges to L if for any $\varepsilon > 0$

$$\{(i,\ j) \in \mathbb{N} \times \mathbb{N} : |S_{ij} - L| \geq \varepsilon\} \in I_u.$$

That is sequence $S = S_{ij}$ uniformly statistically converges to L if $\forall \varepsilon,\ \varepsilon' > 0,\ \exists K > 0$ such that

$$\frac{1}{pq} \left|\{i < p,\ j < q : |S_{n+i, m+j} - L| \geq \varepsilon\}\right| < \varepsilon',\ \forall p,\ q \geq K,\ \forall n,\ m \in \mathbb{N}.$$

We write $U\, st - \lim S_{ij} = L.$

We denote with X a set of all double sequences of 0's and 1's, i.e.,

$$X = \{x = x_{ij} : x_{ij} \in \{0, 1\}, i, j \in \mathbb{N}\}.$$

Let sequence $S = S_{ij}$ and $x \in X$. Then with $S(x)$ we denote a sequence defined following way:

$$S_{ij}(x) = S_{ij}, \text{ for } x_{ij} = 1.$$

The subsequence $S(x)$ of sequence S uniformly statistically converges to L if $\forall \varepsilon, \varepsilon' > 0, \exists K > 0$ such that for $\forall p, q \geq K$ and $\forall n, m \in \mathbb{N}$ provided that $x_{nm} = 1$, we have

$$\frac{|\{i < p, j < q : |S_{n+i, m+j} - L| \geq \varepsilon, x_{n+i, m+j} = 1\}|}{|\{i < p, j < q : x_{n+i, m+j} = 1\}|} \leq \varepsilon'.$$

We write $U\, st - \lim S_{ij}(x) = L$.

By a lacunary sequence we mean an increasing sequence $\Theta = (k_r)$ such that

$$k_0 = 0 \text{ and } h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty.$$

Let

$$I_1 = \{(i, j) \in \mathbb{N} \times \mathbb{N} : i, j \leq k_1\},$$

$$I_2 = \{(i, j) \in \mathbb{N} \times \mathbb{N} : i, j \leq k_2\} \setminus I_1, \ldots,$$

$$I_r = \{(i, j) \in \mathbb{N} \times \mathbb{N} : i, j \leq k_r\} \setminus (I_{r-1} \bigcup I_{r-2} \bigcup \cdots \bigcup I_1), \text{ for } \forall r \in \mathbb{N}.$$

The sequence S_{ij} lacunary statistically converges to L if $\forall \varepsilon > 0,$ we have

$$\lim_{r \to \infty} \frac{1}{|I_r|} |\{(i, j) \in I_r : |S_{ij} - L| \geq \varepsilon\}| = 0.$$

We write $S_\Theta - \lim S_{ij} = L$.

Fridy proved that if $S = S_i$ sequence of real numbers and $\Theta = (k_r)$ lacunary sequence such that

$$1 < \liminf \frac{k_r}{k_{r-1}} \leq \limsup \frac{k_r}{k_{r-1}} < \infty.$$

Then, sequence $S = S_i$ statistically convergent if and only if it lacunary statistically convergent.

Let $S = S_{ij}$ double sequence of real numbers and $\Theta = (k_r)$ lacunary sequence of natural numbers.

A sequence $S = S_{ij}$ lacunary uniformly statistically converges to real number L if $\forall \varepsilon, \varepsilon' > 0, \exists r_0 \in \mathbb{N}$ such that for $\forall r > r_0$ and $\forall n, m \in \mathbb{N}$, we have

$$\frac{1}{|I_r|} \left|\{(i, j) \in I_r : |S_{n+i, m+j} - L| \geq \varepsilon\}\right| \leq \varepsilon',$$

Write $U\, st_\Theta - \lim S_{ij} = L$.

The subset A of the set $\mathbb{N} \times \mathbb{N}$ is lacunary uniformly density zero if $\forall \varepsilon > 0, \exists r_0 \in \mathbb{N}$ such that for $\forall r > r_0$ and $\forall n, m \in \mathbb{N}$, we have

$$\frac{1}{|I_r|} \left|\{(i, j) \in I_r : (n + i, m + j) \in A\}\right| \leq \varepsilon.$$

2. New Results

Theorem 2.1. *Let is* $\Theta = (k_r)$ *lacunary sequence and* $S = S_{ij}$ *double sequence. Then,* $U\, st_\Theta - \lim S_{ij} = L$ *if and only if it* $\exists A \subset \mathbb{N} \times \mathbb{N}$ *lacunary uniformly density zero and* $\lim_{i, j \to \infty} S_{ij}(x) = L$, *in the Pringsheim's sense, for*

$$x_{ij} = \begin{cases} 1, & (i, j) \notin A, \\ 0, & (i, j) \in A. \end{cases}$$

Proof. Let is $U\, st_{\Theta} - \lim S_{ij} = L.$ Then there is a sequence of natural numbers $(u_r)_{r=2}^{\infty}$ such that for $\forall l \geq u_r$ and $\forall n,\, m \in \mathbb{N},$ we have

$$\frac{1}{|I_l|}\left|\left\{(i,\, j) \in I_l : |S_{n+i,\, m+j} - L| \geq \frac{1}{r}\right\}\right| \leq \frac{1}{r}.$$

Let

$$A = \bigcup_{r=2}^{\infty}\bigcup_{n,\, m=1}^{\infty}\left\{(n+i,\, m+j) : (i,\, j) \in \bigcup_{l=u_r}^{u_{r+1}-1} I_l,\, |S_{n+i,\, m+j} - L| \geq \frac{1}{r}\right\}.$$

We define $x \in X$ the following way:

$$x_{ij} = \begin{cases} 1, & (i,\, j) \notin A, \\ 0, & (i,\, j) \in A. \end{cases}$$

For $\forall \varepsilon > 0,\, \exists r_0 \in \mathbb{N}$ such that for $\forall r \geq r_0$ we have $\frac{1}{r} \leq \varepsilon.$ From the definition of the sequence $x,$ such that for $l \geq u_{r_0}$ and $\forall n,\, m \in \mathbb{N}$ provided that $x_{n+i,\, m+j} = 1,$ we have

$$|S_{n+i,\, m+j}(x) - L| = |S_{n+i,\, m+j} - L| \leq \varepsilon.$$

Hence, for $\forall i,\, j \geq k_{u_{r_0}}$ and $\forall i,\, j \in \mathbb{N}$ provided that $x_{ij} = 1,$ we have

$$|S_{ij}(x) - L| \leq \varepsilon.$$

Hence, the subsequence $S(x)$ converges to $L,$ in the Pringsheim's sense. For $\forall l \geq u_{r_0}$ and $\forall n,\, m \in \mathbb{N}$ valid

$$\frac{1}{|I_l|}|\{(i,\, j) \in I_l : (n+i,\, m+j) \in A\}|$$

$$= \frac{1}{|I_l|}\left|\left\{(i,\, j) \in I_l : |S_{n+i,\, m+j} - L| \geq \frac{1}{r}\right\}\right| \leq \frac{1}{r} \leq \varepsilon.$$

Hence,

$$\lim_{l\to\infty}\frac{1}{|I_l|}|\{(i,\, j) \in I_l : (n+i,\, m+j) \in A\}| = 0, \text{ uniformly for } \forall n,\, m \in \mathbb{N}.$$

We assume that there is a subset A of the set $\mathbb{N}\times\mathbb{N}$ such that $\lim_{l\to\infty}\frac{1}{|I_l|}|\{(i, j)\in I_l : (n+i, m+j)\in A\}| = 0,$ uniformly for $\forall n, m\in\mathbb{N}$ and $\lim_{i, j\to\infty} S_{ij}(x) = L,$ in the Pringsheim's sense, for

$$x_{ij} = \begin{cases} 1, & (i, j)\notin A, \\ 0, & (i, j)\in A. \end{cases}$$

For $\forall\varepsilon > 0, \exists N\in\mathbb{N}$ such that for $\forall n, m\geq N,$ we have

$$|S_{nm}(x) - L| \leq \varepsilon.$$

For $\forall l\in\mathbb{N}$ such that $k_{l-1} > N.$ Then,

$$\frac{1}{|I_l|}|\{(i, j)\in I_l : |S_{n+i, m+j} - L| \geq \varepsilon\}|$$

$$= \frac{1}{|I_l|}|\{(i, j)\in I_l : n+i < N \vee m+j < N, |S_{n+i, m+j} - L|\geq\varepsilon\}|$$

$$+ \frac{1}{|I_l|}|\{(i, j)\in I_l : n+i, m+j\geq N, |S_{n+i, m+j} - L|\geq\varepsilon\}|$$

$$\leq \frac{2N(k_l - k_{l-1})}{k_l^2 - k_{l-1}^2} + \frac{1}{|I_l|}|\{(i, j)\in I_l : n+i, m+j\geq N, (n+i, m+j)\in A\}|$$

$$\leq \frac{2N(k_l - k_{l-1})}{k_l^2 - k_{l-1}^2} + \frac{1}{|I_l|}|\{(i, j)\in I_l : (n+i, m+j)\in A\}|.$$

Obviously, the first summand converges to zero uniformly on $n, m\in\mathbb{N}.$ The second summand converges to zero uniformly on $n, m\in\mathbb{N}$ due to the assumption.

So, $U\, st_\Theta - \lim S_{ij} = L.$

Definition 2.2. The subsequence $S(x)$ of S lacunary uniformly statistically converges to L if $\forall \varepsilon, \epsilon > 0, \exists r_0 \in \mathbb{N}$ such that for $\forall l > r_0$ and $\forall n, m \in \mathbb{N}$ provided that $x_{nm} = 1$, we have

$$\frac{|\{(i, j) \in I_l : |S_{n+i, m+j} - L| \geq \varepsilon, x_{n+i, m+j} = 1\}|}{|\{(i, j) \in I_l : x_{n+i, m+j} = 1\}|} \leq \epsilon.$$

Write $U\, st_{\Theta} - \lim S_{ij}(x) = L.$

The characterization is true for subsequences.

Corollary 2.3. *Let* $x \in X$ *and* $S = S_{ij}$ *double sequence and* $\Theta = (k_r)$ *lacunary sequence, then*

$U\, st_{\Theta} - \lim S_{ij}(x) = L$ *if and only if it* $\exists A \subseteq \{(n, m) : x_{nm} = 1\}$ *such that*

$$\lim_{l \to \infty} \frac{|\{(i, j) \in I_l : (n + i, m + j) \in A\}|}{|\{(i, j) \in I_l : x_{n+i, m+j} = 1\}|} = 0,$$

uniformly for $\forall n, m \in \mathbb{N}$ *provided that* $x_{nm} = 1$ *and for subsequence* $S(y)$ *of the sequence S valid*:

$$\lim_{i \to \infty} (\lim_{j \to \infty} S_{ij}(y)) = L,$$

for

$$y_{ij} = \begin{cases} 1, & (i, j) \notin A, x_{ij} = 1, \\ 0, & (i, j) \in A, x_{ij} = 0. \end{cases}$$

Proof. Let $U\, st_{\Theta} - \lim S_{ij}(x) = L.$ Then there is a sequence of natural numbers $(u_r)_{r=2}^{\infty}$ such that for $\forall l > u_r$ and $\forall n, m \in \mathbb{N}$ provided that $x_{nm} = 1$, we have

$$\frac{|\{(i, j) \in I_l : |S_{n+i, m+j} - L| \geq \frac{1}{r}, x_{n+i, m+j} = 1\}|}{|\{(i, j) \in I_l : x_{n+i, m+j} = 1\}|} \leq \frac{1}{r}.$$

Let

$$A = \bigcup_{r=2}^{\infty} \bigcup_{n,m=1}^{\infty}$$

$$\left\{ (n+i, m+j) : (i, j) \in \bigcup_{l=u_r}^{u_{r+1}-1} I_l, \ |S_{n+i,m+j} - L| \geq \frac{1}{r}, \ x_{n+i,m+j} = 1 \right\}.$$

For $\forall \varepsilon > 0$, $\exists r_0 \in \mathbb{N}$ such that for $\forall r > r_0$ we have $\frac{1}{r} \leq \varepsilon$.

Let $u_r \leq l < u_{r+1}$. Then, for $\forall l > u_{r_0}$ and $\forall n, m \in \mathbb{N}$ provided that $x_{nm} = 1$, we have

$$\frac{|\{(i, j) \in I_l : (n+i, m+j) \in A\}|}{|\{(i, j) \in I_l : x_{n+i,m+j} = 1\}|}$$

$$= \frac{|\{(i, j) \in I_l : |S_{n+i,m+j} - L| \geq \frac{1}{r}, \ x_{n+i,m+j} = 1\}|}{|\{(i, j) \in I_l : x_{n+i,m+j} = 1\}|} \leq \frac{1}{r} \leq \varepsilon.$$

Let $l > u_{r_0}$, $(n+i, m+j) \in A^c$, $(i, j) \in I_l$, $x_{n+i,m+j} = 1$, then

$$|S_{n+i,m+j} - L| \leq \frac{1}{r} < \varepsilon.$$

Hence, for $n > k_{u_{r_0}} \vee m > k_{u_{r_0}+1}$ provided that $x_{nm} = 1$, we have

$$|S_{nm} - L| \leq \varepsilon.$$

Hence,

$$\lim_{i\to\infty} (\lim_{j\to\infty} S_{ij}(y)) = L,$$

for

$$y_{ij} = \begin{cases} 1, & (i, j) \notin A, \ x_{ij} = 1, \\ 0, & (i, j) \in A, \ x_{ij} = 0. \end{cases}$$

Let $\exists A \subseteq \{(n, m) : x_{nm} = 1\}$ such that

$$\lim_{l\to\infty} \frac{|\{(i, j) \in I_l : (n+i, m+j) \in A\}|}{|\{(i, j) \in I_l : x_{n+i, m+j} = 1\}|} = 0,$$

uniformly $\forall n, m \in \mathbb{N}$ provided that $x_{nm} = 1$ and for subsequence $S(y)$ of the sequence S valid:

$$\lim_{i\to\infty}(\lim_{j\to\infty} S_{ij}(y)) = L,$$

for

$$y_{ij} = \begin{cases} 1, & (i, j) \notin A,\ x_{ij} = 1, \\ 0, & (i, j) \in A,\ x_{ij} = 0. \end{cases}$$

Then, $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that for $\forall n \geq N \vee \forall m \geq N$ provided that $x_{nm} = 1$ and $(n, m) \in A^c$, we have

$$|S_{nm} - L| < \varepsilon.$$

Then,

$$\frac{|\{(i, j) \in I_l : |S_{n+i, m+j} - L| \geq \varepsilon,\ x_{n+i, m+j} = 1\}|}{|\{(i, j) \in I_l : x_{n+i, m+j} = 1\}|}$$

$$= \frac{|\{(i, j) \in I_l : |S_{n+i, m+j} - L| \geq \varepsilon,\ x_{n+i, m+j} = 1,\ n+i,\ m+j \leq N\}|}{|\{(i, j) \in I_l : x_{n+i, m+j} = 1\}|}$$

$$+ \frac{|\{(i, j) \in I_l : |S_{n+i, m+j} - L| \geq \varepsilon,\ x_{n+i, m+j} = 1,\ n+i > N \vee m+i > N\}|}{|\{(i, j) \in I_l : x_{n+i, m+j} = 1\}|}$$

$$\leq \frac{N^2}{|\{(i, j) \in I_l : x_{n+i, m+j} = 1\}|} + \frac{|\{(i, j) \in I_l : (n+i, m+j) \in A\}|}{|\{(i, j) \in I_l : x_{n+i, m+j} = 1\}|}.$$

Obviously, the first summand converges to zero uniformly on $n, m \in \mathbb{N}$. The second summand converges to zero uniformly on $n, m \in \mathbb{N}$ due to the assumption. So, $U\ st_{\Theta} - \lim S_{ij}(x) = L$.

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