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PROPERTIES OF STRONGLY BALANCED TILINGS BY CONVEX POLYGONS

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Abstract

Every normal periodic tiling is a strongly balanced tiling. The properties of periodic tilings by convex polygons are rearranged from the knowledge of strongly balanced tilings. From the results, we show the properties of representative periodic tilings by a convex pentagonal tile.

1. Introduction

A collection of sets (the "tiles") is a *tiling* (or tessellation) of the plane if their union covers the whole plane but the interiors of different tiles are disjoint. If all the tiles in a tiling are of the same size and shape, then the tiling is *monohedral*. In this study, a polygon that admits a monohedral tiling is a *polygonal tile*. A tiling by convex polygons is *edge-to-edge* if any two convex polygons in a tiling are either disjoint or share one vertex (or an entire edge) in common. A tiling is periodic if it coincides with its

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translation by a nonzero vector. The unit that can generate a periodic tiling by translation only is known as the *fundamental region* [4, 14, 17, 18, 20].

In the classification problem of convex polygonal tiles, only the pentagonal case is open. At present, fifteen types of convex pentagonal tiles are known (see Figure 1) but it is not known whether this list is complete [2-9, 11-14, 17, 21]. However, it has been proved that a convex pentagonal tile that can generate an edge-to-edge tiling belongs to at least one of the eight known types [1, 15, 18, 19]. We are interested in the problem of convex pentagonal tiling (i.e., the complete list of types of convex pentagonal tile, regardless of edge-to-edge and non-edge-to-edge tilings). However, the solution of the problem is not easy. Therefore, we will first treat only convex pentagonal tiles that admit at least one periodic tiling[1]. As such, we consider that the properties of the periodic tilings by convex polygons should be rearranged. From Statement 3.4.8 ("every normal periodic tiling is strongly balanced") in [4], we see that periodic tilings by convex polygonal tiles (i.e., monohedral periodic tilings by convex polygons) are contained in the strongly balanced tilings. The definitions of normal and strongly balanced tilings are given in Section 2. In this paper, the properties of strongly balanced tilings by convex polygons are presented from the knowledge of strongly balanced tilings in general. That is, the properties correspond to those of periodic tilings by a convex polygonal tile.

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[1] We know as a fact that the 15 types of convex pentagonal tile admit at least one periodic tiling. From this, we find that the convex pentagonal tiles that can generate an edge-to-edge tiling admit at least one periodic tiling [4, 9, 14, 15, 17-20]. On the other hand, there is no proof that they admit at least one periodic tiling without using this fact. That is, there is no assurance yet that all convex pentagonal tiles admit at least one periodic tiling. In the solution of the problem of convex pentagonal tiling, it is necessary to consider whether there is a convex polygonal tile that admits infinitely many tilings of the plane, none of which is periodic.

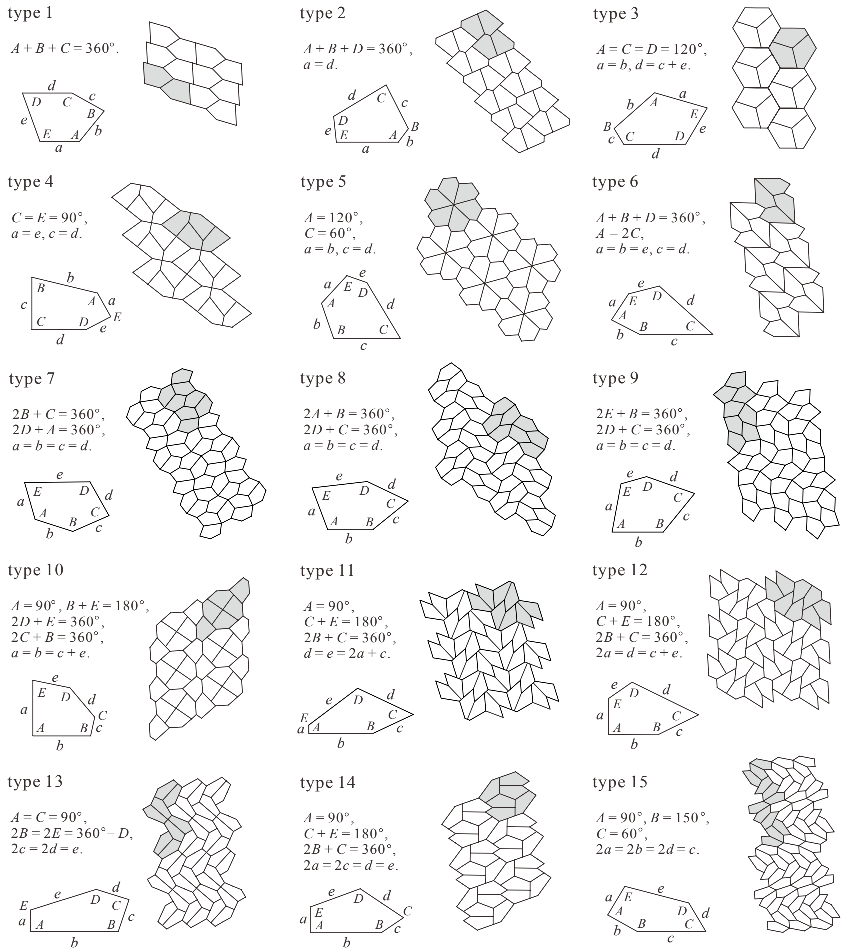

Figure 1. Fifteen types of convex pentagonal tiles. If a convex pentagon can generate a monohedral tiling and is not a new type, it belongs to at least one of types 1-15. Each of the convex pentagonal tiles is defined by some conditions between the lengths of the edges and the magnitudes of the angles, but some degrees of freedom remain. For example, a convex pentagonal tile belonging to type 1 satisfies that the sum of three consecutive angles is equal to 360°. This condition for type 1 is expressed as $A + B + C = 360^\circ$ in this figure. The pentagonal tiles of types 14 and 15 have one degree of freedom, that of size. For example, the value of C of the pentagonal tile of type 14 is $\cos^{-1}((3\sqrt{57} - 17)/16) \approx 1.2099$ rad $\approx 69.32^\circ$.

2. Preparation

Definitions and terms of this section quote from [4].

Terms "vertices" and "edges" are used by both polygons and tilings. In order not to cause confusion, *corners* and *sides* are referred to instead of vertices and the edges of polygons, respectively. At a vertex of a polygonal tiling, corners of two or more polygons meet and the number of polygons meeting at the vertex is called the *valence* of the vertex, and is at least three (see Figure 2). Therefore, an edge-to-edge tiling by polygons is such that the corners and sides of the polygons in a tiling coincide with the vertices and edges of the tiling.

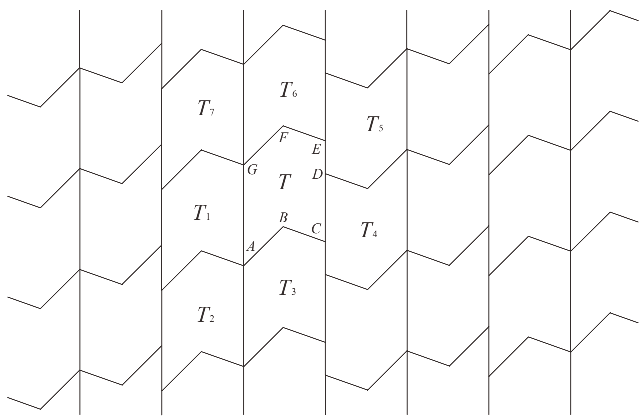

Figure 2. The differences between corners and vertices, sides and edges, adjacents, and neighbors. The points A, B, C, E, F, and G are corners of the tile T; but A, C, D, E, and G are vertices of the tiling (we note that the valence of vertices A and G is four, and the valence of vertices C, D, and E is three). The line segments AB, BC, CE, EF, FG, and GA are sides of T, while AC, CD, DE, EG, and GA are edges of the tiling. The tiles T_1, T_3, T_4, T_5 and T_6 are adjacents (and neighbors) of T, whereas tiles T_2 and T_7 are neighbors (but not adjacents) of T [4].

Two tiles are called *adjacent* if they have an edge in common, and then each is called an adjacent of the other. On the other hand, two tiles are called *neighbors* if their intersection is nonempty (see Figure 2).

There exist positive numbers u and U such that any tile contains a certain disk of radius u and is contained in a certain disk of radius U in which case we say the tiles in tiling are *uniformly bounded*.

A tiling $\Im$ is called *normal* if it satisfies following conditions: (i) every tiles of $\Im$ is a topological disk; (ii) the intersection of every two tiles of $\Im$ is a connected set, that is, it does not consist of two (or more) distinct and disjoint parts; (iii) the tiles of $\Im$ are uniformly bounded.

Let $D(r, M)$ be a closed circular disk of radius r, centered at any point M of the plane. Let us place $D(r, M)$ on a tiling, and let F_1 and F_2 denote the set of tiles contained in $D(r, M)$ and the set of meeting boundary of $D(r, M)$ but not contained in $D(r, M)$, respectively. In addition, let F_3 denote the set of tiles surrounded by these in F_2 but not belonging to F_2. The set $F_1 \cup F_2 \cup F_3$ of tiles is called the patch $A(r, M)$ of tiles generated by $D(r, M)$.

For a given tiling $\Im$, we denote by $v(r, M)$, $e(r, M)$, and $t(r, M)$ the numbers of vertices, edges, and tiles in $A(r, M)$, respectively. The tiling $\Im$ is called *balanced* if it is normal and satisfies the following condition: the limits

$$\lim_{r\to\infty} \frac{v(r, M)}{t(r, M)} \quad \text{and} \quad \lim_{r\to\infty} \frac{e(r, M)}{t(r, M)}$$

exist. Note that $v(r, M) - e(r, M) + t(r, M) = 1$ is called Euler's theorem for planar maps.

Statement 1 (Statement 3.3.13 in [4]). *Every normal periodic tiling is balanced.*

Euler's Theorem for Tilings (Statement 3.3.3 in [4]). *For any normal tiling* $\Im$, *if one of the limits* $v(\Im) = \lim_{r\to\infty} \frac{v(r, M)}{t(r, M)}$ *or* $e(\Im) = \lim_{r\to\infty} \frac{e(r, M)}{t(r, M)}$ *exists and is finite, then so does the other. Thus the tiling is balanced and, moreover,*

$$v(\Im) = e(\Im) - 1. \tag{1}$$

For a given tiling $\Im$, we write $t_h(r, M)$ for the number of tiles with h adjacents in $A(r, M)$, and $v_j(r, M)$ for the numbers of j-valent vertices in $A(r, M)$. Then the tiling $\Im$ is called *strongly balanced* if it is normal and satisfies the following condition: all the limits

$$t_h(\Im) = \lim_{r\to\infty} \frac{t_h(r, M)}{t(r, M)} \quad \text{and} \quad v_j(\Im) = \lim_{r\to\infty} \frac{v_j(r, M)}{t(r, M)}$$

exist. Then,

$$\sum_{h\geq 3} t_h(\Im) = 1 \quad \text{and} \quad v(\Im) = \sum_{j\geq 3} v_j(\Im) \tag{2}$$

hold. Therefore, every strongly balanced tiling is necessarily balanced.

When $\Im$ is strongly balanced, we have

$$2e(\Im) = \sum_{j\geq 3} j \cdot v_j(\Im) = \sum_{h\geq 3} h \cdot t_h(\Im). \tag{3}$$

In addition, as for strongly balanced tiling, following properties are known.

Statement 2 (Statement 3.4.8 in [4]). *Every normal periodic tiling is strongly balanced.*

Statement 3 (Statement 3.5.13 in [4]). *For each strongly balanced tiling* $\Im$, *we have*

$$\frac{1}{\sum_{j\geq 3} j \cdot w_j(\Im)} + \frac{1}{\sum_{h\geq 3} h \cdot t_h(\Im)} = \frac{1}{2}, \tag{4}$$

where

$$w_j(\Im) = \frac{v_j(\Im)}{v(\Im)}.$$

Thus $w_j(\Im)$ can be interpreted as that fraction of the total number of vertices in $\Im$ which have valence j, and $\sum_{j\geq 3} j \cdot w_j(\Im)$ is the *average* valence taken over all the vertices. Since $\sum_{h \geq 3} t_h(\Im) = 1$ there is a similar interpretation of $\sum_{h \geq 3} h \cdot t_h(\Im)$: it is the *average* number of adjacents of the tiles, taken over all the tiles in $\Im$. Since the valence of the vertex is at least three,

$$\sum_{j \geq 3} j \cdot w_j(\Im) \geq 3. \tag{5}$$

Statement 4 (Statement 3.5.6 in [4]). *In every strongly balanced tiling* $\Im$, *we have*

$$2\sum_{j \geq 3}(j-3) \cdot v_j(\Im) + \sum_{h \geq 3}(h-6) \cdot t_h(\Im) = 0,$$

$$\sum_{j \geq 3}(j-4) \cdot v_j(\Im) + \sum_{h \geq 3}(h-4) \cdot t_h(\Im) = 0,$$

$$\sum_{j \geq 3}(j-6) \cdot v_j(\Im) + 2\sum_{h \geq 3}(h-3) \cdot t_h(\Im) = 0.$$

3. Consideration and Discussion

A polygon with n sides and n corners is referred to as an *n-gon*. For the discussion below, note that n-gons in a strongly balanced tiling do not need to be congruent (i.e., the tiling does not need to be monohedral).

3.1. Case of convex n-gons

Let $\Im_n^{sb}$ be a strongly balanced tiling by convex n-gons.

Proposition 1. $\sum_{h \geq 3} h \cdot t_h(\Im_n^{sb}) \leq 6.$

Proof. From (4), we have that

$$\frac{2\sum_{h \geq 3} h \cdot t_h(\Im_n^{sb})}{\sum_{h \geq 3} h \cdot t_h(\Im_n^{sb}) - 2} = \sum_{j \geq 3} j \cdot w_j(\Im_n^{sb}).$$

Since the valence of the vertex is at least three, i.e., $\sum_{j \geq 3} j \cdot w_j(\Im_n^{sb}) \geq 3$,

$$\frac{2\sum_{h \geq 3} h \cdot t_h(\Im_n^{sb})}{\sum_{h \geq 3} h \cdot t_h(\Im_n^{sb}) - 2} \geq 3.$$

Therefore, we obtain Proposition 1. □

From Proposition 1, there is no strongly balanced tiling that is formed by convex n-gons for $n \geq 7$, since the average number of adjacents is greater than six. Note that the number of sides of all convex polygons in $\Im_n^{sb}$ does not have the same necessity. For example, there is no strongly balanced tiling by convex 6-gons and convex 8-gons, and there is no strongly balanced tiling by only convex 5-gons whose number of adjacents is seven or more.

Note that Proposition 1 is not a proof that there is no convex polygonal tile with seven or more sides. If there were a proof that all convex polygonal tiles admit at least one periodic tiling, it could be used to prove that there is no convex polygonal tile with seven or more sides from Proposition 1.

Proposition 2. $3 \leq \sum_{j \geq n} j \cdot w_j(\Im_n^{sb}) \leq \frac{2n}{n-2}$.

Proof. From (4) and $\sum_{h \geq 3} h \cdot t_h(\Im_n^{sb}) \geq n$,

$$\frac{2\sum_{j \geq 3} j \cdot w_j(\Im_n^{sb})}{\sum_{j \geq 3} j \cdot w_j(\Im_n^{sb}) - 2} = \sum_{h \geq n} h \cdot t_h(\Im_n^{sb}) \geq n.$$

Therefore, from the above inequality and (5), we obtain Proposition 2. □

Let $\Im_n^{sbe}$ be a strongly balanced edge-to-edge tiling by convex n-gons.

Proposition 3. $\sum_{j \geq 3} j \cdot w_j(\Im_n^{sbe}) = \frac{2n}{n-2}$.

Proof. The number of adjacents of all convex n-gons in $\Im_n^{sb}$ is equal to n. That is, $\sum_{h \geq n} h \cdot t_h(\Im_n^{sbe}) = n$. Then, (4) is $\frac{1}{\sum_{j \geq 3} j \cdot w_j(\Im_n^{sbe})} + \frac{1}{n} = \frac{1}{2}$.

Therefore, we obtain Proposition 3. □

3.2. Case of convex hexagons

As for $\Im_6^{sb}$ (i.e., a strongly balanced tiling by convex hexagons (6-gon)), the number of adjacents of each convex hexagon should be greater than or equal to six (i.e., $h \geq 6$). Therefore,

$$\sum_{h \geq 6} t_h(\Im_6^{sb}) = t_6(\Im_6^{sb}) + \sum_{h \geq 7} t_h(\Im_6^{sb}) = 1. \tag{6}$$

On the other hand, from Proposition 1, we have

$$\sum_{h \geq 6} h \cdot t_h(\Im_6^{sb}) = 6 \cdot t_6(\Im_6^{sb}) + \sum_{h \geq 7} h \cdot t_h(\Im_6^{sb}) \leq 6. \tag{7}$$

Proposition 4. $\sum_{j \geq 3} j \cdot w_j(\Im_6^{sb}) = 3$.

Proof. From (6) and (7),

$$6\left(1 - \sum_{h \geq 7} t_h(\Im_6^{sb})\right) + \sum_{h \geq 7} h \cdot t_h(\Im_6^{sb}) \leq 6.$$

Hence, we obtain $\sum_{h \geq 7} (h - 6) \cdot t_h(\Im_6^{sb}) \leq 0$. On the other hand, $\sum_{h \geq 7} (h - 6) \cdot t_h(\Im_6^{sb}) \geq 0$ holds because $t_h(\Im_6^{sb}) = \lim_{r \to \infty} \frac{t_h(r, M)}{t(r, M)} \geq 0$. From these inequalities,

$$\sum_{h \geq 7} (h - 6) \cdot t_h(\Im_6^{sb}) = 0.$$

Therefore, $t_h(\Im_6^{sb}) = 0$ for $h \neq 6$. That is, we have

$$t_6(\Im_6^{sb}) = 1, \sum_{h \geq 6} h \cdot t_h(\Im_6^{sb}) = 6 \quad \text{and} \quad \sum_{h \geq 7} t_h(\Im_6^{sb}) = \sum_{h \geq 7} h \cdot t_h(\Im_6^{sb}) = 0.$$

Thus, from these relationships and (4), we arrive at Proposition 4. □

From Statement 2, a monohedral periodic tiling by a convex hexagon is strongly balanced. If a fundamental region in a monohedral periodic tiling by a convex hexagon has vertices with valences of four or more, $\sum_{j \geq 3} j \cdot w_j(\Im_6^{sb}) > 3$ and $\sum_{h \geq 6} h \cdot t_h(\Im_6^{sb}) < 6$ from (4), which is a contradiction of Proposition 4. Thus, we have the following corollary.

Corollary 1. *A monohedral periodic tiling by a convex hexagon is an edge-to-edge tiling with only* 3-*valent vertices*.

It is well known that convex hexagonal tiles (i.e., convex hexagons that admit a monohedral tiling) belong to at least one of the three types shown in Figure 3. That is, convex hexagonal tiles admit at least one periodic edge-to-edge tiling whose valence is three at all vertices. In fact, the representative tilings of the three types in Figure 3 are periodic edge-to-edge tilings whose valence is three at all vertices. From Corollary 1 and the fact that the valence of vertices is at least three, it might be considered that the valence of all vertices in monohedral tilings by convex hexagons is three; however, that is not true. For example, as shown Figure 4, there are monohedral tilings by convex hexagons with vertices of valence equal to four (note that a monohedral tiling is not always a periodic tiling). However, it is clear that the convex hexagonal tiles of Figure 4 can generate a periodic edge-to-edge tiling in which the valence of all vertices is three.

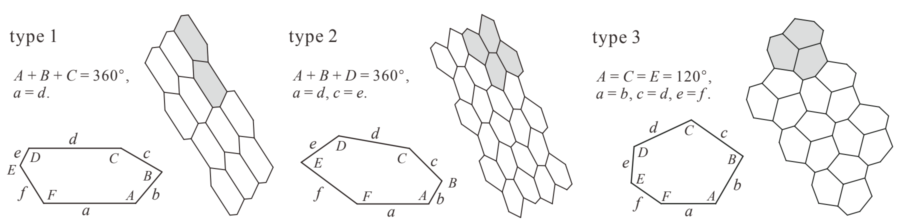

Figure 3. Three types of convex hexagonal tiles. If a convex hexagon can generate a monohedral tiling, it belongs to at least one of types 1-3. The pale gray hexagons in each tiling indicate the fundamental region.

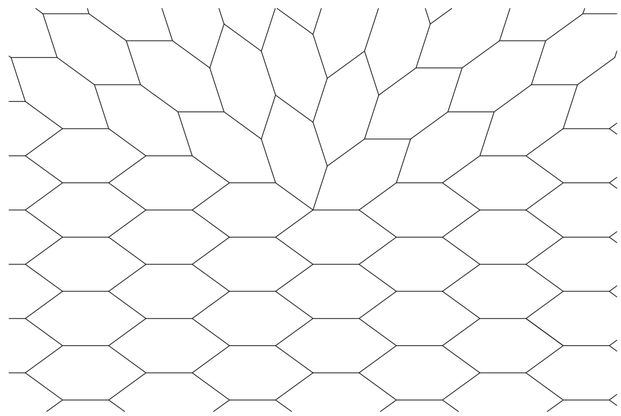

Figure 4. Monohedral tilings by convex hexagons with vertices of valence equal to four.

3.3. Case of convex pentagons

As for $\Im_5^{sb}$ (i.e., a strongly balanced tiling by convex pentagons (5-gons)), the number of adjacents of each convex pentagon should be greater than or equal to five. From Proposition 1, we obtain the following.

Proposition 5. $5 \leq \sum_{h \geq 5} h \cdot t_h(\Im_5^{sb}) \leq 6.$

Since $\sum_{h \geq 5} h \cdot t_h(\Im_5^{sb})$ is the average number of adjacents of the convex pentagons in $\Im_5^{sb}$, we obtain the following theorem [16].

Theorem 1. *A tiling* $\Im_5^{sb}$ *contains a convex pentagon whose number of adjacents is five or six.*

Then, we obtain the following propositions:

Proposition 6. $\frac{5}{2} \leq e(\Im_5^{sb}) \leq 3.$

Proposition 7. $\frac{3}{2} \leq v(\Im_5^{sb}) \leq 2.$

Proof of Propositions 6 and 7. From (3) and Proposition 5, we have that $5 \leq 2e(\Im_5^{sb}) \leq 6$. Since each strongly balanced tiling is necessarily balanced, $v(\Im_5^{sb}) = e(\Im_5^{sb}) - 1$ holds from Euler's Theorem for tilings.

Therefore, we have that $5 \le 2(v(\Im_5^{sb}) + 1) \le 6$. Thus, we obtain Propositions 6 and 7. □

Proposition 8. $3 \le \sum_{j \ge 3} j \cdot w_j(\Im_5^{sb}) \le \frac{10}{3}$.

Proof. It is clear from Proposition 2. □

Proposition 9. $0 \le \sum_{h \ge 7} (h - 6) \cdot t_h(\Im_5^{sb}) \le t_5(\Im_5^{sb}) \le 1$.

Proof. From Proposition 5,

$$5 \le \sum_{h \ge 5} h \cdot t_h(\Im_5^{sb}) = 5t_5(\Im_5^{sb}) + 6t_6(\Im_5^{sb}) + \sum_{h \ge 7} h \cdot t_h(\Im_5^{sb}) \le 6. \tag{8}$$

From (2),

$$\sum_{h \ge 3} t_h(\Im_5^{sb}) = t_5(\Im_5^{sb}) + t_6(\Im_5^{sb}) + \sum_{h \ge 7} t_h(\Im_5^{sb}) = 1,$$

$$t_6(\Im_5^{sb}) = 1 - t_5(\Im_5^{sb}) - \sum_{h \ge 7} t_h(\Im_5^{sb}). \tag{9}$$

From (8), (9) and $0 \le t_h(\Im_5^{sb}) \le 1$, we obtain the inequality of Proposition 9. □

From Proposition 9, we obtain the following theorem [16].

Theorem 2. *A tiling* $\Im_5^{sb}$ *that satisfies* $\sum_{h \ge 7} t_h(\Im_5^{sb}) > 0$ *contains a convex pentagon whose number of adjacents is five.*

As for $\Im_5^{sbe}$ (i.e., a strongly balanced edge-to-edge tiling by convex pentagons), we have the following proposition.

Proposition 10. $t_5(\Im_5^{sbe}) = 1$, $v(\Im_5^{sbe}) = \frac{3}{2}$, $e(\Im_5^{sbe}) = \frac{5}{2}$, *and* $\sum_{j \ge 3} j \cdot w_j(\Im_6^{sbe}) = \frac{10}{3}$.

Proof. It is clear from Proposition 3, Euler's Theorem for tilings and $t_h(\Im_5^{sbe}) = 0$ for $h \ne 5$. □

Next, we have other propositions as follows.

Proposition 11. $v_3(\Im_5^{sb}) = 2 + \sum_{j\geq 4}(2-j)\cdot v_j(\Im_5^{sb})$.

Proof. From (1) and the definition of $v(\Im)$, we have

$$e(\Im_5^{sb}) = v(\Im_5^{sb}) + 1 = \sum_{j\geq 3} v_j(\Im_5^{sb}) + 1 = v_3(\Im_5^{sb}) + \sum_{j\geq 4} v_j(\Im_5^{sb}) + 1. \quad (10)$$

On the other hand, from (3),

$$2e(\Im_5^{sb}) = \sum_{j\geq 3} j\cdot v_j(\Im_5^{sb}) = 3v_3(\Im_5^{sb}) + \sum_{j\geq 4} j\cdot v_j(\Im_5^{sb}) \quad (11)$$

holds. Therefore, from (10) and (11), we have

$$2\left(v_3(\Im_5^{sb}) + \sum_{j\geq 4} v_j(\Im_5^{sb}) + 1\right) = 3v_3(\Im_5^{sb}) + \sum_{j\geq 4} j\cdot v_j(\Im_5^{sb}).$$

Thus, we obtain Proposition 11. □

Proposition 12. $v_3(\Im_5^{sbe}) = \sum_{j\geq 4}(3j-10)\cdot v_j(\Im_5^{sbe})$.

Proof. From Proposition 10 and the definitions of $w_j(\Im)$ and $v(\Im)$,

$$\sum_{j\geq 3} j\cdot w_j(\Im_5^{sbe}) = \frac{\sum_{j\geq 3} j\cdot v_j(\Im_5^{sbe})}{v(\Im_5^{sbe})} = \frac{3v_3(\Im_5^{sbe}) + \sum_{j\geq 4} j\cdot v_j(\Im_5^{sbe})}{v_3(\Im_5^{sbe}) + \sum_{j\geq 4} v_j(\Im_5^{sbe})} = \frac{10}{3}.$$

Thus, we obtain Proposition 12. □

As for $v(\Im_5^{sb})$, we have the following propositions:

Proposition 13. $v(\Im_5^{sb}) = \sum_{j\geq 3} v_j(\Im_5^{sb}) = \frac{1}{2}\sum_{h\geq 5}(h-2)\cdot t_h(\Im_5^{sb})$.

Proposition 14. $v(\Im_5^{sb}) = \frac{1}{2} + \frac{1}{2}\sum_{h\geq 5}(h-3)\cdot t_h(\Im_5^{sb})$.

Proposition 15. $v(\Im_5^{sb}) = 2 + \frac{1}{2}\sum_{h\geq 5}(h-6)\cdot t_h(\Im_5^{sb})$.

Proposition 16. $v(\Im_5^{sb}) = 2 - \sum_{j \geq 4} (j - 3) \cdot v_j(\Im_5^{sb})$.

Proof of Propositions 13, 14, 15, and 16. From the first equation in Statement 4,

$$2 \sum_{j \geq 4} (j - 3) \cdot v_j(\Im_5^{sb}) + \sum_{h \geq 5} (h - 6) \cdot t_h(\Im_5^{sb}) = 0.$$

Note that $t_3(\Im_5^{sb}) = t_4(\Im_5^{sb}) = 0$, since $\Im_5^{sb}$ is $h \geq 5$. The above equation is rearranged as

$$\sum_{j \geq 4} (j - 3) \cdot v_j(\Im_5^{sb}) = -\frac{1}{2} \sum_{h \geq 5} (h - 6) \cdot t_h(\Im_5^{sb}). \tag{12}$$

From the second equation in Statement 4,

$$-v_3(\Im_5^{sb}) + \sum_{j \geq 5} (j - 4) \cdot v_j(\Im_5^{sb}) + \sum_{h \geq 5} (h - 4) \cdot t_h(\Im_5^{sb}) = 0.$$

The above equation is rearranged as

$$\sum_{j \geq 5} (j - 4) \cdot v_j(\Im_5^{sb}) = v_3(\Im_5^{sb}) - \sum_{h \geq 5} (h - 4) \cdot t_h(\Im_5^{sb}). \tag{13}$$

Then,

$$\sum_{j \geq 4} (j - 3) \cdot v_j(\Im_5^{sb}) = v_4(\Im_5^{sb}) + \sum_{j \geq 5} (j - 4 + 1) \cdot v(\Im_5^{sb})$$

$$= v_4(\Im_5^{sb}) + \sum_{j \geq 5} (j - 4) \cdot v(\Im_5^{sb}) + \sum_{j \geq 5} v_j(\Im_5^{sb}). \tag{14}$$

By replacing $\sum_{j \geq 4} (j - 3) \cdot v_j(\Im_5^{sb})$ of (12) and $\sum_{j \geq 5} (j - 4) \cdot v_j(\Im_5^{sb})$ of (13) in (14), the latter becomes

$$-\frac{1}{2} \sum_{h \geq 5} (h - 6) \cdot t_h(\Im_5^{sb}) = v_4(\Im_5^{sb}) + v_3(\Im_5^{sb}) - \sum_{h \geq 5} (h - 4) \cdot t_h(\Im_5^{sb})$$

$$+ \sum_{j \geq 5} v_j(\Im_5^{sb}).$$

Simplifying both sides,

$$\sum_{j\geq 3} v_j(\Im_5^{sb}) = \frac{1}{2}\sum_{h\geq 5}(-h+6+2h-8)\cdot t_h(\Im_5^{sb}) = \frac{1}{2}\sum_{h\geq 5}(h-2)\cdot t_h(\Im_5^{sb}).$$

Thus, we obtain Proposition 13.

Next, from $\sum_{j\geq 4}(j-3)\cdot v_j(\Im_5^{sb})$ and Proposition 11,

$$\begin{aligned}\sum_{j\geq 4}(j-3)\cdot v_j(\Im_5^{sb}) &= -\sum_{j\geq 4}(2-j)\cdot v_j(\Im_5^{sb}) - \sum_{j\geq 4} v_j(\Im_5^{sb}) \\ &= 2 - v_3(\Im_5^{sb}) - \sum_{j\geq 4} v_j(\Im_5^{sb}) \\ &= 2 - \sum_{j\geq 3} v_j(\Im_5^{sb}).\end{aligned}$$

Thus, we obtain Proposition 16.

From (12) and Proposition 16,

$$2 - \sum_{j\geq 3} v_j(\Im_5^{sb}) = -\frac{1}{2}\sum_{h\geq 5}(h-6)\cdot t_h(\Im_5^{sb}).$$

Thus, we obtain Proposition 15.

From the third equation in Statement 4,

$$-3v_3(\Im_5^{sb}) + \sum_{j\geq 4}(j-6)\cdot v_j(\Im_5^{sb}) + 2\sum_{h\geq 5}(h-3)\cdot t_h(\Im_5^{sb}) = 0.$$

By replacing $v_3(\Im_5^{sb})$ of Proposition 11 in the above equation, it becomes

$$4\sum_{j\geq 4}(j-3)\cdot v_j(\Im_5^{sb}) = 6 - 2\sum_{h\geq 5}(h-3)\cdot t_h(\Im_5^{sb}). \tag{15}$$

Form Proposition 11 and (15),

$$v_3(\Im_5^{sb}) = 2 - \sum_{j \geq 4} v_j(\Im_5^{sb}) - \sum_{j \geq 4} (j-3) \cdot v_j(\Im_5^{sb})$$

$$= \frac{1}{2} - \sum_{j \geq 4} v_j(\Im_5^{sb}) + \frac{1}{2} \sum_{h \geq 5} (h-3) \cdot t_h(\Im_5^{sb}).$$

Simplifying,

$$v_3(\Im_5^{sb}) + \sum_{j \geq 4} v_j(\Im_5^{sb}) = \frac{1}{2} + \frac{1}{2} \sum_{h \geq 5} (h-3) \cdot t_h(\Im_5^{sb}).$$

Thus, we obtain Proposition 14. □

Here, we consider the case of $v(\Im_5^{sb}) = \frac{3}{2}$ (i.e., the minimum case of $v(\Im_5^{sb})$). From Proposition 15, we have

$$2 + \frac{1}{2} \sum_{h \geq 5} (h-6) \cdot t_h(\Im_5^{sb}) = 2 - \frac{1}{2} t_5(\Im_5^{sb}) + \frac{1}{2} \sum_{h \geq 6} (h-6) \cdot t_h(\Im_5^{sb}) = \frac{3}{2}.$$

Simplifying this equation,

$$t_5(\Im_5^{sb}) = 1 + \sum_{h \geq 6} (h-6) \cdot t_h(\Im_5^{sb}). \tag{16}$$

Since $\sum_{h \geq 5} t_h(\Im_5^{sb}) = t_5(\Im_5^{sb}) + \sum_{h \geq 6} t_h(\Im_5^{sb}) = 1$, $\sum_{h \geq 6} (h-6) \cdot t_h(\Im_5^{sb})$ in (16) is equal to zero. That is, in the case of $v(\Im_5^{sb}) = \frac{3}{2}$, $\sum_{h \geq 6} t_h(\Im_5^{sb}) = 0$. Therefore, $v(\Im_5^{sb}) = \frac{3}{2}$ if and only if $\Im_5^{sb}$ is $\Im_5^{sbe}$.

3.4. Properties of representative periodic tilings by a convex pentagonal tile

Let $\Im_5^{r(x)}$ be a representative periodic tiling by a convex pentagonal tile of type x. That is, $\Im_5^{r(x)}$ for $x = 1, \ldots, 15$ is a strongly balanced tiling by a convex pentagonal tile. Representative tilings of types 1 or 2 are generally non-edge-to-edge, as shown in Figure 1. However, in special cases, the convex pentagonal tiles of types 1 or 2 can generate edge-to-edge tilings, as shown in Figure 5. Here, the convex pentagonal tiles of (a) and (b) in Figure 5 are referred to as those of types 1e and 2e, respectively. Then, let $\Im_5^{r(1e)}$ and $\Im_5^{r(2e)}$ be representative edge-to-edge periodic tilings by a convex pentagonal tile of types 1e and 2e, respectively.

The properties of each tiling $\Im_5^{r(x)}$ are obtained from the results of Subsection 3.3, etc. Table 1 summarizes the results [16]. We can check that the representative periodic tilings of each type of convex pentagonal tile that can generate an edge-to-edge tiling satisfy Proposition 10.

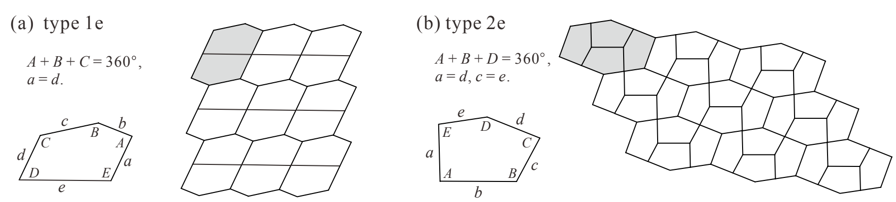

Figure 5. Examples of edge-to-edge tilings by convex pentagonal tiles that belong to types 1 or 2. The pale gray pentagons in each tiling indicate the fundamental region.

4. Conclusion

In this paper, although it is accepted as a fact, it is proved that the only convex polygonal tiles that admit at least one periodic tiling are triangles, quadrangles, pentagons, and hexagons. As for the fact (proof) that the only convex polygonal tiles are triangles, quadrangles, pentagons, and hexagons, note that it is necessary to take except periodic tilings also into consideration.

The properties in Section 3 could lead to a solution to the problem of classifying the types of convex pentagonal tile. On the other hand, in May 2017, Michaël Rao proposed a proof of the complete list of types of convex pentagonal tiles [10, 22]. He declared that there are only the known 15 types of convex pentagonal tiles (see Figure 1).

Table 1. Properties of $\Im_5^{r(x)}$

x of $\Im_5^{r(x)}$	$t_h\left(\Im_5^{r(x)}\right)$	$v_j\left(\Im_5^{r(x)}\right)$	$2e\left(\Im_5^{r(x)}\right)$	$w_j\left(\Im_5^{r(x)}\right)$	$\sum_{j\geq 3} j\cdot w_j\left(\Im_5^{r(x)}\right)$
1e, 2e, 4, 6, 7, 8, 9	$t_5 = 1$, $t_h = 0$ for $h \neq 5$	$v_3 = 1$, $v_4 = \frac{1}{2}$, $v_j = 0$ for $j \neq 3$ or 4	5	$w_3 = \frac{2}{3}$, $w_4 = \frac{1}{3}$, $w_j = 0$ for $j \neq 3$ or 4	$\frac{10}{3} = 3.\dot{3}$
5	$t_5 = 1$, $t_h = 0$ for $h \neq 5$	$v_3 = \frac{4}{3}$, $v_6 = \frac{1}{6}$, $v_j = 0$ for $j \neq 3$ or 6	5	$w_3 = \frac{8}{9}$, $w_6 = \frac{1}{9}$, $w_j = 0$ for $j \neq 3$ or 6	$\frac{10}{3} = 3.\dot{3}$
1, 2, 3, 12	$t_6 = 1$, $t_h = 0$ for $h \neq 6$	$v_3 = 2$, $v_j = 0$ for $j \neq 3$	6	$w_3 = 1$, $w_j = 0$ for $j \neq 3$	3
10	$t_5 = \frac{2}{3}$, $t_7 = \frac{1}{3}$, $t_h = 0$ for $h \neq 5$ or 7	$v_3 = \frac{5}{3}$, $v_4 = \frac{1}{6}$, $v_j = 0$ for $j \neq 3$ or 4	$\frac{17}{3} = 5.\dot{6}$	$w_3 = \frac{10}{11}$, $w_4 = \frac{1}{11}$, $w_j = 0$ for $j \neq 3$ or 4	$\frac{34}{11} = 3.\dot{0}\dot{9}$
11	$t_5 = t_7 = \frac{1}{2}$, $t_h = 0$ for $h \neq 5$ or 7	$v_3 = 2$, $v_j = 0$ for $j \neq 3$	6	$w_3 = 1$, $w_j = 0$ for $j \neq 3$	3
13	$t_5 = t_6 = \frac{1}{2}$, $t_h = 0$ for $h \neq 5$ or 6	$v_3 = \frac{3}{2}$, $v_4 = \frac{1}{4}$, $v_j = 0$ for $j \neq 3$ or 4	$\frac{11}{2} = 5.5$	$w_3 = \frac{6}{7}$, $w_4 = \frac{1}{7}$, $w_j = 0$ for $j \neq 3$ or 4	$\frac{22}{7} \approx 3.142...$
14	$t_5 = t_6 = t_7 = \frac{1}{3}$, $t_h = 0$ for $h \neq 5$, 6 or 7	$v_3 = 2$ $v_j = 0$ for $j \neq 3$	6	$w_3 = 1$, $w_j = 0$ for $j \neq 3$	3
15	$t_5 = \frac{2}{3}$, $t_6 = \frac{1}{3}$, $t_h = 0$ for $h \neq 5$ or 6	$v_3 = \frac{4}{3}$, $v_4 = \frac{1}{3}$, $v_j = 0$ for $j \neq 3$ or 4	$\frac{16}{3} = 5.\dot{3}$	$w_3 = \frac{4}{5}$, $w_4 = \frac{1}{5}$, $w_j = 0$ for $j \neq 3$ or 4	$\frac{16}{5} = 3.2$

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