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CONVEX POLYGONS FOR APERIODIC TILING

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Abstract

If all tiles in a tiling are congruent, the tiling is called monohedral. Tiling by convex polygons is called edge-to-edge if any two convex polygons are either disjoint or share one vertex or one entire edge in common. In this paper, we prove that a convex polygon that can generate an edge-to-edge monohedral tiling must be able to generate a periodic tiling.

1. Introduction

A *tiling* of the plane is an exact covering of the plane by a collection of sets without gaps or overlaps (except for the boundaries of the sets). More precisely, a collection of sets (the "tiles") is a tiling of the plane if their union is the entire plane, but the interiors of different tiles are disjoint. The tiles are frequently polygons and are all congruent to one another or at least congruent to one of a small number of *prototiles* [4, 5]. If all tiles in the tiling are of the same size and shape (i.e., congruent), the tiling is called *monohedral* [4]. Therefore, a prototile of a monohedral tiling by

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polygons is single polygon in the tiling, and we call the polygon the *polygonal tile* [4, 14-19].

A tiling of the plane is *periodic* if the tiling can be translated onto itself in two nonparallel directions. More precisely, a tiling is periodic if it coincides with its translation by two linearly independent vectors. A set of prototiles is called *aperiodic* if congruent copies of the prototiles admit infinitely many tilings of the plane, none of which are periodic. It must be emphasized that no periodic tilings are permitted at all, even using just one of the prototiles [1, 4, 5, 12]. (A tiling that has no periodicity is called nonperiodic. On the other hand, a tiling by aperiodic sets of prototiles is called aperiodic. Note that, although an aperiodic tiling is a nonperiodic tiling, a nonperiodic tiling is not necessarily an aperiodic tiling.) The Penrose tiling, which is known as a quasiperiodic tiling, is a nonperiodic tiling, and it can also be considered an aperiodic tiling that is generated by the aperiodic set of prototiles with a matching condition (see Section 2 for details).

For aperiodic tilings, the critical problem is to find sets of aperiodic prototiles that are essentially different from the ones already known. In particular, there are following two problems [5]:

(i) Is there a single aperiodic prototile (with or without a matching condition), that is, one that admits only aperiodic tilings by congruent copies?

(ii) It is well known that there is a set of three convex polygons that are aperiodic with no matching condition on the edges. Is there a set of prototiles with a size less than or equal to two that is aperiodic?

Problem (i) is called the Einstein Problem (ein stein = one stone), and the affirmative solution (a convex hexagon with a matching condition) was shown in recent years [12]. The elements of an aperiodic set of three convex polygons with no matching condition on the edges in Problem (ii) are one convex hexagon and two convex pentagons [4, 5]. Then, the aperiodic tiling by the three convex polygons is based on the Penrose tiling (see Subsection 2.2).

Tiling by convex polygons is called *edge-to-edge* if any two convex polygons are either disjoint or share one vertex or one entire edge in common [4, 14-19]. We have studied the convex pentagonal tiles that can generate an edge-to-edge tiling. As the result, we find the following [15, 18].

Theorem 1. *Without matching conditions other than "edge-to-edge", no single convex polygon can be an aperiodic prototile.*

In Section 2, we explain the aperiodic set of prototiles from the Penrose tiling and introduce the aperiodic tiling by one convex hexagon and two convex pentagons with no matching condition on the edges. In Section 3, we present the proof of Theorem 1. Section 4 presents an estimate for the aperiodic tiling by a convex polygonal tile from the known facts on convex polygonal tiles.

2. Aperiodic Tiling and Aperiodic Set of Prototiles

2.1. Penrose tiling

Here, an aperiodic set of prototiles is introduced using the Penrose tiling. This is the topic relevant to Problem (i) on aperiodic tiling mentioned in Section 1.

The rhombuses in Figure 1(a) and (b) (called Penrose rhombuses) are the prototiles of the Penrose tiling. We note that all vertices of the two rhombuses are colored; there exist edges with orientations, and the length of the edges of the rhombus in Figure 1(a) is equal to that of the rhombus in Figure 1(b). To obtain the tiling that is generated by the rhombuses in Figure 1(a) and (b), the vertices always meet with the same color; it is edge-to-edge, and the edges with orientations must match the direction of the orientations. The way of matching that is required for the generation of the tiling is called a *matching condition* [4, 5]. The set of the two rhombuses in Figure 1(a) and (b) is an aperiodic set of prototiles with a matching condition [4]. Therefore, the Penrose tiling in Figure 1(c) is an aperiodic tiling in which the rhombuses in Figure 1(a) and (b) are generated according to the matching condition. In addition, there is another way of choosing the two prototiles of the Penrose tiling: the kite (see Figure 2(a)) and the dart (see Figure 2(b)). The edges of the kite and the dart have two lengths in the ratio $1 : \tau,$ where $\tau = (1 + \sqrt{5})/2 \approx 1.1618...$ is the golden number. The vertices are colored with two colors, say black and white, as shown. To obtain the Penrose tiling by using the kite and the dart, we must place equal edges together and also match the colors at the vertices. The set of the kite and the dart in Figure 2 is an aperiodic set of prototiles with a matching condition [4, 5]. Note that each prototile in Figures 1 and 2 cannot independently generate a tiling when it is placed according to a matching condition.

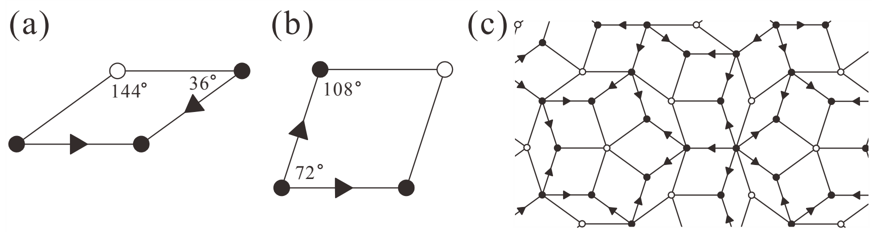

Figure 1. Penrose tiling and Penrose rhombuses of an aperiodic set.

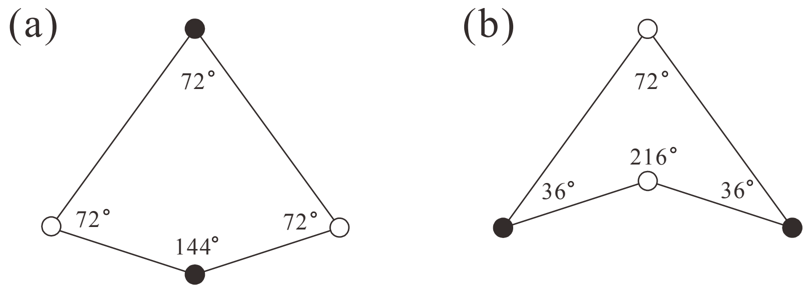

Figure 2. The kite and the dart of the Penrose aperiodic set.

In the above explanation for a matching condition, we use vertices with color conditions. If the edges are admitted to use a color, the vertices do not need to have a matching condition. That is, a matching condition is a condition concerning edge matching. It must be emphasized that the matching condition can be represented by assigning colors and orientations to some of the edges of the prototile.

Next, in order to understand the aperiodic set of prototiles, the prototiles of the Penrose tilings shown in Figure 3(a) and (b) are introduced. Although the rhombuses in Figure 3 have respectively the same shape as the rhombuses in Figure 1, their matching conditions are different. For the two rhombuses in Figure 3, the vertices are not colored, and all edges have orientations. To obtain the Penrose tiling (Figure 3(c)) that is generated by the rhombuses in Figure 3(a) and (b), the lengths and orientations of the edges of the rhombuses must match. On the other hand, the rhombus in Figure 3(b) can independently generate a periodic tiling, as shown in Figure 3(d), according to a matching condition. That is, although the two prototiles in Figure 3(a) and (b) can generate a nonperiodic tiling, one of them can also generate a periodic tiling. Therefore, the set of the rhombuses in Figure 3(a) and (b) is not an aperiodic set of prototiles. Based on the above discussion, we note that a set of prototiles of the Penrose tiling is not necessarily an aperiodic set of prototiles.

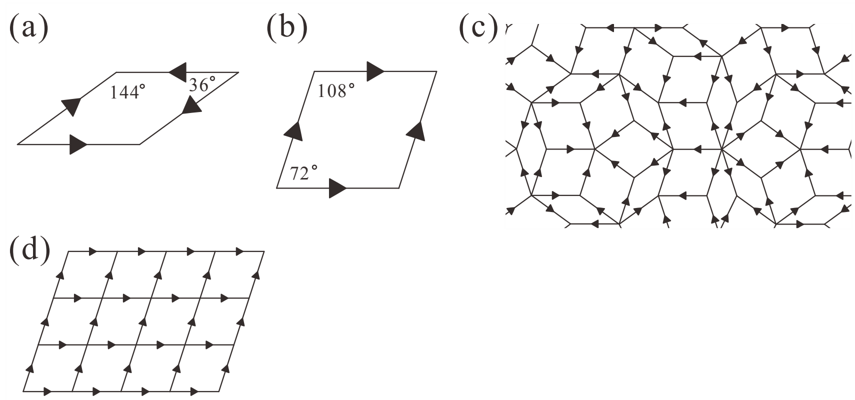

Figure 3. Prototiles of the Penrose tiling, which is not an aperiodic set.

2.2. Aperiodic tiling due to Ammann that uses three unmarked convex polygons as prototiles

In this subsection, we introduce the prototiles and tilings relevant to Problem (ii) on aperiodic tiling mentioned in Section 1. Problem (ii) is the problem of finding the smallest sets of aperiodic prototiles, each of which is an unmarked convex polygon (i.e., a convex polygon with no matching condition). Ammann has produced a remarkable example of such a set by recomposition from the set of rhombuses in Figure 1(a) and (b) [4]. If the inside of each rhombus in Figure 1(a) and (b) is divided as shown in

Figure 4(a) and (b), the Penrose tiling in Figure 1(c) becomes the tiling shown in Figure 4(c). If the edges (broken lines) of the rhombuses are eliminated from the tiling in Figure 4(c), the aperiodic tiling with one unmarked convex hexagon and two convex pentagons is obtained (see Figure 4(d)). That is, the set of one convex hexagon and two convex pentagons in the edge-to-edge tiling in Figure 4(d) is the set of three convex polygons that are aperiodic with no matching condition on the edges.

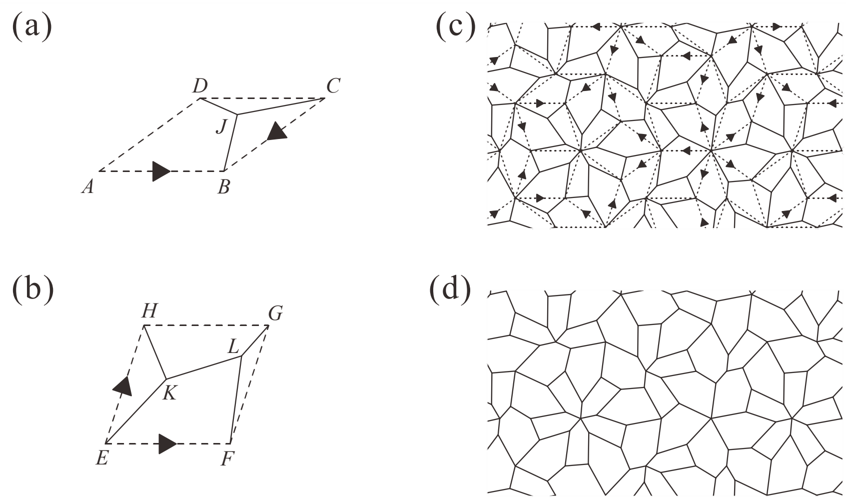

Figure 4. Aperiodic tiling due to Ammann, which uses three unmarked convex polygons as prototiles–one hexagon and two pentagons. This tiling is a recomposition by Penrose rhombuses and is obtained from it using the markings shown in (a) and (b). These markings are completely determined by the choice of point *J*, as we must have $GL = DJ$, $FL = CJ = EK$, and $HK = BJ$ [4] points out that the aperiodicity *J* must be chosen so that DJ, CJ, BJ, and KL are of different lengths.

3. Proof of Theorem 1

A unit that can only generate a periodic tiling by translation is called a *fundamental region*.

No convex polygon with seven or more edges can generate a monohedral tiling [3-7, 10]. All convex hexagons that can generate a monohedral tiling are categorized into three types[1] and can generate a periodic edge-to-edge tiling (see Figure 5) [3-6, 10]. Note that, a regular convex hexagon belongs to all of these three types.

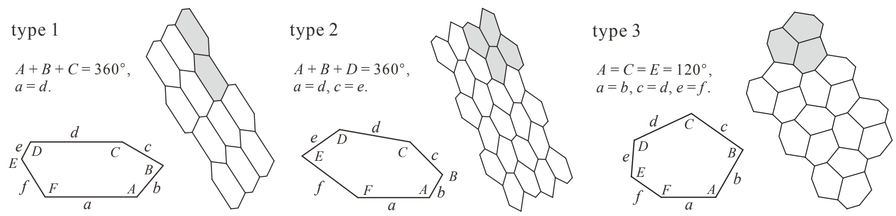

Figure 5. Three types of convex hexagonal tiles. If a convex hexagon can generate a monohedral tiling, it belongs to at least one of types 1-3. The pale gray hexagons in each tiling indicate a fundamental region.

At present, essentially 15 different types of convex pentagonal tiles are known (see Figure 6), but it is not known whether this list is complete [3-6, 8, 10, 11, 13-21]. There are edge-to-edge and non-edge-to-edge tilings by convex pentagonal tiles of the 15 types. A convex pentagonal tile belonging only to types 3, 10, 11, 12, 13, 14, or 15 in Figure 6 cannot generate an edge-to-edge tiling. In Figure 6, non-edge-to-edge tilings are shown for type 1 or type 2 as general representative tilings, but the set of convex pentagonal tiles of type 1 or type 2 contains convex pentagonal tiles that can generate edge-to-edge tilings (see Figure 7). The remaining six types can generate an edge-to-edge tiling [14, 16]. From the results in [2, 15, 17, 18], we know that all convex pentagons that can generate an edge-to-edge monohedral tiling belong to at least one of eight types (i.e., types 1, 2, and 4-9 in Figure 6) among the 15 types and they must be able to generate a periodic tiling.

[1]The classification of types of convex polygonal tiles is based on the essentially different properties of polygons. The classification problem of types of convex polygonal tiles and the classification problem of polygonal tilings are quite different.

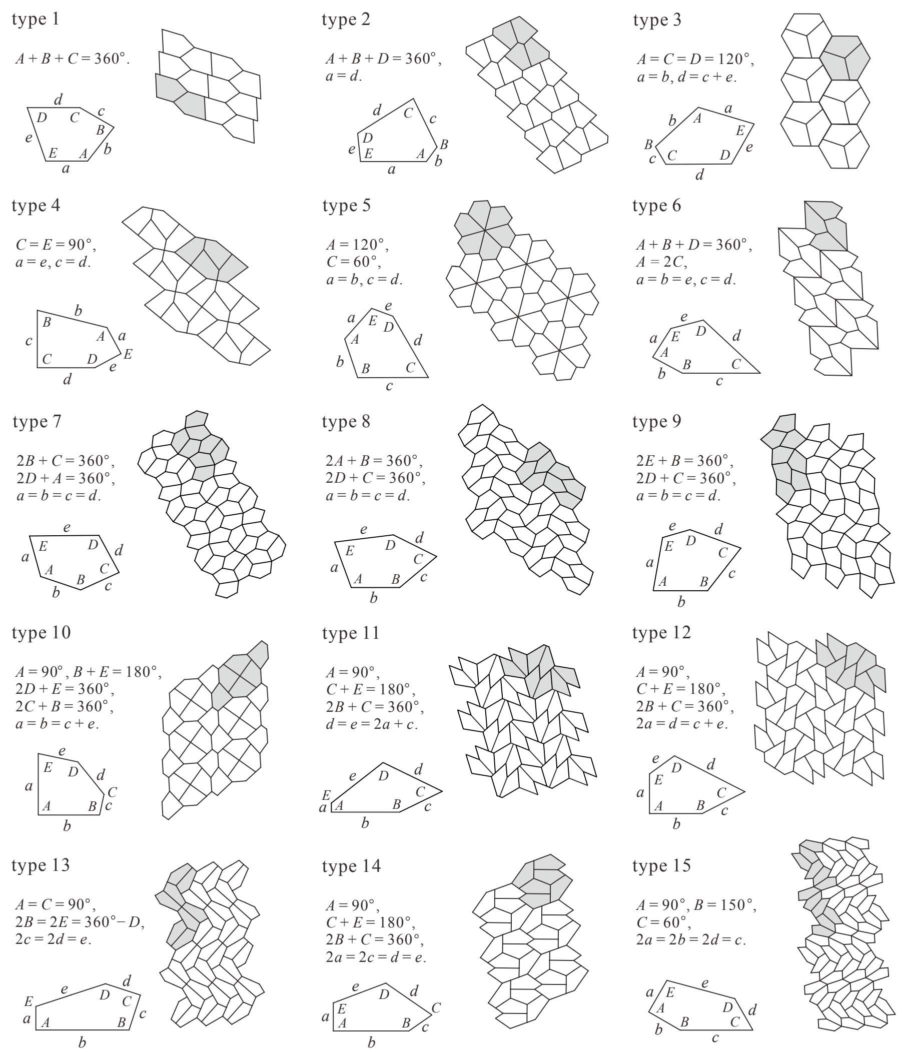

Figure 6. Fifteen types of convex pentagonal tiles. Each of the convex pentagonal tiles is defined by some conditions between the lengths of the edges and the magnitudes of the angles, but some degrees of freedom remain. For example, a convex pentagonal tile belonging to type 1 satisfies that the sum of three consecutive angles is equal to 360°. This condition for type 1 is expressed as $A + B + C = 360^\circ$ in this figure. The pentagonal tiles of types 14 and 15 have one degree of freedom, that of size. For example, the value of C of the pentagonal tile of type 14 is $\cos^{-1}((3\sqrt{57} - 17)/16) \approx 1.2099$ rad $\approx 69.32^\circ$.

Any convex quadrilateral can generate a periodic edge-to-edge tiling by using a fundamental region of a convex hexagon with edges that have parallel opposite sides and are equal in length, which is made by the two quadrilaterals. Any triangle can generate a periodic edge-to-edge tiling by using a fundamental region of a parallelogram that is made by the two triangles. Thus, we obtain Theorem 1.

□

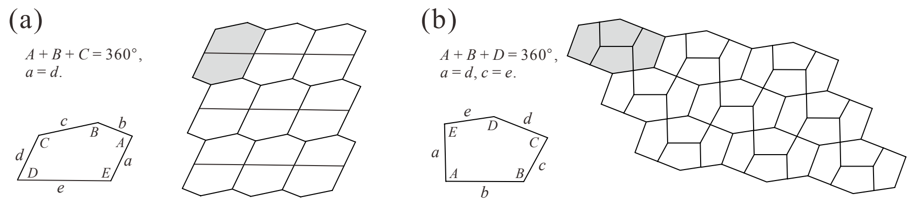

Figure 7. Examples of edge-to-edge tilings by convex pentagonal tiles that belong to type 1 or type 2. The pale gray pentagons in each tiling indicate the fundamental region. (a) Convex pentagonal tiles that belong to type 1. (b) Convex pentagonal tiles that belong to type 2 [14, 16].

4. Further Consequences

From the known results of convex polygonal tiles, if there is a single convex polygon such as an aperiodic prototile with no matching condition on the edges, the tile is a convex pentagonal tile that can generate an aperiodic non-edge-to-edge tiling. Further, if the complete list of all types of convex pentagonal tiles is obtained and if all convex pentagonal tiles in the complete list can generate a periodic tiling, no single convex polygon can be an aperiodic prototile.

In 2012, we found Theorem 1 at the same time we presented Theorem in [17]. In May 2017, Michaël Rao declared that the complete list of types of convex pentagonal tiles was obtained [9, 21]. That is, he proposed a proof of the property that there is not a single convex polygon such as an aperiodic prototile with no matching condition on the edges. If Rao's approach is valid, the fact "no single convex polygon can be an aperiodic prototile without matching conditions" is decided.

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